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Research paper

Bayesian and E-Bayesian estimation for the Kumaraswamy distribution based on type-ii censoring

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Abstract

This paper introduces the Bayesian and E-Bayesian estimation for the shape parameter of the Kumaraswamy distribution based on type-II censored schemes. These estimators are derived under symmetric loss function [squared error loss (SELF))] and three asymmetric loss functions [LINEX loss function (LLF), Degroot loss function (DLF) and Quadratic loss function (QLF)]. Monte Carlo simulation is performed to compare the E-Bayesian estimators with the associated Bayesian estimators in terms of Mean Square Error (MSE).

Keywords: *Censored Sampling; E-Bayes Estimates; Kumaraswamy Distribution; Type-II Censored; Loss Functions; Monte Carlo Simulation.*

1. Introduction

The Kumaraswamy distribution is similar to the Beta distribution, but much simpler to use especially in simulation studies due to the simple closed form of both its probability density function and cumulative distribution function. Kumaraswamy [1], [2] proved that the ordinary probability distribution functions such as normal, log-normal, beta and empirical distributions such as Johnson and polynomial-transformed-normal, etc., have not great accuracy in fitting hydrological random variables such as daily rainfall, daily stream flow, etc. and developed a new probability density function known as the sinepower probability density function to fit up random processes which are bounded at the lower and upper ends. Furthermore, Kumaraswamy [3] introduced a new probability distribution for double bounded random processes with hydrological applications, which is known as Kumaraswamy distribution. The continuous part of Kumaraswamy distribution, denoted Kum (λ,θ) has probability density function (pdf) and the cumulative distribution function (cdf) specified by

$$f(x;\lambda,\theta)=\lambda\theta x^{\lambda-1}(1-x^{\lambda})^{\theta-1}, \quad 0<x<1, \quad \lambda,\theta>0 \tag{1-1}$$

And

$$F(x;\lambda,\theta)=1-(1-x^{\lambda})^{\theta}, \quad 0<x<1, \quad \lambda,\theta>0 \tag{1-2}$$

Where λ and θ are the shape parameters. Based on different values of λ and θ, Kumaraswamy [3] and Ponnambalam, et al [4], have referred to that the Kumaraswamy distribution can be used as an approximation for many distributions, such as uniform, triangular, and can also be reproduce result of beta distribution. Nadarajah [5] considered that the Kumaraswamy distribution is special case of the three parameter beta distribution. Jones [6] has obtained the main properties of the Kumaraswamy distribution. Furthermore, a few number of authors deals with the Kumaraswamy distribution under Bayesian procedure, for example; Sindhu et al [7] obtained Bayesian and non-Bayesian estimators for the shape parameter of the Kumaraswamy distribution under type-II censoring. Also, Eldin et al [8] produced a study in estimating the parameters of the Kumaraswamy distribution based on general progressive type- II censoring.

