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Research Paper

Necessary and sufficient conditions for oscillations of first order neutral delay difference equations with constant coefficients

A. Murugesan *, P. Sowmiya

Department of Mathematics, Government Arts College (Autonomous), Salem-636 007, Tamil Nadu, India
**Corresponding author E-mail: amurugesan3@gmail.com*

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Abstract

In this paper, we establish the necessary and sufficient conditions for oscillation of the following first order neutral delay difference equation

$$\Delta[x(n) + px(n-\tau)] + qx(n-\sigma) = 0, \qquad n \geq n_0, \qquad (*)$$

where τ and σ are positive integers, $p \neq 0$ is a real number and q is a positive real number. We proved that every solution of $(*)$ oscillates if and only if its characteristic equation

$$(\lambda - 1)(1 + p\lambda^{-\tau}) + q\lambda^{-\sigma} = 0 \qquad (**)$$

has no positive roots.

Keywords: *Neutral, delay difference equation, oscillatory properties.*

1. Introduction

In this paper, we consider the first order neutral delay difference equation of the form

$$\Delta[x(n) + px(n-\tau)] + qx(n-\sigma) = 0, \qquad n \geq n_0, \qquad (1)$$

where Δ is the forward difference operator given by $\Delta x(n) = x(n+1) - x(n)$, τ and σ are positive integers, $p \neq 0$ is a real number and q is a positive real number.

Let $s = \max\{\tau, \sigma\}$ and n_0 be a fixed nonnegative integer. By a solution of (1), we mean a nontrivial real sequence $\{x(n)\}$ which is defined for all positive integer $n \geq n_0 - s$ and satisfies (1) for $n \geq n_0$. A solution $\{x(n)\}$ of (1) is said to be oscillatory if for every positive integer $N > n_0$, there exists $n \geq N$ such that $x(n)x(n+1) \leq 0$, otherwise $\{x(n)\}$ is said to be nonoscillatory.

The oscillation theory of neutral delay difference equation has been extensively developed during the past few years. See, for example, [4,8,9,11] and the references cited therein. For the general background on difference equations, see the monographs [2,3,5-7].

In [10], Xiaohui Gong et al. proved that if $-1 < p < 0, \tau - \sigma > 1$, then $q - \frac{\tau}{\tau-\sigma-1}(1 - p^{\frac{1}{\tau}}) > 0$ is a sufficient conditions for oscillation of Eq. (1).

In [12], Zhou et al. established the necessary and sufficient conditions for oscillation of all solutions of Eq. (1) by investigating the nature of the roots of its characteristic equation.

Our aim in this paper is to obtain a necessary and sufficient condition under which all solutions of the Eq.(1) oscillate. Indeed we prove that every solution of Eq. (1) oscillates if and only if its characteristic equation

$$(\lambda - 1)(1 + p\lambda^{-\tau}) + q\lambda^{-\sigma} = 0 \tag{2}$$

has no positive real roots. That is, the oscillatory character of the solution is determined by the roots of the characteristic equation. It is to be noted however, that in general the behavior of solutions of neutral delay difference equations exhibit features which are not true for nonneutral delay difference equations.

In the sequel all functional inequalities that we write are assumed to hold eventually, that is for all sufficiently large n.

2. Some useful lemmas

In this section we establish some useful lemmas which will be used in the proof of our main theorem.

Lemma 2.1 *Let $\{x(n)\}$ be an eventually positive solution of (1).*

Set

$$z(n) = x(n) + px(n - \tau) \tag{3}$$

If $p > -1$, then $\{z(n)\}$ is eventually positive solution of (1), decreasing and $\lim_{n\to\infty} z(n) = 0$.

Proof. We see that $z(n)$ is a linear combination of solutions of Eq. (1). By the linearity of Eq. (1) and its autonomous nature, $\{z(n)\}$ is also a solution of Eq. (1).

From (1) and (3), we have

$$\Delta z(n) = -qx(n - z) < 0.$$

This shows that $\{z(n)\}$ is strictly decreasing sequence. We will prove that $z(n) > 0$. For otherwise, $z(n) < 0$. This implies that $x(n) < -px(n-\tau)$. This is possible only when $-1 < p < 0$. Inductively, we have $x(n+k\tau) < (-p)^k x(n)$. This shows that $x(n) \to 0$ as $n \to \infty$ which leads to $z(n) \to 0$ as $n \to \infty$. This is a contradiction.

