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Research Paper

Partial orders On $C = D + Di$ and $H = D + Di + Dj + Dk$

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Abstract

Let D be a totally ordered integral domain. We study partial orders on the rings $C = D + Di$ and $H = D + Di + Dj + Dk$, where $i^2 = j^2 = k^2 = -1$.

Keywords: *Complex number, directed partial order, lattice order, partial order, quaternion.*

1. Introduction

Throughout the paper, D denotes a totally ordered integral domain, $C = D+Di = \{a+bi \mid a, b \in D\}$ with $i^2 = -1$, and

$$H = D + Di + Dj + Dk = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_0, a_1, a_2, a_3 \in D\},$$

with $i^2 = j^2 = k^2 = -1$. C and H may be called the ring of *complex numbers over* D and the ring of *quaternions over* D. If $D = \mathbb{R}$, the field of real numbers, then $\mathbb{C} = \mathbb{R} + \mathbb{R}i$ and $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ are the field of complex numbers and the division ring of real quaternions, respectively. Describing the directed partial orders on $\mathbb{C}$ and $\mathbb{H}$ is an open question [2, Problem 31, p.212]. Recently some directed partial orders on $\mathbb{H}$ have been constructed [4]. We notice that the same directed partial order can be constructed for complex numbers and quaternions over non-archimedean totally ordered integral domains.

A *partially ordered algebra* R *over* D (*po-algebra over* D) is a partially ordered ring (*po*-ring) R and an algebra over D such that $D^+R^+ \subseteq R^+$, where $R^+ = \{r \in R \mid r \geq 0\}$ and $D^+ = \{a \in D \mid a \geq 0\}$. A *po*-algebra R is called a *directed algebra* if the partial order is a directed partial order, that is, any element in R is a difference of two positive elements; and a *po*-algebra R is called a lattice-ordered algebra (*ℓ-algebra*) if the partial order is a lattice order. In this article, we study partial orders on C and H to make them into a *po*-algebra over D. For undefined terminologies and background information on *po*-rings and ℓ-rings, the reader is referred to [1, 2, 3].

2. Partial orders on C and H

For $a, b \in D^+$, $a \ll b$ (or $b \gg a$) means that $na \leq b$ for all positive integer n.

Theorem 1 *Define the positive cone* P_C *of* C *as follows.*

$$P_C = \{a + bi \mid a \geq 0 \text{ and } |b| \ll a \text{ in } D\}.$$

(1) *P_C is the positive cone of a partial order on C such that (C, P_C) is a po-algebra over D.*
(2) *If there is an element $z \in D^+$ such that $1 \ll z$, then (C, P_C) is a directed algebra.*

Proof. (1) It is clear that $P_C \cap -P_C = \{0\}$, $P_C + P_C \subseteq P_C$, and $D^+ P_C \subseteq P_C$. We show that $P_C P_C \subseteq P_C$. Suppose that $a + bi, x + yi \in P_C$. We need that $(a + bi)(x + yi) = (ax - by) + (ay + bx)i \in P_C$. From $|b| \ll a$ and $|y| \ll x$, we have $by \leq |b||y| \leq ax$, so $ax - by \geq 0$. Also for all positive integer n, we have

$$3n|ay + bx| + 3by \leq 3na|y| + 3n|b|x + 3|b||y| \leq ax + ax + ax = 3(ax),$$

and hence $n|ay+bx| \leq ax-by$ for all positive integer n, that is, $|ay+bx| \ll (ax-by)$. Therefore $(a+bi)(x+yi) \in P_C$, and P_C is a partial order on C.

(2) Suppose that $1 \ll z$ for some $z \in D$. Let $a \in D$ and $a \geq 0$. Then $a \in P_C$. If $a \in D$ with $a < 0$, then $-a \in P_C$. Thus each element in D is a difference of two elements in P_C. For $b \in D$ with $b > 0$, take $w = bz \in D^+$. Then $b \ll w$, and $bi = (w + 2bi) - (w + bi)$ is a difference of two elements in P_C. If $b \in D$ with $b < 0$, then $-b > 0$ and so $-bi$ is a difference of two elements in P_C by previous argument. Hence bi is a difference of two positive elements. Now it is easy to see that any $a + bi \in C$ is a difference of two elements in P_C. Therefore P_C is directed.