The expected Bayesian estimation or briefly E-Bayesian estimation is a new approach of Bayesian estimation first introduced by Han [9]. Han [10] obtained the E-Bayes and hierarchical Bayes estimates of the reliability parameter for testing data from products with exponential distribution under type-I censoring and by considering the quadratic loss function. He showed that by using simulation study, the E-Bayesian estimator is efficient and easy to operate. Yin and Liu [11] constructed the E-Bayesian estimation and hierarchical Bayesian estimation techniques for estimating the reliability parameter of the geometric distribution based on scaled squared loss function in complete samples. They deducted that the E-Bayes method is more stability and convenient in terms of calculation complexity than the hierarchical Bayes method. Wei et al [12] applied the minimum risk equivariant estimation and E-Bayes estimation techniques for estimating the parameter of the Burr-XII distribution under entropy loss function in complete samples. They deducted that E-Bayes estimates have most accuracy. Jaheen and Okasha [13] compared the Bayesian and E-Bayesian estimators for the parameters and reliability function of the Burr Burr-XII distribution under type-II censoring and by considering the squared error loss and LINEX loss functions. They pointed out that the overall performance of the E-Bayes estimates are better than the similar obtained by using the Bayes criteria. Cai et al [14] used the E-Bayesian estimation technique for forecasting of security investment. Okasha [15] constructed the maximum likelihood, Bayesian and E-Bayesian methods for estimating the scale parameter, reliability and hazard functions of the Weibull distribution under type-2 censored samples and by considering the squared error loss function. He deducted that the E-Bayes estimates were more efficient than the maximum likelihood estimates or the Bayes estimates. Azimi et al [16] estimated the parameter and reliability function of the generalized half Logistic distribution by using the Bayes and E-Bayes techniques under progressively type-II censoring and by considering the squared error loss and LINEX loss functions. They deducted that the E-Bayes criteria generally is more efficient than the Bayes method. Javadkani et al [17] constructed the Bayes, empirical Bayes and E-Bayes methods for estimating the shape parameter and the reliability function of the two parameter bathtub-shaped lifetime distribution under on progressively first-failure-censored samples and by considering the minimum expected loss and LINEX loss functions. Okasha [18] applied the Bayesian and the E-Bayesian techniques for estimating the scale parameter, reliability and hazard functions of the Lomax distribution under type-2 censored and by considering the balanced squared error loss function. He deducted that the performance of the E-Bayes estimates is generally better than the Bayes estimates. Reyad and Othman [19] derived the Bayesian and E-Bayesian estimates for the shape parameter of the Gumbell type-II distribution based on type-II censoring and by considering squared error, LINEX, Degroot, Quadratic and minimum expected loss functions. They deducted that the E-Bayes estimates were generally much better than the other estimates.

The main object of this paper is to introduce a statistical comparison between the Bayesian and E-Bayesian procedures for estimating the shape parameter of the Kumaraswamy distribution based on type-II censoring. The resulting estimators are obtained based on symmetric and different asymmetric loss functions and the results obtained in this paper can be generalized to use in complete sample.

The layout of the paper is as follow. In Section 2 and 3 respectively, the Bayesian and E-Bayesian estimates of the parameter θ based on type-II censored sample are derived under squared, LINEX, Degroot and quadratic loss functions. In Section 4, the properties of the E-Bayesian estimators are discussed. Simulation study has been performed to compare the resulting estimators in Section 5. Some concluding remarks have been given in the last Section.

2. Bayesian estimation

This section spotlights on the derivation of the Bayes estimates for the shape parameter θ of the Kum (λ,θ) under symmetric loss function [squared error loss (SELF)) and three asymmetric loss functions (LINEX loss function (LLF), Degroot loss function (DLF) and quadratic loss function (QLF)].

In a typical life test, n item is placed under observation as each failure occurs. In type-II censored technique, the test is terminated when the number of failure units r is completed which is a predetermined condition. In this case the data collected consists of observations $x_{(1)},x_{(2)},x_{(3)},...x_{(r)}$ plus the information that $(n-r)$ items survived beyond the condition of termination. The likelihood function for $x_{(1)},x_{(2)},x_{(3)},...x_{(r)}$ failed observations is given by

$$L(\lambda,\theta|\underline{x})=\frac{n!}{(n-r)!}\prod_{i=1}^{r}f(x_{(i)})\left[1-F(x_{(r)})\right]^{n-r} \tag{2-1}$$

Substituting (1-1) and (1-2) in (2-1), the later function can be obtained to be,

$$\begin{aligned}L(\lambda,\theta|\underline{x})&=\frac{n!}{(n-r)!}\prod_{i=1}^{r}\lambda\theta x_{(i)}^{\lambda-1}\left[1-x_{(i)}^{\lambda}\right]^{\theta-1}\left[(1-x_{(r)}^{\lambda})^{\theta}\right]^{n-r}\\&=\frac{n!}{(n-r)!}\lambda^{r}\theta^{r}\prod_{i=1}^{r}\left[\frac{x_{(i)}^{\lambda-1}}{1-x_{(i)}^{\lambda-1}}\right]e^{-\theta w}\end{aligned} \tag{2-2}$$

Where

$$H=-\left[\sum_{i=1}^{r}\ln(1-x_{(i)}^{\lambda})+(n-r)\ln(1-x_{(r)}^{\lambda})\right] \tag{2-3}$$