Hence $\{z(n)\}$ is decreasing, eventually positive solution of (1). Then we have $\lim_{n\to\infty} z(n) = l$ where $l \geq 0$.

We will prove that $l = 0$. For otherwise

$$\lim_{n\to\infty} \Delta[z(n) + pz(n - \tau)] = \lim_{n\to\infty}(-qz(n - \sigma)) = -ql.$$

Then

$$\lim_{n\to\infty} [z(n) + pz(n - \tau)] = -\infty.$$

This is a contradiction.

Hence

$$\lim_{n\to\infty} z(n) = 0.$$

Lemma 2.2 *Let $\{x(n)\}$ be an eventually positive solution of (1).*
Set $z(n) = -x(n) - px(n-\tau)$.
If $p < -1$, then $\{z(n)\}$ is an eventually positive solution of (1), increasing and

$$\lim_{n\to\infty} z(n) = +\infty.$$

The proof of the lemma is similar to that of the Lemma 2.1 and so the proof is omitted.

3. Main results

Our main result is the following:

Theorem 3.1 *A necessary and sufficient condition for all solutions of Eq. (1) to oscillate is that the characteristic equation (2) has no positive roots.*

Proof. Observe that for $p = 0$, Eq. (1) leads to

$$\Delta x(n) + qx(n-\sigma) = 0, \quad n \geq n_0 \tag{4}$$

and

$$q\frac{(\sigma+1)^{\sigma+1}}{\sigma^\sigma} > 1 \tag{5}$$

is a necessary and sufficient condition for all solutions of (4) to oscillate (*see*, [5]).
Note that in this case the characteristic equation is

$$\lambda - 1 + q\lambda^{-\sigma} = 0 \tag{6}$$

and (5) is equivalent to the fact that (6) has no positive roots. Also for $p = -1$, Eq. (1) leads to

$$\Delta[x(n) - x(n-\tau)] + qx(n-\sigma) = 0, \quad n \geq n_0 \tag{7}$$

and every solution of (7) oscillates (*see*, [1]).

From these observations, it is clear that to prove the theorem we have to consider the following three cases for p:

(i) $-1 < p < 0$,

(ii) $p > 0$, and

(iii) $p < -1$.

The theorem will be proved in the contrapositive form: There is a nonoscillatory solution of Eq. (1) if and only if the characteristic equation (2) has a positive root.

Assume first that (2) has a positive root λ_0. Then Eq. (1) has the nonoscillatory solution

$$x(n) = \{\lambda_0^n\}.$$

Assume, conversely, that there is a nonoscillatory solution $\{y(n)\}$ of (1) which, without loss of generality, can be considered, eventually positive.
Consider now the following cases:

(i) *The case $-1 < p < 0$.*

Set

$$x(n) = y(n) + py(n-\tau) \tag{8}$$

and

$$z(n) = x(n) + px(n-\tau). \tag{9}$$

Then $\{x(n)\}$ and $\{z(n)\}$ are eventually positive solutions of (1) and decreasing sequences. It is easy to see that $z(n) < x(n-\sigma)$. Define the set

$$\wedge(z) = \{\lambda > 1 : \Delta z(n) + (\lambda - 1)z(n) < 0 \quad \textit{eventually}\}. \tag{10}$$

From (1) we have

$$0 = \Delta z(n) + qx(n-\sigma) > \Delta z(n) + qz(n), \quad \textit{eventually},$$

so that $1 + q \in \wedge(z)$. That is, $\wedge(z)$ is nonempty.
On the other hand, it follows that

$$0 = \Delta z(n) + qx(n-\sigma) = \Delta z(n) + qz(n-\sigma) - pqx(n-\tau-\sigma),$$

and summing from n to $n+\tau-1$ we find

$$z(n+\tau) - z(n) + q\sum_{s=n}^{n+\tau-1} z(s-\sigma) - pq\sum_{s=n}^{n+\tau-1} x(s-\tau-\sigma) = 0.$$

Taking into account that both $\{z(n)\}$ and $\{x(n)\}$ are positive and decreasing, the last equation yields.

$$z(n+\tau) - z(n) + q\tau z(n+\tau-\sigma) - pq\tau x(n-\sigma) \leq 0,$$

which implies that

$$-pq\tau x(n-\sigma) \leq z(n).$$

Thus

$$0 = \Delta z(n) + qx(n-\sigma) \leq \Delta z(n) + \left(\frac{1}{-p\tau}\right) z(n), \quad \textit{eventually}.$$