The identity element of C is denoted by 1. Clearly $1 \in P_C$. It is clear that D is an archimedean totally ordered integral domain if and only if $P_C = D^+$. We note that if D is a totally ordered field, then $1 \ll z$ for some $z \in D^+$ is equivalent to that D is non-archimedean. P_C is not a lattice order, for instance, $i \vee 0$ does not exist with respect to P_C. The verification of this fact is left to the reader.

It turns out that the positive cone P_C defined in Theorem 1 is the largest partial order on C to make it into a *po*-algebra over D.

Theorem 2 *Suppose that C is a po-algebra over D. If $a + bi \geq 0$ in C, then $a \geq 0$ and $|b| \ll a$ in D.*

Proof. Suppose that $z = a + bi \geq 0$ in C. We first show that $a \geq 0$ in D. Assume $a < 0$ in D and we derive a contradiction. Since $-a > 0$ in D and C is a *po*-algebra over D, we have $-az \geq 0$ in C. Then $z^2 - 2az = -(a^2 + b^2) \geq 0$ in C. Thus $-(a^2 + b^2)z \geq 0$ in C. On the other hand, $(a^2 + b^2) \in D^+$ and $z \geq 0$ in C implies that $(a^2 + b^2)z \geq 0$ in C. Therefore we have $(a^2 + b^2)z = 0$, which is a contradiction. Thus $a \geq 0$ in D.

Now assume that $z = a + bi \geq 0$ in C and $a \geq 0$ in D. We show that $|b| \ll a$ in D. If $a = 0$, then $z = bi \geq 0$ in C implies that $b = 0$ by a similar argument in the previous paragraph. For the following, we assume $a > 0$. Then $z^2 = a^2 + 2abi - b^2 \geq 0$ in C implies that

$$z^3 + b^2 z = (a^2 + 2abi)z = a^3 + 3a^2 bi - 2ab^2 \geq 0.$$

Let $z_1 = a^3 + 3a^2bi - 2ab^2$. We have

$$\begin{aligned} & z_2 = (z_1 + 2ab^2)z = a^4 + 4a^3bi - 3a^2b^2 \geq 0 \\ \Rightarrow\quad & z_3 = (z_2 + 3a^2b^2)z = a^5 + 5a^4bi - 4a^3b^2 \geq 0 \\ \vdots\quad & \\ \Rightarrow\quad & z_n = (z_{n-1} + na^{n-1}b^2)z = a^{n+2} + (n+2)a^{n+1}bi - (n+1)a^n b^2 \geq 0 \end{aligned}$$

Then we have $a^{n+2} - (n+1)a^n b^2 \geq 0$ in D for all positive integer n since the real part of a positive element in C is positive in D, and hence $(n+1)b^2 \leq a^2$ for all positive integer n. Thus for all positive integer m, $(mb)^2 = m^2b^2 \leq a^2$, so $-a \leq mb \leq a$. Therefore $m|b| \leq a$ for all positive integer m, that is, $|b| \ll a$.

Another important property of the positive cone P_C is that if $z = a + bi \in P_C$, then $\bar{z} = a - bi \in P_C$. Recall that a *po*-ring R is called *division closed* if for any $a, b \in R$, $ab > 0$ and one of a and $b > 0$, then so is the other [2]. It follows that P_C is division closed since $z \in P_C$ implies that $\bar{z} \in P_C$. In the case that D is a totally ordered field, this fact implies that each nonzero positive element in (C, P_C) has a positive inverse. We notice that D also has this property since D is totally ordered.

Following is an example of a partial order on C in which the identity element is not positive.

Example 3 *For an element $z = a + bi \in C$, define the positive cone*

$$P = \{z \in C \mid z = 0 \text{ or } a > 0, b > 0 \text{ and } b \ll a\}.$$

It is straightforward to check that C is po-algebra with respect to P and 1 is not positive. Clearly $P \cap D^+ = \{0\}$ and D is archimedean if and only if $P = \{0\}$. Similarly if there is a positive element z in D such that $1 \ll z$, then P is directed.

Now we consider lattice orders on C.

Theorem 4 *If D is a totally ordered field, then there is no lattice orders on C to make it into an ℓ-algebra over D.*

Proof. Suppose that C is an ℓ-algebra over D. We derive a contradiction. Since C cannot be totally ordered, there are $u, v \in C$, $u > 0$, $v > 0$ and $u \wedge v = 0$. Then u, v forms a basis of C as a vector space over D, since C is two-dimensional over D, and an element $au + bv \geq 0$ with $a, b \in D$ if and only if $a \geq 0$ and $b \geq 0$ in D.