Assuming λ is known, we can use the Gamma distribution as an conjugate prior distribution of θ with shape and scale parameter a and b respectively and its pdf given by

$$g(\theta|a,b)=\frac{b^{a}}{\Gamma(a)}\theta^{a-1}e^{-b\theta},\qquad \theta>0,\ a,b>0 \tag{2-4}$$

Combining (2-2) and (2-4), from Bayesian theorem the posterior density function of θ can be obtained as

$$\begin{aligned}\pi(\theta|\underline{x})&=\frac{L(\theta|\underline{x})g(\theta|a,b)}{\int_0^{\infty}L(\theta|\underline{x})g(\theta|a,b)d\theta}\\&=\frac{(H+b)^{r+a}}{\Gamma(r+a)}\theta^{r+a-1}e^{-(H+b)},\qquad \theta>0\end{aligned} \tag{2-5}$$

That mean, the posterior distribution of θ obeys $\Gamma(r+a,H+b)$.

2.1. Bayesian estimation under squared error loss function (SELF)

A commonly used loss function is the square error loss function (SELF) defined as follows:

$$L_1(\hat{\theta},\theta)=(\hat{\theta}-\theta)^2 \tag{2-6}$$

Where $\hat{\theta}$ is an estimator of θ. The Bayes estimator of θ denoted by $\hat{\theta}_{BS}$ can be obtained as

$$\hat{\theta}_{BS}=E_{\pi}(\theta|\underline{x}) \tag{2-7}$$

Where E_{π} indicated to the expectation of the posterior distribution. We can derived $\hat{\theta}_{BS}$ by using (2-5) in (2-7) to be

$$\hat{\theta}_{BS}=\frac{r+a}{H+b} \tag{2-8}$$

2.2. Bayesian estimation under LINEX loss function (LLF)

Zellner [20] represent the LINEX (linear-exponential) loss function (LLF) to be

$$L_2(\hat{\theta},\theta)=m\left\{\exp\left[s(\hat{\theta}-\theta)\right]-s(\hat{\theta}-\theta)-1\right\} \tag{2-9}$$

With two parameters $m>0,\ s\neq 0,$ where m is the scale of the loss function and s determines its shape. Without loss of generality, we assume $m=1$. The Bayes estimator relative to LLF denoted by $\hat{\theta}_{BL}$ can be obtained as

$$\hat{\theta}_{BL}=\left(\frac{-1}{s}\right)\ln\left[E_{\theta}\left(e^{-s\theta}\middle|\underline{x}\right)\right] \tag{2-10}$$

We can obtain $\hat{\theta}_{BL}$ by using (2-5) in (2-10) to be

$$\hat{\theta}_{BL}=\left(\frac{r+a}{s}\right)\ln\left[1+\frac{s}{H+b}\right] \tag{2-11}$$

2.3. Bayesian estimation under degroot loss function (DLF)

The Degroot loss function (DLF) is defined by Degroot [21] to be

$$L_3(\hat{\theta},\theta)=\left(\frac{\theta-\hat{\theta}}{\hat{\theta}}\right) \quad (2\text{-}12)$$

The Bayes estimator relative to DLF denoted by $\hat{\theta}_{BD}$ can be obtained as

$$\hat{\theta}_{BD}=\frac{E_\pi(\theta^2|\underline{x})}{E_\pi(\theta|\underline{x})} \quad (2\text{-}13)$$

We can get $\hat{\theta}_{BD}$ by using (2-5) in (2-13) to be

$$\hat{\theta}_{BD}=\frac{r+a+1}{H+b} \quad (2\text{-}14)$$

2.4. Bayesian estimation under quadratic loss function (QLF)

Bhuiyan et al [22] defined the quadratic loss function (QLF) as follows:

$$L_4(\hat{\theta},\theta)=\left(\frac{\theta-\hat{\theta}}{\theta}\right) \quad (2\text{-}15)$$