Therefore $\lambda_0 = 1 - \frac{1}{p\tau} > 1$ is an upper bound of $\wedge(z)$ which does not depend on z. Thus $\wedge(z)$ is nonempty and bounded from above.
Let $\lambda \in \wedge(z)$ and consider the function

$$w(n) = Tz = z(n) + pz(n-\tau). \tag{11}$$

Set

$$m = \inf_{\lambda > 1}\left\{\left(\frac{1}{\lambda} - 1\right)(1 + p\lambda^{\tau}) + q\lambda^{\sigma}\right\}. \tag{12}$$

Assume, for the sake of contradiction, that (2) has no positive roots. Furthermore

$$\lim_{\lambda\to\infty}\left\{\left(\frac{1}{\lambda} - 1\right)(1 + p\lambda^{\tau}) + q\lambda^{\sigma}\right\} = +\infty$$

and therefore m is positive. We will show that $2 - \frac{1}{\lambda} + m \in \wedge(w)$. From (1) and (11) and the fact that both $\{z(n)\}$ and $\{w(n)\}$ are solutions of (1), we obtain

$$\Delta w(n) = -qz(n-\sigma). \tag{13}$$

Define $u(n) = \lambda^n z(n)$. Then

$$\begin{aligned}\Delta u(n) &= \lambda^n \Delta z(n) + z(n+1)\Delta\lambda^n \\ &= \lambda^n \Delta z(n) + z(n+1)\left(\lambda^{n+1} - \lambda^n\right)\end{aligned}$$

$$= \lambda^n \left[\Delta z(n) + (\lambda - 1) z(n+1)\right]$$
$$\leq \lambda^n \left[\Delta z(n) + (\lambda - 1) z(n)\right] < 0, \quad \textit{eventually}$$

and therefore $\{u(n)\}$ is eventually decreasing. Since $z(n) = \lambda^{-n} u(n)$, (11) and (13) give, respectively

$$w(n) = \lambda^{-n} u(n) + p\lambda^{-n}\lambda^{\tau} u(n-\tau)$$

and

$$\Delta w(n) = -q\lambda^{-n}\lambda^{\sigma} u(n-\sigma).$$

Now using the fact that $\{u(n)\}$ is decreasing, we obtain

$$\Delta w(n) + \left(1 - \tfrac{1}{\lambda} + m\right) w(n)$$
$$= -q\lambda^{-n}\lambda^{\sigma} u(n-\sigma) + \left(1 - \frac{1}{\lambda} + m\right)\left\{\lambda^{-n} u(n) + p\lambda^{-n}\lambda^{\tau} u(n-\tau)\right\} \tag{14}$$
$$= \lambda^{-n}\left\{-q\lambda^{\sigma} u(n-\sigma) + \left(1 - \frac{1}{\lambda} + m\right) u(n) + \left(1 - \frac{1}{\lambda} + m\right) p\lambda^{\tau} u(n-\tau)\right\}$$
$$\leq \lambda^{-n} u(n)\left\{-q\lambda^{\sigma} + \left(1 - \frac{1}{\lambda}\right) + m + \left(1 - \frac{1}{\lambda}\right) p\lambda^{\tau} + mp\lambda^{\tau}\right\}$$
$$< \lambda^{-n} u(n)\left\{-\left[\left(\frac{1}{\lambda} - 1\right)(1 + p\lambda^{\tau}) + q\lambda^{\sigma}\right] + m\right\}$$
$$\leq \lambda^{-n} u(n)[-m + m] = 0,$$

which implies that $2 - \frac{1}{\lambda} + m \in \wedge(w)$. Now set $z_0(n) = z(n)$, $z_1(n) = Tz_0 = w(n)$, $z_2(n) = Tz_1$ and in general $z_n = Tz_{n-1}$, $n = 1, 2, 3, \ldots$ and observe that for $\lambda \in \wedge(z) \equiv \wedge(z_0) \Rightarrow 2 - \frac{1}{\lambda} + nm \in \wedge(z_n)$, which is a contradiction, since λ_0 is a common upper bound for all $\wedge(z_n)$.

(ii) *The case* $p > 0$.

