International Journal of Advanced Mathematical Sciences, *3 (1) (2015) 1-5*
www.sciencepubco.com/index.php/IJAMS
©Science Publishing Corporation
doi: 10.14419/ijams.v3i1.3835
Research Paper

Blow-up result in a Cauchy problem for the nonlinear viscoelastic Petrovsky equation

Erhan Pişkin

Dicle University, Department of Mathematics, 21280 Diyarbakır, Turkey
E-mail: episkin@dicle.edu.tr

Abstract

In this paper, we consider a Cauchy problem for the nonlinear viscoelastic Petrovsky equation. We obtain the blow up of solutions by applying a lemma due to Zhou.

Keywords: *Blow Up; Cauchy Problem; Nonlinear Viscoelastic Petrovsky Equation.*

1. Introduction

In [5], Li et al. considered the following nonlinear viscoelastic Petrovsky problem

$$\begin{cases} u_{tt}+\Delta^2 u-\int_0^t g(t-\tau)\Delta^2 u(t,\tau)d\tau-\Delta u_{tt}-\Delta u_t+|u_t|^{m-1}u_t=|u|^{p-1}u, & x\in\Omega,\ t>0, \\ u(x,t)=\dfrac{\partial u(x,t)}{\partial \upsilon}=0, & x\in\Omega, \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & x\in\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $R^n\,(n\geq 1)$ with a smooth boundary $\partial\Omega$, $m,p\geq 1$; υ is the unit outer normal on $\partial\Omega$; and g is a nonnegative memory term. They established some asymptotic behavior and blow up results for solutions with positive initial energy.
Guesmia [3] studied the problem

$$u_{tt}+\Delta^2 u+q(x)u+g(u_t)=0, \tag{2}$$

where $q:\Omega\to R^+$ is a bounded function? Under some assumptions, he showed the solution of (2) decay results by using the semigroup method. In [7], Messaoudi investigated the semilinear Petrovsky equation

$$u_{tt}+\Delta^2 u+|u_t|^{m-1}u_t=|u|^{p-1}u. \tag{3}$$

He showed that the solution blows up in finite time if $p>m$ and while it exists globally if $p\leq m$. in [9], Wu and Tsai showed that the solution of (3) is global under some conditions. Also, Chen and Zhou [2] studied the blow up of the solution of (3).
Recently, Li et al. [6] considered the following Petrovsky equation

$$u_{tt}+\Delta^2 u-\Delta u_t+|u_t|^{m-1}u_t=|u|^{p-1}u. \tag{4}$$

The authors obtained global existence, decay and blow up of the solution. Very recently, Pişkin and Polat [8] studied the decay of the solution of the problem (4).
In this paper, our aim is to extend the result of [5], established in bounded domains, to the problem in unbounded domains. Namely, we consider the following Cauchy problem

$$\begin{cases} u_{tt}+\Delta^2 u-\int_0^t g(t-\tau)\Delta^2 u(t,\tau)d\tau-\Delta u_{tt}-\Delta u_t+|u_t|^{m-1}u_t=|u|^{p-1}u, & x\in R^n,\ t>0, \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & x\in R^n \end{cases} \tag{5}$$

where g, u_0, u_1 are functions to be specified later.

This paper is organized as follows. In section 2, we present some notations, lemmas, and the local existence theorem. In section 3, under suitable conditions on the initial data, we prove a finite time blow up result.

2. Preliminary notes

In this section, we give some assumptions and lemmas which will be used throughout this work. Hereafter we denote by $\|.\|$ and $\|.\|_p$ the norm of $L^2(R^n)$ and $L^p(R^n)$, respectively. First, we make the following assumptions

(G) $g: R^+ \to R^+$ is a nonincreasing differentiable function such that

$1-\int_0^\infty g(\tau)d\tau = l > 0,\ g'(t)\le 0,\ t\ge 0.$

Next, we state the local existence theorem of the problem (5), which can be established by combining the arguments of [1], [7].