The Bayes estimator of θ based on QLF denoted by $\hat{\theta}_{BQ}$ can be obtained as

$$\hat{\theta}_{BQ}=\frac{E_\pi(\theta^{-1}|\underline{x})}{E_\pi(\theta^{-2}|\underline{x})} \quad (2\text{-}16)$$

We can derived $\hat{\theta}_{BQ}$ by using (2-5) in (2-16) to be

$$\hat{\theta}_{BQ}=\frac{r+a-2}{H+b} \quad (2\text{-}17)$$

3. E-Bayesian estimation

In this section, we consider the E-Bayes estimates of the shape parameter θ of the Kum (λ,θ) under symmetric loss function [squared error loss (SELF)) and three asymmetric loss functions (LINEX loss function (LLF), Degroot loss function (DLF) and quadratic loss function (QLF)].

Based on Han [23], the prior parameters a and b must be choose to guarantee that $g(\theta|a,b)$ given in (2-4) is a decreasing function of θ. The derivative of $g(\theta|a,b)$ with respect to θ is

$$\frac{dg(\theta|a,b)}{d\theta}=\frac{b^a}{\Gamma(a)}\theta^{a-2}e^{b\theta}\left[(a-1)-b\theta\right] \quad (3\text{-}1)$$

Note that $a>0, b>0$ and $\theta>0$ leads to $0<a<1, b>0$ due to $\frac{dg(\theta|a,b)}{d\theta}<0$, and therefore $g(\theta|a,b)$ is a decreasing function of θ. Suppose that a and b are independent with bivariate density function

$$\pi(a,b)=\pi_1(a)\pi_2(b) \quad (3\text{-}2)$$

Then, the E-Bayesian estimate of θ (expectation of the Bayesian estimate of θ) can be written as

$$\hat{\theta}_{EB}=E(\theta|\underline{x})=\iint_\Omega \hat{\theta}_B(a,b)\,\pi(a,b)\,da\,db \quad (3\text{-}3)$$

Where $\hat{\theta}_B(a,b)$ is the Bayes estimate θ of given by (2-8), (2-11), (2-14) and (2-17). For more details see (Han [9, 24]).

3.1. E-Bayesian Estimation under Squared Error Loss Function (SELF)

The E-Bayes estimates of θ are derived depending on three different distributions of the hyper-parameters a and b. These distributions are used to study the impact of the different prior distributions on the E-Bayesian estimation of θ. The following distributions of a and b may be used:

$$\pi_1(a,b)=\frac{2(c-b)}{c^2}, \qquad 0<a<1,\ 0<b<c \quad (3\text{-}4)$$

$$\pi_2(a,b)=\frac{1}{c}, \qquad 0<a<1,\ 0<b<c \quad (3\text{-}5)$$

$$\pi_3(a,b)=\frac{2b}{c^2}, \qquad 0<a<1,\ 0<b<c \quad (3\text{-}6)$$

We can obtained the E-Bayes estimates of θ relative to SELF based on $\pi_1(a,b)$ which is denoted as $\hat{\theta}_{EBS1}$ by using (2-8) and (3-4) in (3-3) to be

$$\begin{aligned}\hat{\theta}_{EBS1}&=\int_0^1\int_0^c\left(\frac{r+a}{H+b}\right)\left[\frac{2(c-b)}{c^2}\right]db\,da\\&=\left(\frac{2r+1}{c}\right)\left[\left(1+\frac{H}{c}\right)\ln\left(1+\frac{c}{H}\right)-1\right]\end{aligned} \quad (3\text{-}7)$$

Similarly, we can derive the E-Bayesian estimates of θ relative to SELF based on $\pi_2(a,b)$ and $\pi_3(a,b)$ which are denoted as $\hat{\theta}_{EBS2},\hat{\theta}_{EBS3}$ by using (2-8), (3-5) in (3-3) and (2-8), (3-6) in (3-3) respectively to be

$$\hat{\theta}_{EBS2}=\int_0^1\int_0^c\left(\frac{r+a}{H+b}\right)\left[\frac{1}{c}\right]db\,da=\left(\frac{2r+1}{2c}\right)\left[\ln\left(1+\frac{c}{H}\right)\right] \quad (3\text{-}8)$$