As in case (i) we set

$$z(n) = x(n) + px(n-\tau) \tag{9}$$

and

$$\wedge(z) = \left\{\lambda > 1 : \Delta z(n) + (\lambda - 1) z(n) < 0 \quad \textit{eventually}\right\}. \tag{10}$$

In this case using the characteristic equation of (1).

$$F(\lambda) = (\lambda - 1)(1 + p\lambda^{-\tau}) + q\lambda^{-\sigma} = 0.$$

We see that if $\tau \geq \sigma > 0$, $F(1) = q$ while $\lim_{\lambda \to 0} F(\lambda) = -\infty$ and therefore (1) always has nonoscillatory solutions. Thus $\sigma > \tau$ is a necessary condition for all solutions of (1) to oscillate. By the Lemma 2.1, $\{x(n)\}$ and $\{z(n)\}$ are eventually positive solution of (1) and decreasing. Thus

$$z(n) = x(n) + px(n-\tau) < x(n-\sigma) + px(n-\sigma) = (1+p)x(n-\sigma).$$

That is,

$$x(n-\sigma) > \frac{1}{1+p} z(n).$$

From (1) we have

$$0 = \Delta z(n) + qx(n-\sigma) > \Delta z(n) + \frac{q}{1+p} z(n), \quad \textit{eventually},$$

so that $1+\frac{q}{1+p}\in\wedge(z)$. That is $\wedge(z)$ is nonempty.

Now we will show that $\wedge(z)$ is bounded from above. Observe that Eq. (1) is autonomous and $z(n)$, given by (9), as a linear combinations of solutions of (1) is itself a solution of (1) and therefore

$$\Delta z(n)+p\Delta z(n-\tau)+qz(n-\sigma)=0.$$

Also $\Delta z(n)=-qx(n-\sigma)<0$ and $\Delta^2 z(n)=-q\Delta x(n-\sigma)>0$, that is, $\{\Delta z(n)\}$ is increasing and therefore $\Delta z(n)>\Delta z(n-\tau)$. Thus the last equation yields

$$(1+p)\Delta z(n-\tau)+qz(n-\sigma)\leq 0,$$

or

$$\Delta z(n)+\frac{q}{1+p}z(n-(\sigma-\tau))\leq 0. \tag{15}$$

Let $\kappa=\left[\frac{\sigma-\tau}{2}\right]$ where [.] denotes the greatest integer function.

Summing the inequality (15) from $n-\kappa$ to n, we have

$$\sum_{s=n-\kappa}^{n}\Delta z(s)+\frac{q}{1+p}\sum_{s=n-\kappa}^{n}z(s-(\sigma-\tau))\leq 0,$$

or

$$z(n+1)-z(n-\kappa)+\frac{q}{1+p}(\kappa+1)z(n-(\sigma-\tau))\leq 0,$$

or

$$\frac{q}{1+p}(\kappa+1)z(n-(\sigma-\tau))\leq z(n-\kappa),$$

or

$$z(n-(\sigma-\tau))\leq\frac{1+p}{q(\kappa+1)}z(n-\kappa). \tag{16}$$

Summing the inequality (15) from n to $n+\kappa$, we have

$$\sum_{s=n}^{n+\kappa}\Delta z(s)+\frac{q}{1+p}\sum_{s=n}^{n+\kappa}z(s-(\sigma-\tau))\leq 0,$$

or

$$z(n+\kappa+1)-z(n)+\frac{q}{1+p}(\kappa+1)z(n+\kappa-\sigma+\tau)\leq 0,$$

or

$$-z(n)+\frac{q}{1+p}(\kappa+1)z(n+\kappa-\sigma+\tau)\leq 0,$$

or

$$z(n+\kappa-\sigma+\tau)\leq\left(\frac{1+p}{q(\kappa+1)}\right)z(n). \tag{17}$$

Using the fact that $n+\kappa-(\sigma-\tau)\leq n-\kappa$ and $\{z(n)\}$ is decreasing,

$$z(n+\kappa-(\sigma-\tau))\geq z(n-\kappa). \tag{18}$$

Thus from (17) and (18), we have

$$z(n-\kappa)\leq z(n+\kappa-\sigma+\tau)\leq\frac{1+p}{q(\kappa+1)}z(n),$$

or

$$z(n-\kappa) \leq \left(\frac{1+p}{q(\kappa+1)}\right) z(n). \tag{19}$$

Using (19) in (16), we have

$$z\left(n-(\sigma-\tau)\right) \leq \left(\frac{1+p}{q(\kappa+1)}\right)\left(\frac{1+p}{q(\kappa+1)}\right) z(n),$$

or

$$z\left(n-(\sigma-\tau)\right) \leq \left(\frac{1+p}{q(\kappa+1)}\right)^2 z(n). \tag{20}$$