Theorem 1: *(Local existence). Suppose that (G) holds, and* $1<p<\infty$ *if* $n=1,2,3,4,$ *and* $1<p<\frac{n}{n-4}$ *if* $n\ge 5.$ *Then for any initial data* $(u_0,u_1)\in H^2(R^n)\times H^1(R^n),$ *with compact support, the problem (5) has a unique local solution*

$u\in C\left([0,T);H^2(R^n)\right)\cap C^1\left([0,T);L^2(R^n)\right),$

$u_t\in L^2\left([0,T);H^1(R^n)\right)\cap L^2(\Omega\times[0,T))$

for T small enough.

To obtain the result of this paper, we will introduce the modified energy functional

$$E(t)=\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla u\|^2+\frac{1}{2}\left(1-\int_0^t g(\tau)d\tau\right)\|\Delta u\|^2+\frac{1}{2}(g\circ\Delta u)(t)-\frac{1}{p+1}\|u\|_{p+1}^{p+1}, \tag{6}$$

where

$(g\circ v)(t)=\int_0^t g(t-\tau)\|v(t)-v(\tau)\|d\tau.$

The next lemma shows that our energy functional (6) is a nonincreasing function along the solution of (5).

Lemma 2: $E(t)$ *is a nonincreasing function for* $t\ge 0$ *and*

$$E'(t)=-\left(\|\nabla u_t\|^2+\|u_t\|^2\right)\le 0. \tag{7}$$

Proof: By multiplying the equation in (5) by u_t and integrating over R^n, we obtain (7).

3. Blow up of solutions

In this section, we shall show that the solution of the problem (5) blow up in finite time, by the similar arguments as in [4]. For the purpose, we give the lemma.

Lemma 3: *[10] Suppose that* $\psi(t)$ *is a twice continuously differentiable function satisfying*

$$\begin{cases} \psi''(t)+\psi'(t)\ge C_0(t+L)^\beta \psi^{1+\alpha}(t),\ t>0, \\ \psi(0)>0,\ \psi'(0)\ge 0, \end{cases}$$

where $C_0, L>0, -1<\beta\le 0, \alpha>0$ are constants. Then, $\psi(t)$ blow up in finite time.

Theorem 4: *Suppose that (G) holds, and* $1<p<\infty$ *if* $n=1,2,3,4,$ *and* $1<p<\frac{n}{n-4}$ *if* $n\ge 5.$ *Assume further that*

$$\int_0^t g(\tau)d\tau<\frac{p^2-1}{p^2}. \tag{8}$$

Then for any initial data $(u_0,u_1)\in H^2\left(R^n\right)\times H^1\left(R^n\right)$, with compact support, satisfying

$E(0)\le 0,\ \int_{R^n} u_0u_1dx\ge 0,\ \|\nabla u_0\|^2\le\|u_0\|^2,$

Then the corresponding solution blows up in finite time. In other words, there exists a positive constant T^* such that $\lim_{t\to T^*}\|u\|^2=\infty.$

Proof: By multiplying the equation in (5) by u_t and integrating over R^n, using integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\int_{R^n}|u_t|^2dx+\int_{R^n}|\Delta u|^2dx+\int_{R^n}|\nabla u_t|^2dx\right)+\int_{R^n}|\nabla u_t|^2dx+\int_{R^n}|u_t|^2dx-\int_0^t g(t-\tau)\int_{R^n}\Delta u(\tau)\Delta u_t(t)dxd\tau$$
$$=\frac{1}{p+1}\frac{d}{dt}\int_{R^n}|u|^{p+1}dx, \tag{9}$$

the last term on the left side of (9) can be estimated as follows

$$\begin{aligned}\int_0^t g(t-\tau)\int_{R^n}\Delta u\Delta u_t dxd\tau&=\int_0^t g(t-\tau)\int_{R^n}\left[\Delta u(\tau)-\Delta u(t)\right]\Delta u_t(t)dxd\tau+\int_0^t g(t-\tau)\int_{R^n}\Delta u(t)\Delta u_t(t)dxd\tau\\&=-\frac{1}{2}\int_0^t g(t-\tau)\frac{d}{dt}\int_{R^n}\left[\Delta u(\tau)-\Delta u(t)\right]^2dxd\tau+\int_0^t g(t-\tau)\left(\frac{1}{2}\frac{d}{dt}\int_{R^n}|\Delta u(t)|^2dx\right)d\tau\\&=-\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(t-\tau)\int_{R^n}\left[\Delta u(\tau)-\Delta u(t)\right]^2dxd\tau\right]+\frac{1}{2}\int_0^t g'(t-\tau)\left(\int_{R^n}\left[\Delta u(\tau)-\Delta u(t)\right]^2dx\right)d\tau\\&\quad+\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(\tau)\int_{R^n}|\Delta u(t)|^2dxd\tau\right]-\frac{1}{2}g(t)\int_{R^n}|\Delta u(t)|^2dx.\end{aligned} \tag{10}$$