And

$$\hat{\theta}_{EBS3}=\int_0^1\int_0^c\left(\frac{r+a}{H+b}\right)\left[\frac{2b}{c^2}\right]db\,da=\left(\frac{2r+1}{c}\right)\left[1-\frac{H}{c}\ln\left(1+\frac{c}{H}\right)\right] \quad (3\text{-}9)$$

3.2. E-Bayesian estimation under LINEX loss function (LLF)

We can get the E-Bayes estimate of θ relative to LLF based on $\pi_1(a,b)$ which is denoted as $\hat{\theta}_{EBL1}$ by using (2-11) and (3-4) in (3-3) to be

$$\hat{\theta}_{EBL1}=\int_0^1\int_0^c\left(\frac{r+a}{s}\right)\ln\left[1+\frac{s}{H+b}\right]\left[\frac{2(c-b)}{c^2}\right]db\,da$$

$$=\left(\frac{2r+1}{2}\right)\left\{\begin{aligned}&\left[\left(\frac{-(H+c)^2}{c^2s}\right)\ln\left(1+\frac{c}{H}\right)\right]\\&+\left[\left(\frac{(H+s+c)^2}{c^2s}\right)\ln\left(1+\frac{c}{H+s}\right)\right]\\&+\left[\frac{1}{s}\ln\left(1+\frac{s}{H}\right)\right]-\left[\frac{1}{c}\right]\end{aligned}\right\} \quad (3\text{-}10)$$

By the same way, we can derive the E-Bayes estimates of θ relative to LLF based on $\pi_2(a,b)$ and $\pi_3(a,b)$ which are denoted as

$\hat{\theta}_{EBL2}, \hat{\theta}_{EBL3}$ by using (2-11), (3-5) in (3-3) and (2-11), (3-6) in (3-3) respectively to be

$$\hat{\theta}_{EBL2} = \int_0^1\int_0^c \left(\frac{r+a}{s}\right)\ln\left[1+\frac{s}{H+b}\right]\left[\frac{1}{c}\right]db\,da$$

$$= \left(\frac{2r+1}{2s}\right)\left\{\begin{array}{l}\left[\ln\left(1+\frac{s}{H+c}\right)\right]+\left[\left(\frac{H+s}{c}\right)\ln\left(1+\frac{c}{H+s}\right)\right]\\ -\left[\left(\frac{H}{c}\right)\ln\left(1+\frac{c}{H}\right)\right]\end{array}\right\} \quad (3\text{-}11)$$

And

$$\hat{\theta}_{EBL3} = \int_0^1\int_0^c \left(\frac{r+a}{s}\right)\ln\left[1+\frac{s}{H+b}\right]\left[\frac{2b}{c^2}\right]db\,da$$

$$= \left(\frac{2r+1}{2}\right)\left\{\begin{array}{l}\left[\left(\frac{-(H+c)^2}{c^2 s}\right)\ln\left(1+\frac{c}{H+s}\right)\right]\\ +\left[\left(\frac{H^2}{c^2 s}\right)\ln\left(1+\frac{c}{H}\right)\right]\\ +\left[\frac{1}{s}\ln\left(1+\frac{s}{H+c}\right)\right]+\left[\frac{1}{c}\right]\end{array}\right\} \quad (3\text{-}12)$$

3.3. E-Bayesian estimation under DeGroot loss function (DLF)

We can obtained the E-Bayes estimates of θ relative to DLF based on $\pi_1(a,b)$ which is denoted as $\hat{\theta}_{EBD1}$ by using (2-14) and (3-4) in (3-3) to be

$$\begin{aligned}\hat{\theta}_{EBD1} &= \int_0^1\int_0^c \left(\frac{r+a+1}{H+b}\right)\left[\frac{2(c-b)}{c^2}\right]db\,da \\ &= \left(\frac{2r+3}{c}\right)\left[\left(1+\frac{H}{c}\right)\ln\left(1+\frac{c}{H}\right)-1\right]\end{aligned} \quad (3\text{-}13)$$