Next summing $\Delta z(n) + qx(n-\sigma) = 0$ from $n-\sigma+\tau$ to $n-1$, we obtain

$$z(n) - z(n-\sigma+\tau) + q(\sigma-\tau)x(n-\sigma-1) = 0.$$

Since $\{x(n)\}$ is positive and decreasing and $z(n)$ is positive, we have

$$-z(n-\sigma+\tau) + q(\sigma-\tau)x(n-\sigma) \leq 0,$$

or

$$q(\sigma-\tau)x(n-\sigma) \leq z(n-(\sigma-\tau)). \tag{21}$$

Combining (20) and (21), we obtain

$$q(\sigma-\tau)x(n-\sigma) \leq \left(\frac{1+p}{q(\kappa+1)}\right)^2 z(n),$$

or

$$x(n-\sigma) \leq \frac{(1+p)^2}{q^3(\sigma-\tau)(\kappa+1)^2} z(n).$$

Thus

$$0 = \Delta z(n) + qx(n-\sigma) \leq \Delta z(n) + q\frac{(1+p)^2}{q^3(\sigma-\tau)(\kappa+1)^2} z(n), \quad \textit{eventually},$$

or

$$0 \leq \Delta z(n) + \left(\frac{(1+p)^2}{q^2(\sigma-\tau)(\kappa+1)^2}\right) \Delta z(n), \quad \textit{eventually}.$$

Therefore $\lambda_0 = 1 + \frac{(1+p)^2}{q^2(\sigma-\tau)(\kappa+1)^2}$ is an upper bound of $\wedge(z)$ which does not depend on z. Thus $\wedge(z)$ is nonempty and bounded from above.

Now if we follow the same procedure as in the last part of the proof of case (i), considering $\lambda \in \wedge(z)$ and defining $w(n)$, m as in (11) and (12), we will show that $2 - \frac{1}{\lambda} + \mu \in \wedge(w)$, where $\mu = \frac{m}{1+p\lambda_0^\tau} > 0$.
Define the sequence $\{u(n)\}$ as in the case (i). Now

$$\Delta w(n) + \left(1 - \frac{1}{\lambda} + \mu\right) w(n) = -qz(n-\sigma) + \left(1 - \frac{1}{\lambda} + \mu\right)[z(n) + pz(n-\tau)]$$

$$= -q\lambda^{-n+\sigma} u(n-\sigma) + \left(1 - \frac{1}{\lambda} + \mu\right)\lambda^{-n} u(n) + \left(1 - \frac{1}{\lambda} + \mu\right) p\lambda^{-n+\tau} u(n-\tau)$$

(since $\{u(n)\}$ is decreasing and $\sigma > \tau$)

$$< \lambda^{-n} u(n-\sigma)\left\{-q\lambda^\sigma + \left(1 - \frac{1}{\lambda} + \mu\right) + \left(1 - \frac{1}{\lambda}\right) p\lambda^\tau + \mu p\lambda^\tau\right\}$$

$$= \lambda^{-n} u(n-\sigma) \left\{ -q\lambda^{\sigma} + \left(1 - \frac{1}{\lambda}\right)(1 + p\lambda^{\tau}) + \mu\,(1 + p\lambda^{\tau}) \right\}$$

$$\Delta w(n) + (1 - \frac{1}{\lambda} + \mu) w(n) < \lambda^{-n} u(n-\sigma) \left\{ -\left(\left(\frac{1}{\lambda} - 1 \right)(1 + p\lambda^{\tau}) + q\lambda^{\sigma} \right) + \mu\,(1 + p\lambda^{\tau}) \right\}$$

$$\leq \lambda^{-n} u(n-\sigma) \left\{ -m + \mu\,(1 + p\lambda_0^{\tau}) \right\} = 0$$

which implies that $2 - \frac{1}{\lambda} + \frac{m}{1+p\lambda_0^{\tau}} \in \wedge(w)$. Thus (as in case (i)) leads to a contradiction

(iii) *The case* $p < -1$.