Let $iu = au + bv$ with $a, b \in D$. Since iu is not comparable with 0, a and b cannot be zero and must be in opposite sign. We may assume that $a > 0$ and $b < 0$. Then $-bv = (a - i)u > 0$ and $u \wedge (a - i)u = u \wedge (-bv) = 0$ since D is a totally ordered field. Let $w = a - i$. Then $w^2 = a^2 - 2ai - 1 = -(a^2 + 1) + 2aw$. Since u^2 and $wu^2 = (wu)u = (-bv)u$ both are positive,

$$u^2 = a_1 u + b_1(wu) \text{ and } wu^2 = a_2 u + b_2(wu) \text{ with } a_1, a_2, b_1, b_2 \text{ being positive in } D.$$

Then we also have

$$\begin{aligned} wu^2 &= a_1(wu) + b_1(w^2 u) \\ &= a_1(wu) + b_1(-(a^2+1) + 2aw)u \\ &= a_1(wu) - b_1(a^2+1)u + 2b_1 a(wu) \\ &= -b_1(a^2+1)u + (a_1 + 2b_1 a)(wu). \end{aligned}$$

Thus we have $a_2 = -b_1(a^2 + 1)$ and $b_2 = a_1 + 2b_1 a$. So $b_1 = a_2 = 0$ and $b_2 = a_1$, and hence $u = a_1 \in D$ from $u^2 = a_1 u$ and C being a field. It follows that $wu = a_1 a - a_1 i$, so $a_1 i = a_1 a - wu$. Take square of the both sides of the previous equation, we have

$$-a_1^2 = a_1^2 a^2 - 2a_1 a wu + w^2 u^2,$$

and hence

$$(wu)^2 = -a_1^2(a^2 + 1) + 2a_1 a(wu) = -a_1(a^2 + 1)u + 2a_1 a(wu).$$

From $u \wedge wu = 0$, we must have $-a_1(a^2 + 1) \geq 0$ in D, which is a contradiction. This completes the proof.

For a totally ordered integral domain D, although we believe $C = D + Di$ cannot be made into an ℓ-algebra over D, we are unable to prove it except for some special cases. For instance, if D is archimedean, C cannot even be a directed algebra over D. Another special case is given below.

Let F be a totally ordered quotient field of D. Then $F + Fi$ is a quotient field of $C = D + Di$. It is still an open question weather or not a lattice order on an integral domain can be extended to its quotient field. Suppose that $C = D + Di$ is an ℓ-algebra over D. Let

$$\overline{f}(C) = \{a \in C \mid \forall x, y \in C, x \wedge y = 0 \Rightarrow ax \wedge y = 0\}.$$

Then $\overline{f}(C)$ is a subring of C. We may call elements in $\overline{f}(C)$ as *generalized f-element* which may not be positive. By [3, Theorem 4.33], C cannot be made into an ℓ-algebra over D which is algebraic over $\overline{f}(C)$ since in this case, the lattice order on C can be extended to its quotient field $F + Fi$, which is not possible by Theorem 4.

Next we consider quaternions over D defined as

$$H = D + Di + Dj + Dk = \{a + bi + cj + dk \mid a, b, c, d \in D\}$$

with the coordinatewise addition and the multiplication as follows.

$$\begin{aligned} & (a_0 + a_1 i + a_2 j + a_3 k)(b_0 + b_1 i + b_2 j + b_3 k) \\ = & (a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3) + (a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2) i \\ & + (a_0 b_2 + a_2 b_0 + a_3 b_1 - a_1 b_3) j + (a_0 b_3 + a_3 b_0 + a_1 b_2 - a_2 b_1) k. \end{aligned}$$

The proof of following theorem is similar to Theorem 1. We omit the proof and leave the verification of it to the reader.

Theorem 5 *Define the positive cone P_H of H as follows.*

$$P_H = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_0 \geq 0 \text{ and } |a_1| \ll a_0, |a_2| \ll a_0, |a_3| \ll a_0\}$$

(1) *P_H is the positive cone of a partial order on H such that (H, P_H) is a po-algebra over D.*
(2) *If there is an element $z \in D^+$ such that $1 \ll z$, then (H, P_H) is a directed algebra.*

Like P_C on C, P_H is also the largest partial order on H to make it into a *po*-algebra over D.