Also, we can derive the E-Bayesian estimates of θ relative to DLF based on $\pi_2(a,b)$ and $\pi_3(a,b)$ which are denoted as $\hat{\theta}_{EBS2}, \hat{\theta}_{EBS3}$ by using (2-14), (3-5) in (3-3) and (2-8), (3-6) in (3-3) respectively to be

$$\hat{\theta}_{EBD2} = \int_0^1\int_0^c \left(\frac{r+a+1}{H+b}\right)\left[\frac{1}{c}\right]db\,da = \left(\frac{2r+3}{2c}\right)\left[\ln\left(1+\frac{c}{H}\right)\right] \quad (3\text{-}14)$$

And

$$\begin{aligned}\hat{\theta}_{EBD3} &= \int_0^1\int_0^c \left(\frac{r+a+1}{H+b}\right)\left[\frac{2b}{c^2}\right]db\,da \\ &= \left(\frac{2r+3}{c}\right)\left[1-\frac{H}{c}\ln\left(1+\frac{c}{H}\right)\right]\end{aligned} \quad (3\text{-}15)$$

3.4. Bayesian estimation under quadratic loss function (QLF)

We can obtained the E-Bayes estimates of θ relative to QLF based on $\pi_1(a,b)$ which is denoted as $\hat{\theta}_{EBD1}$ by using (2-17) and (3-4) in (3-3) to be

$$\begin{aligned}\hat{\theta}_{EBQ1} &= \int_0^1\int_0^c \left(\frac{r+a-2}{H+b}\right)\left[\frac{2(c-b)}{c^2}\right]db\,da \\ &= \left(\frac{2r-3}{c}\right)\left[\left(1+\frac{H}{c}\right)\ln\left(1+\frac{c}{H}\right)-1\right]\end{aligned} \quad (3\text{-}16)$$

By the same way, we can derive the E-Bayesian estimates of θ relative to QLF based on $\pi_2(a,b)$ and $\pi_3(a,b)$ which are denoted as $\hat{\theta}_{EBS2}, \hat{\theta}_{EBS3}$ by using (2-17), (3-5) in (3-3) and (2-17), (3-6) in (3-3) respectively to be

$$\hat{\theta}_{EBQ2} = \int_0^1\int_0^c \left(\frac{r+a-2}{H+b}\right)\left[\frac{1}{c}\right]db\,da = \left(\frac{2r-3}{2c}\right)\left[\ln\left(1+\frac{c}{H}\right)\right] \quad (3\text{-}17)$$

And

$$\begin{aligned}\hat{\theta}_{EBQ3} &= \int_0^1\int_0^c \left(\frac{r+a-2}{H+b}\right)\left[\frac{2b}{c^2}\right]db\,da \\ &= \left(\frac{2r-3}{c}\right)\left[1-\frac{H}{c}\ln\left(1+\frac{c}{H}\right)\right]\end{aligned} \quad (3\text{-}18)$$

4. Properties of E-Bayesian estimation

This section investigated the relations among the E-Bayesian estimators $\hat{\theta}_{EBSi}, \hat{\theta}_{EBLi}, \hat{\theta}_{EBDi}, \hat{\theta}_{EBQi}\ (i=1,2,3)$

4.1. Relations between $\hat{\theta}_{EBSi}\ (i=1,2,3)$

Proposition 1: It follows from (3-7), (3-8) and (3-9) that

i) $\hat{\theta}_{EBS1} < \hat{\theta}_{EBS2} < \hat{\theta}_{EBS3}$

ii) $\lim_{H\to\infty}\hat{\theta}_{EBS1} = \lim_{H\to\infty}\hat{\theta}_{EBS2} = \lim_{H\to\infty}\hat{\theta}_{EBS3}$

Proof: See Appendix.

4.2. Relations between $\hat{\theta}_{EBLi}\ (i=1,2,3)$

Proposition 2: It follows from (3-10), (3-11) and (3-12) that

i) $\hat{\theta}_{EBL1} < \hat{\theta}_{EBL2} < \hat{\theta}_{EBL3}$

ii) $\lim_{H\to\infty}\hat{\theta}_{EBL1} = \lim_{H\to\infty}\hat{\theta}_{EBL2} = \lim_{H\to\infty}\hat{\theta}_{EBL3}$

Proof: See Appendix.