First, we assume $\tau \geq \sigma$. Set

$$x(n) = -y(n) - py(n-\tau) \tag{22}$$

and

$$z(n) = -x(n) - px(n-\tau). \tag{23}$$

Then by Lemma 2.2 $\{x(n)\}$ and $\{z(n)\}$ are eventually positive solutions of (1) and increasing. From (23) we have

$$x(n-\sigma) > \frac{1}{-p} z(n).$$

Define the set

$$\wedge(z) = \left\{ \lambda > 1 : -\Delta z(n) + (\lambda - 1) z(n) < 0 \quad \textit{eventually} \right\}. \tag{24}$$

From (1)

$$0 = -\Delta z(n) + q x(n-\sigma) > -\Delta z(n) - \frac{q}{p} z(n),$$

so that $1 - \frac{q}{p} \in \wedge(z)$. That is , $\wedge(z)$ is nonempty.

Now we will show that $\wedge(z)$ is bounded from above. From (23) we have

$$x(n-\tau) = \frac{1}{-p}(x(n) + z(n))$$

and therefore

$$\Delta z(n) = q x(n-\sigma) = \frac{q}{-p}\left[x(n+\tau-\sigma) + z(n+\tau-\sigma) \right].$$

Summing the last equation from $n-\tau$ to $n-1$ and taking into account that $\{x(n)\}$ and $\{z(n)\}$ are eventually positive and increasing, we obtain

$$\sum_{s=n-\tau}^{n-1} \Delta z(s) = \frac{-q}{p} \sum_{s=n-\tau}^{n-1} (x(s+z-\sigma) + z(s+z-\sigma)),$$

or

$$z(n) - z(n-\tau) \geq \frac{-q\tau}{p}(x(n-\sigma) + z(n-\sigma)),$$

which implies that

$$x(n-\sigma) \leq \frac{-p}{q\tau} z(n), \quad \textit{eventually}.$$

Thus

$$0 = -\Delta z(n) + q x(n-\sigma) \leq -\Delta z(n) + \left[\left(1 - \frac{p}{\tau}\right) - 1 \right] z(n), \quad \textit{eventually}.$$

Therefore $\lambda_0 = 1 - \frac{p}{\tau}$ is an upper bound of $\wedge(z)$ which does not depend on z. Thus $\wedge(z)$ is nonempty and bounded from above.

Next we follow a procedure analogous to that given in the last part of the proof of case (i). Since $\wedge(z)$ is nonempty, there exist $\lambda \in \wedge(z)$.

Consider the sequence $\{w(n)\}$ where

$$w(n) = -z(n) - pz(n-\tau) \tag{25}$$

which is also a solution of (1) and therefore from (1), $\Delta w(n) = qz(n-\sigma)$.

Assume for the sake of contradiction, that (2) has no positive roots and set

$$m = \inf_{\lambda>1} \left\{(\lambda-1)(1+p\lambda^{-\tau}) + q\lambda^{-\sigma})\right\}. \tag{26}$$

Then m is positive. We shall show that $\lambda + \mu \in \wedge(w)$ where $\mu = \frac{m}{-p} > 0$. Define $u(n) = \lambda^{-n}z(n)$. Then

$$\Delta u(n) = \lambda^{-(n+1)}\left[\Delta z(n) - (\lambda-1)z(n)\right] > 0, eventually$$

and therefore $\{u(n)\}$ is eventually increasing sequence. Since $z(n) = \lambda^n u(n)$, similarly as in (14), we obtain

$$\begin{aligned}
&-\Delta w(n) + (\lambda - 1 + \mu)w(n) \\
&\quad = -q\lambda^{n-\sigma}u(n-\sigma) + (\lambda-1+\mu)\left[-\lambda^n u(n) - p\lambda^{n-\tau}u(n-\tau)\right] \\
&\quad \le \lambda^n u(n-\sigma)\left\{-q\lambda^{-\sigma} - (\lambda-1+\mu) + (\lambda-1+\mu)(-p)\lambda^{-\tau}\right\} \\
&\quad = \lambda^n u(n-\sigma)\left\{-(\lambda-1)\left(1+p\lambda^{-\tau}\right) - q\lambda^{-\sigma} - \mu - \mu p\lambda^{-\tau}\right\} \\
&\quad < \lambda^n u(n-\sigma)\left\{-m - \mu p\lambda^{-\tau}\right\} \\
&\quad < \lambda^n u(n-\sigma)\left\{-m + \mu(-p)\right\} = 0
\end{aligned}$$

which implies that $\lambda + \mu \in \wedge(w)$ and, as before, we are lead to a contradiction.