Inserting (10) into (9), to get

$$\frac{1}{2}\frac{d}{dt}\left[\int_{R^n}|u_t|^2dx+\int_{R^n}|\Delta u|^2dx+\int_{R^n}|\nabla u_t|^2dx-\frac{2}{p+1}\int_{R^n}|u|^{p+1}dx\right.$$
$$\left.+\int_0^t g(t-\tau)\int_{R^n}\left[\Delta u(\tau)-\Delta u(t)\right]^2dxd\tau-\int_0^t g(\tau)\int_{R^n}|\Delta u(t)|^2dxd\tau\right]$$
$$=\frac{1}{2}\int_0^t g'(t-\tau)\int_{R^n}\left[\Delta u(\tau)-\Delta u(t)\right]^2dxd\tau-\frac{1}{2}g(t)\int_{R^n}|\Delta u(t)|^2dx$$
$$-\int_{R^n}|u_t|^2dx-\int_{R^n}|\nabla u_t|^2dx.$$

To apply Lemma 3, we define

$$\psi(t)=\frac{1}{2}\int_{R^n}\left(|u|^2+|\nabla u|^2\right)dx. \tag{11}$$

Therefore

$$\psi'(t)=\int_{R^n}(uu_t+\nabla u\nabla u_t)dx, \tag{12}$$

and

$$\psi''(t)=\int_{R^n}\left(uu_{tt}+|u_t|^2+\nabla u\nabla u_{tt}+|\nabla u_t|^2\right)dx. \tag{13}$$

Then, eq (5) is used to estimate (13) as follows

$$\begin{aligned}\psi''(t)&=\int_{R^n}\left(uu_{tt}+|u_t|^2+\nabla u\nabla u_{tt}+|\nabla u_t|^2\right)dx\\&=-\int_{R^n}|\Delta u|^2dx-\int_0^t\Delta u(t)\int_{R^n}g(t-\tau)\Delta u(\tau)dxd\tau-\int_{R^n}\nabla u\nabla u_tdx\\&\quad-\int_{R^n}uu_tdx+\int_{R^n}u|u|^{p-1}udx+\int_{R^n}\left(|u_t|^2+|\nabla u_t|^2\right)dx.\end{aligned} \tag{14}$$

On using

$$\int_0^t\Delta u(t)\int_{R^n}g(t-\tau)\Delta u(\tau)dxd\tau=\int_0^t g(t-\tau)\int_{R^n}\Delta u(t)\left[\Delta u(\tau)-\Delta u(t)\right]dxd\tau+\int_0^t g(\tau)d\tau\int_{R^n}|\Delta u(t)|^2dx.$$

Eq. (14) becomes

$$\begin{aligned}\psi''(t)&=-\left(1-\int_0^t g(\tau)d\tau\right)\int_{R^n}|\Delta u(t)|^2dx-\int_0^t g(t-\tau)\int_{R^n}\Delta u(t)\left[\Delta u(\tau)-\Delta u(t)\right]dxd\tau\\&\quad-\int_{R^n}\nabla u\nabla u_tdx-\int_{R^n}uu_tdx+\int_{R^n}|u|^{p+1}dx+\int_{R^n}\left(|u_t|^2+|\nabla u_t|^2\right)dx.\end{aligned} \tag{15}$$

We then use Young inequality to estimate the second term in (15). Namely,

$$\int_{R^n}\Delta u(t)\int_0^t g(t-\tau)\left[\Delta u(\tau)-\Delta u(t)\right]d\tau dx\le\delta\int_{R^n}|\Delta u(t)|^2dx+\frac{1}{4\delta}\int_{R^n}\left|\int_0^t g(t-\tau)\left(\Delta u(\tau)-\Delta u(\tau)\right)d\tau\right|^2dx$$

$$\le \delta \int_{R^n} |\Delta u(t)|^2 dx + \frac{1}{4\delta}\left(\int_0^t g(\tau)d\tau\right)(g \circ \Delta u). \tag{16}$$