4.3. Relations between $\hat{\theta}_{EBDi}\ (i=1,2,3)$

Proposition 3: It follows from (3-13), (3-14) and (3-15) that

i) $\hat{\theta}_{EBD3} < \hat{\theta}_{EBD2} < \hat{\theta}_{EBD1}$

ii) $\lim_{H\to\infty}\hat{\theta}_{EBD1} = \lim_{H\to\infty}\hat{\theta}_{EBD2} = \lim_{H\to\infty}\hat{\theta}_{EBD3}$

Proof: See Appendix.

4.4. Relations between $\hat{\theta}_{EBQi}\ (i=1,2,3)$

Proposition 4: It follows from (3-16), (3-17) and (3-18) that

i) $\hat{\theta}_{EBQ1} < \hat{\theta}_{EBQ2} < \hat{\theta}_{EBQ3}$

ii) $\lim_{H\to\infty}\hat{\theta}_{EBQ1}=\lim_{H\to\infty}\hat{\theta}_{EBQ2}=\lim_{H\to\infty}{}_{EBQ3}$

Proof: See Appendix.

5. Monte Carlo simulation study

In order to assess the statistical performances of these estimators, we conducted a Monte Carlo simulation in the following steps:

Step (1): Simulation were performed under different censoring schemes (different values of n, r) and for $\lambda = 2$ and $s=-1, c=2.$

Step (2): We generate a and b from uniform priors distributions (0, 1) and (0, c) respectively given in (3-4), (3-5) and (3-6).

Step (3): For given values of a, b we generate θ from the gamma prior distribution given in (2-4).

Step (4): For known values of λ, type-II censored samples are generated from Kum (λ,θ) with pdf and cdf given in (1-1) and (1-2) respectively through the adoption of inverse transformation method, by using the formula

$$t_i = F^{-1}(U_i) = \left[1-\left(1-F\right)^{\frac{1}{\theta}}\right]^{\frac{1}{\lambda}},\ i=1,2,\dots n$$

Where U is a random variable obeys uniform distribution on the interval (0, 1)

Step (5): Under the SELF, we compute the estimates $\hat{\theta}_{BS}, \hat{\theta}_{EBS1}, \hat{\theta}_{EBS2}$ and $\hat{\theta}_{EBS3}$ of θ from (2-8), (3-7), (3-8) and (3-9) respectively.

Step (6): Under the LLF, we compute the estimates $\hat{\theta}_{BL}, \hat{\theta}_{EBL1}, \hat{\theta}_{EBL2}$ and $\hat{\theta}_{EBL3}$ of θ from (2-11), (3-10), (3-11) and (3-12) respectively.

Step (7): Under the DLF, we compute the estimates $\hat{\theta}_{BD}, \hat{\theta}_{EBD1}, \hat{\theta}_{EBD2}$ and $\hat{\theta}_{EBD3}$ of θ from (2-14), (3-13), (3-14) and (3-15) respectively.

Step (8): Under the QLF, we compute the estimates $\hat{\theta}_{BQ}, \hat{\theta}_{EBQ1}, \hat{\theta}_{EBQ2}$ and $\hat{\theta}_{EBQ3}$ of θ from (2-17), (3-16), (3-17) and (3-18) respectively.

Step (9): We repeat the above steps 10000 times. We then compute the Mean Square Error (MSE) for the estimates for different censoring schemes and given values of c, s, λ where

$$MSE(\hat{\theta}) = \frac{1}{10000}\sum_{i=1}^{10000}(\hat{\theta}_i-\theta)^2$$

And $\hat{\theta}$ stands for an estimator of θ. The simulation results are displayed in Table 1.