To complete the proof in this case that $p < -1$ we have to assume that $\sigma > \tau$. Here we set

$$z(n) = -x(n) - px(n-\tau) + q\sum_{s=n-\sigma}^{n-\tau-1} x(s)$$

which is a solution of (1) eventually positive.

Set

$$x(n) = -y(n) - py(n-\tau)$$

and

$$v(n) = -x(n) - px(n-\tau). \tag{27}$$

Then

$$z(n) = v(n) + q\sum_{s=n-\sigma}^{n-\tau-1} x(s) \tag{28}$$

Then $\{v(n)\}$ and $\{x(n)\}$ are eventually positive solution and increasing. Since $z(n)$ is a linear combination of eventually positive solutions $\{x(n)\}$ and $\{v(n)\}$, $\{z(n)\}$ is also an eventually positive solution of (1). From (1), we have $\Delta z(n) = qx(n-\tau) > 0$ which implies that $\{\Delta z(n)\}$ is increasing. Therefore (28) yields.

$$z(n) < -px(n-\tau) + q\sum_{s=n-\sigma}^{n-\tau-1} x(s),$$

or

$$z(n) < -px(n-\tau) + q(\sigma-\tau)x(n-\tau),$$

or

$$z(n) < (q(\sigma-\tau)-p)\, x(n-\tau),$$

or

$$x(n-\tau) > \frac{1}{(q(\sigma-\tau)-p)} z(n).$$

As before, we define the set

$$\wedge(z) = \{\lambda > 1 : -\Delta z(n) + (\lambda-1)z(n) < 0 \quad \textit{eventually}\}. \tag{29}$$

In view of the last inequality, we have

$$0 = -\Delta z(n) + qx(n-\tau) > -\Delta z(n) + \frac{q}{q(\sigma-\tau)-p} z(n), \quad \textit{eventually}$$

so that $\lambda_1 = 1 + \frac{q}{q(\sigma-\tau)-p} \in \wedge(z)$. That is, $\wedge(z)$ is nonempty.

Next, we will show that $\wedge(z)$ is bounded from above. Since $\{z(n)\}$ is a solution of (1) it follows that

$$\Delta z(n) + p\Delta z(n-\tau) + qz(n-\sigma) = 0$$

which implies

$$\Delta z(n) + p\Delta z(n-\tau) \le 0.$$

But $\Delta z(n) = qx(n-\tau)$ and thus

$$qx(n-\tau) + p\Delta z(n-\tau) \le 0.$$

Summing the last inequality from n to $n+\tau-1$ and taking into account that $\{x(n)\}$ is positive and increasing and $\{z(n)\}$ is eventually positive, we obtain

$$0 \ge q\sum_{s=n}^{n+\tau-1} x(s-\tau) + p\sum_{s=n}^{n+\tau-1} \Delta z(s-\tau)$$

$$\ge q\tau x(n-\tau) + p[z(n) - z(n-\tau)]$$

$$\ge q\tau x(n-\tau) + pz(n)$$

which implies

$$q\tau x(n-\tau) + pz(n) \le 0, \quad \textit{eventually}$$

or

$$x(n-\tau) \le \frac{(-p)}{q\tau} z(n), \quad \textit{eventually}.$$

Thus

$$0 = -\Delta z(n) + qx(n-\tau) \le -\Delta z(n) + \frac{(-p)}{\tau} z(n), \quad \textit{eventually}$$

which implies that $\lambda_0 = 1 - \frac{p}{\tau}$ is an upper bound of $\wedge(z)$ which does not depend on z. Thus $\wedge(z)$ is nonempty and bounded from above.

Now let $\lambda \in \wedge(z)$ with $\lambda \leq \lambda_1$. Consider the sequence $\{w(n)\}$ where

$$w(n) = -z(n) - pz(n-\tau) + q \sum_{s=n-\sigma}^{n-\tau-1} z(s).$$

Using the fact that $\{z(n)\}$ and $\{w(n)\}$ are solutions of (1), we obtain

$$\Delta[z(n) + pz(n-\tau)] = -qz(n-\sigma),$$

or

$$\Delta\left[w(n) - q \sum_{s=n-\sigma}^{n-\tau-1} z(s)\right] = qz(n-\sigma).$$