Theorem 6 *Suppose that $H = D + Di + Dj + Dk$ is a po-algebra over D. If $a_0 + a_1 i + a_2 j + a_3 k \geq 0$ in H, then $a_0 \geq 0$ and $|a_1| \ll a_0, |a_2| \ll a_0, |a_3| \ll a_0$ in D.*

Proof. Suppose that $w = a_0 + a_1 i + a_2 j + a_3 k \geq 0$ in H. We first show that $a_0 \geq 0$ in D. Since $w^2 - 2a_0 w = -(a_0^2 + a_1^2 + a_2^2 + a_3^2)$, we have

$$w^3 - 2a_0 w^2 = -(a_0^2 + a_1^2 + a_2^2 + a_3^2) w.$$

If $a_0 < 0$ in D, then since H is a *po*-algebra over D, we have $w^3 - 2a_0 w^2 \geq 0$ in H. It follows that $-(a_0^2 + a_1^2 + a_2^2 + a_3^2) w \geq 0$ in H, which contradicts with $(a_0^2 + a_1^2 + a_2^2 + a_3^2) w \geq 0$. Thus $a_0 \geq 0$ in D.

If $a_0 = 0$, then $w^2 = -(a_1^2 + a_2^2 + a_3^2) \geq 0$ in H implies that $(a_1^2 + a_2^2 + a_3^2) w = 0$, and hence $a_1 = a_2 = a_3 = 0$, so $|a_i| \ll a_0$ is true, $i = 1, 2, 3$. For the following, assume $a_0 > 0$. Let $v = a_1 i + a_2 j + a_3 k$. Then $v^2 = -(a_1^2 + a_2^2 + a_3^2)$ and $w^2 = a_0^2 + 2a_0 v + v^2 \geq 0$ in H implies that

$$w^3 - v^2 w = (a_0^2 + 2a_0 v) w = a_0^3 + 3a_0^2 v + 2a_0 v^2 \geq 0.$$

Let $w_1 = a_0^3 + 3a_0^2 v + 2a_0 v^2$. We have

$$\begin{aligned} & w_2 = (w_1 - 2a_0 v^2) w = a_0^4 + 4a_0^3 v + 3a_0^2 v^2 \geq 0 \\ \Rightarrow \ & w_3 = (w_2 - 3a_0^2 v^2) z = a_0^5 + 5a_0^4 v + 4a_0^3 v^2 \geq 0 \\ & \vdots \\ \Rightarrow \ & w_n = (w_{n-1} - n a_0^{n-1} v^2) z = a_0^{n+2} + (n+2) a_0^{n+1} v + (n+1) a_0^n v^2 \geq 0 \end{aligned}$$

Then we have $a_0^{n+2} + (n+1) a_0^n v^2 \geq 0$ in D for all positive integer n since the real part of w_n is positive, and hence $-(n+1) v^2 \leq a_0^2$ for all positive integer n. From $-v^2 = (a_1^2 + a_2^2 + a_3^2)$, for all positive integer m and $i = 1, 2, 3$, $(m a_i)^2 \leq a_0^2$, so $-a_0 \leq m a_i \leq a_0$. Therefore $m|a_i| \leq a_0$ for all positive integer m, that is, $|a_i| \ll a_0$, $i = 1, 2, 3$.

As a direct consequence of Theorem 6, H cannot be a direct algebra over an archimedean totally ordered domain D. We believe that if $D = F$ is a totally ordered field, then H cannot be an ℓ-algebra over F. However we lack ability to provide a proof of it in general. What we do know is that if $D = F$ is a totally ordered field, then H cannot be an ℓ-algebra over F in which $1 > 0$.

Theorem 7 *Let $H = F + Fi + Fj + Fk$, where F is a totally ordered field. Then H cannot be made into an ℓ-algebra over F with $1 > 0$.*

Proof. Suppose that H is an ℓ-algebra over F with $1 > 0$. Since H cannot be totally ordered, there is an element $0 \neq u \in H$ such that $1 \wedge u = 0$. Suppose that $u = b_0 + b_1 i + b_2 j + b_3 k$. Then $u^2 = 2b_0 u - (b_0^2 + b_1^2 + b_2^2 + b_3^2) > 0$, so $-(b_0^2 + b_1^2 + b_2^2 + b_3^2) \geq 0$ by $1 \wedge u = 0$. Therefore $b_0^2 + b_1^2 + b_2^2 + b_3^2 = 0$, which contradicts with $u \neq 0$.

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