By combining (15) and (16), we get

$$\psi''(t) \ge -\left(1-\int_0^t g(\tau)d\tau\right)\int_{R^n} |\Delta u(t)|^2 dx - \delta\int_{R^n} |\Delta u(t)|^2 dx - \frac{1}{4\delta}\left(\int_0^t g(\tau)d\tau\right)(g \circ \Delta u)$$

$$-\int_{R^n} \nabla u \nabla u_t dx - \int_{R^n} u u_t dx + \int_{R^n} |u|^{p+1} dx + \int_{R^n}\left(|u_t|^2 + |\nabla u_t|^2\right)dx. \tag{17}$$

From (12), (13) and (17), we obtain

$$\psi''(t) + \psi'(t) \ge -\left(1+\delta-\int_0^t g(\tau)d\tau\right)\int_{R^n} |\Delta u(t)|^2 dx - \frac{1}{4\delta}\left(\int_0^t g(\tau)d\tau\right)(g \circ \Delta u)$$

$$+\int_{R^n} |u|^{p+1} dx + \int_{R^n}\left(|u_t|^2 + |\nabla u_t|^2\right)dx. \tag{18}$$

Now, we exploit (6) to substitute for $(g \circ \Delta u)(t)$;

$$(g \circ \Delta u)(t) = 2E(t) - \|u_t\|^2 - \|\nabla u\|^2 - \left(1-\int_0^t g(\tau)d\tau\right)\|\Delta u\|^2 + \frac{2}{p+1}\|u\|_{p+1}^{p+1}.$$

Thus (18) takes the form

$$\psi''(t) + \psi'(t) \ge -\frac{1}{2\delta}\left(\int_0^t g(\tau)d\tau\right)E(t) + \left[\frac{1}{4\delta}\left(\int_0^t g(\tau)d\tau\right)\left(1-\int_0^t g(\tau)d\tau\right) - \left(1+\delta-\int_0^t g(\tau)d\tau\right)\right]\|\Delta u\|^2$$

$$+\left[1+\frac{1}{4\delta}\left(\int_0^t g(\tau)d\tau\right)\right]\|u_t\|^2 + \frac{1}{4\delta}\left(\int_0^t g(\tau)d\tau\right)\|\nabla u\|^2 + \|\nabla u_t\|^2 + \left[1-\frac{1}{4\delta}\left(\int_0^t g(\tau)d\tau\right)\frac{2}{p+1}\right]\|u\|_{p+1}^{p+1}. \tag{19}$$

At this point we choose $\delta > 0$ so that

$$\frac{1}{4\delta}\left(\int_0^t g(\tau)d\tau\right)\left(1-\int_0^t g(\tau)d\tau\right) - \left(1+\delta-\int_0^t g(\tau)d\tau\right) \ge 0$$

and

$$1-\frac{1}{4\delta}\left(\int_0^t g(\tau)d\tau\right)\frac{2}{p+1} > 0.$$

This is, of course, possible by (8). We then conclude, from (19), that

$$\psi''(t) + \psi'(t) \ge \gamma \|u\|_{p+1}^{p+1}. \tag{20}$$

Now, we use Hölder inequality to estimate $\|u\|_{p+1}^{p+1}$ as follows

$$\int_{R^n} |u|^2 dx \le \left(\int_{R^n} |u|^{p+1} dx\right)^{\frac{2}{p+1}} \left(\int_{B(t+L)} 1 dx\right)^{\frac{p-1}{p+1}},$$

where $L > 0$ is such that

$$\operatorname{supp}\{u_0(x), u_1(x)\} \subset B(L),$$

and $B(t+L)$ is the ball, with radius $t+L$, centered at the origin. If we call W_n the volume of the unit ball then

$$\int_{R^n} |u|^{p+1} dx \ge \left(\int_{R^n} |u|^2 dx\right)^{\frac{p+1}{2}} \left(W_n (t+L)^n\right)^{\frac{1-p}{2}}. \tag{21}$$