This implies

$$\Delta w(n) = qz(n-\tau).$$

We define m as in (26) and $u(n) = \lambda^{-n} z(n)$. We will show that $(\lambda - 1) + \mu \in \wedge(w)$ where

$$\mu = \frac{m\lambda_1^{\tau}}{\left(\frac{q}{\lambda_1 - 1} - p\right)}.$$

We have

$$-\Delta w(n) + (\lambda - 1 + \mu)w(n)$$

$$= [-qz(n-\tau)] + (\lambda - 1 + \mu)\left[-z(n) - pz(n-\tau) + q \sum_{s=n-\sigma}^{n-\tau-1} z(s)\right]$$

$$= -q\lambda^{n-\tau}u(n-\tau) + (\lambda - 1 + \mu)\left[-\lambda^n u(n) - p\lambda^{n-\tau}u(n-\tau) + q \sum_{s=n-\sigma}^{n-\tau-1} \lambda^s u(s)\right]$$

(and since $\{u(n)\}$ is increasing and $\sigma > \tau$)

$$= \lambda^n u(n-\tau)\left\{-q\lambda^{-\tau} + (\lambda - 1 + \mu)\left[-1 - p\lambda^{-\tau} + q\lambda^{-n} \sum_{s=n-\sigma}^{n-\tau-1} \lambda^s\right]\right\}$$

$$= \lambda^n u(n-\tau)\left\{-\left[(\lambda-1)(1+p\lambda^{-\tau}) + q\lambda^{-\sigma}\right] + \mu(-1 - p\lambda^{-\tau}) + \mu q\left(\frac{\lambda^{-\tau} - \lambda^{-\sigma}}{\lambda - 1}\right)\right\}$$

$$< \lambda^n u(n-\tau)\left\{-\left[(\lambda-1)(1+p\lambda^{-\tau}) + q\lambda^{-\sigma}\right] - \mu p\lambda^{-\tau} + \left[\frac{\mu q\lambda^{-\tau}}{\lambda - 1}\right]\right\}$$

$$= \lambda^n u(n-\tau)\left\{-\left[(\lambda-1)(1+p\lambda^{-\tau}) + q\lambda^{-\sigma}\right] + \mu\lambda^{-\tau}\left(\frac{q}{\lambda - 1} - p\right)\right\}$$

$$\leq \lambda^n u(n-\tau)\left\{-m + \mu\lambda_1^{-\tau}\left(\frac{q}{\lambda_1 - 1} - p\right)\right\} = 0$$

$$\leq \lambda^n u(n-\tau)\left\{-m + \mu\lambda_1^{-\tau}\left(\frac{q}{\lambda_1 - 1} - p\right)\right\} = 0$$

which implies that $\lambda + \mu \in \wedge(w)$ and we are lead to a contradiction.

The proof of the theorem is complete.

4. Remarks

In [1,10], several sufficient conditions involving p,τ, q and σ have been obtained under which all solutions of (1) oscillate. In the present paper we can very easily derive sufficient conditions in terms of p,τ, q and σ by comparing element of the set $\wedge(z)$ in each case. For example, in the case where $-1 < p < 0$ we found that $1 + q \in \wedge(z)$ while $1 - \frac{1}{p\tau}$ is an upper bound of $\wedge(z)$. Therefore if we assume that $q > -\frac{1}{p\tau}$ we are lead to a contradiction and every solution must oscillate. Summarizing we have the following:

Consider the first order neutral delay difference equation

$$\Delta[x(n)+px(n-\tau)]+qx(n-\sigma) = 0, \qquad n \geq n_0, \tag{1}$$

where τ and σ are positive integers, q is a positive constant and p is a real parameter. Then each of the following conditions

$$(-p)q\tau > 1 \quad when \quad -1 < p < 0, \tag{30}$$

$$\left(\frac{q}{1+p}\right)^3 (\sigma - \tau)(\kappa + 1)^2 > 1 \quad when \quad p > 0, \quad where \quad \kappa = \left[\frac{\sigma - \tau}{2}\right], \tag{31}$$

$$\frac{q\tau}{p^2} > 1. \quad when \quad p < -1 \quad and \quad \tau \geq \sigma, \tag{32}$$

or

$$\frac{q\tau}{p^2 - pq(\sigma - \tau)} > 1 \quad when \quad p < -1 \quad and \quad \sigma > \tau \tag{33}$$

implies that every solution of (1) oscillates.

From the above result it follows that each of the conditions (30)-(33) implies that Eq. (2) has no positive roots something that cannot be so easily determined by investigating directly the equation (2).

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