From the definition of $\psi(t)$, we get

$$[2\psi(t)]^{\frac{p+1}{2}} = \left[\int_{R^n}\left(|u|^2 + |\nabla u|^2\right)dx\right]^{\frac{p+1}{2}}$$

$$\le 2^{\frac{p-1}{2}}\left[\left(\int_{R^n} |u|^2 dx\right)^{\frac{p+1}{2}} + \left(\int_{R^n} |\nabla u|^2 dx\right)^{\frac{p+1}{2}}\right]. \tag{22}$$

Combining (20)-(21), we have

$$\psi''(t) + \psi'(t) \ge \gamma\left[2\psi^{\frac{p+1}{2}}(t) - \left(\|\nabla u\|^2\right)^{\frac{p+1}{2}}\right]\left(W_n (t+L)^n\right)^{\frac{1-p}{2}}.$$

From assumptions of Theorem, we deduce by continuity that there exists $T_* \le T$ such that

$$\|\nabla u\|^2 - \|u\|^2 \le 0, \ \forall t \in [0, T_*),$$

so

$$\psi^{\frac{p+1}{2}}(t)-\left(\|\nabla u\|^2\right)^{\frac{p+1}{2}}\geq 0.$$

Consequently, (22) implies that

$$\psi''(t)+\psi'(t)\geq \gamma\psi^{\frac{p+1}{2}}(t)(W_n)^{\frac{1-p}{2}}(t+L)^{n\frac{1-p}{2}}.$$

It is easy to verify that the requirements of Lemma 3 are satisfied by

$$C_0=\gamma(W_n)^{\frac{1-p}{2}}>0,\ \beta=n\frac{1-p}{2},\ \alpha=\frac{p+1}{2}>0.$$

Therefore $\psi(t)$ blow up in finite.

References

[1] N. E. Amroun and A. Benaissa, Global existence and energy decay of solutions to a Petrovsky equation with general nonlinear dissipation and source term, Georgian Math J, 13(3) (2006) 397-410.

[2] W. Chen and Y. Zhou, Global nonexistence for a semilinear Petrovsky equation, Nonlinear Analysis, 70 (2009) 3203-3208. http://dx.doi.org/10.1016/j.na.2008.04.024.

[3] A. Guesmia, Existence globale ET stabilisation interne nonlin’eaire d'un syst`eme dePetrovsky, Bulletin of the Belgian Mathematical Society, 5(4) (1998) 583-594.

[4] M. Kafini and M.I. Mustafa, Blow up result in a Cauchy viscoelastic problem with strong damping and dispersive, Nonlinear Analysis: RWA, 20 (2014) 14-20. http://dx.doi.org/10.1016/j.nonrwa.2014.04.005.

[5] G. Li, Y. Sun and W. Liu, On asymptotic behavior and blow-up of solutions for a nonlinear viscoelastic Petrovsky equation with positive initial energy, J Function Spaces Appl, (2013) 1-7.

[6] G. Li, Y. Sun and W. Liu, Global existence and blow up of solutions for a strongly damped Petrovsky system with nonlinear damping, Appl. Anal, 91(3) (2012) 575-586. http://dx.doi.org/10.1080/00036811.2010.550576.

[7] S.A. Messaoudi, Global existence and nonexistence in a system of Petrovsky, J Math Anal Appl, 265(2) (2002) 296-308. http://dx.doi.org/10.1006/jmaa.2001.7697.

[8] E. Pişkin and N. Polat, on the decay of solutions for a nonlinear Petrovsky equation, Mathematical Sciences Letters, 3(1) (2014) 43-47. http://dx.doi.org/10.12785/msl/030107.

[9] S.T. Wu and L.Y. Tsai, on global solutions and blow-up of solutions for a nonlinearly damped Petrovsky system, Taiwanese J. Math. 13 (2A) (2009) 545-558.

[10] Y. Zhou, A blow-up result for a nonlinear wave equation with damping and vanishing initial energy in, Appl. Math. Lett, 18 (2005) 281-286. http://dx.doi.org/10.1016/j.aml.2003.07.018.