Table 1: Averaged Values of Mses. for Estimates of the Parameter θ

n	r	$\hat{\theta}_{BS}$	$\hat{\theta}_{EBS}$	$\hat{\theta}_{BL}$	$\hat{\theta}_{EBL}$	$\hat{\theta}_{BD}$	$\hat{\theta}_{EBD}$	$\hat{\theta}_{BQ}$	$\hat{\theta}_{EBQ}$
			0.041169		0.041034		0.038029		0.051310
25	15	0.041319	0.041325	0.041107	0.041116	0.037960	0.037973	0.051905	0.051898
			0.041518		0.041243		0.037959		0.052425
			0.019775		0.019921		0.019083		0.024211
25	20	0.019724	0.019734	0.019811	0.019818	0.018846	0.018856	0.024482	0.024480
			0.019717		0.019749		0.018664		0.024780
			0.055128		0.054761		0.051566		0.064023
40	25	0.055392	0.055390	0.055003	0.055002	0.051751	0.051751	0.064420	0.064414
			0.055666		0.055257		0.051951		0.064817
			0.010677		0.010674		0.102657		0.012577
40	35	0.010693	0.010693	0.010673	0.010674	0.010224	0.010221	0.012707	0.012706
			0.010721		0.010684		0.010188		0.012844
			0.053963		0.053654		0.051549		0.059339
70	45	0.054174	0.054172	0.053860	0.053859	0.051740	0.051739	0.059586	0.059584
			0.054385		0.054067		0.051932		0.059831
			0.015862		0.015709		0.014932		0.018092
70	55	0.015968	0.015968	0.015811	0.015810	0.015019	0.015019	0.018230	0.018229
			0.016076		0.015915		0.015110		0.018369
			0.018910		0.018765		0.018058		0.020795
100	75	0.019008	0.019008	0.018861	0.018861	0.018148	0.018148	0.020908	0.020908
			0.019107		0.018959		0.018240		0.021022
			0.005894		0.005838		0.005871		0.006672
100	90	0.005934	0.005934	0.005875	0.005875	0.005618	0.005618	0.006728	0.006727
			0.005975		0.005914		0.005607		0.006784

6. Conclusion remarks

- We can conclude based on the results shown in Table 1, the E-Bayesian estimates of θ under SELF, DLF and QLF have smaller MSE as compared with the corresponding Bayes estimates in all cases. On the other hand, the E-Bayesian estimates of θ under LLF are more efficient the associated Bayes estimates in nearly all cases except for $n=40, r=35$ where the Bayes estimates based on LLF are the best.
- In comparing the different E-Bayesian estimates, we can deducted from the results shown in Table 1, that the efficiency of the E-Bayesian estimates of $\hat{\theta}_i (i=1,2,3)$ under SELF, LLF, DLF and QLF can be ordered due to having smaller MSE to be $\hat{\theta}_{EBD} > \hat{\theta}_{EBL} > \hat{\theta}_{EBS} > \hat{\theta}_{EBQ}$.

7. Appendix

Proof of Proposition 1

i) From (3-7), (3-8) and (3-9), we get

$$\hat{\theta}_{EBS2}-\hat{\theta}_{EBS1}=\hat{\theta}_{EBS3}-\hat{\theta}_{EBS2} = \left(\frac{2r+1}{2c}\right)\left[2-\left(1+\frac{2H}{c}\right)\ln\left(1+\frac{c}{H}\right)\right] \qquad (A.1)$$

For $-1<x<1$, we have:

$$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}.$$

Assuming $x=\frac{c}{H}$ when $0<c<H$, $0<\frac{c}{H}<1$, we get

$$2-\left(1+\frac{2H}{c}\right)\ln\left(1+\frac{c}{H}\right)$$

$$=2-\left(1+\frac{2H}{c}\right)\left[\begin{array}{l}\frac{c}{H}-\frac{1}{2}\left(\frac{c^2}{H^2}\right)+\frac{1}{3}\left(\frac{c^3}{H^3}\right)\\ -\frac{1}{4}\left(\frac{c^4}{H^4}\right)+\frac{1}{5}\left(\frac{c^5}{H^5}\right)-\frac{1}{6}\left(\frac{c^6}{H^6}\right)+...\end{array}\right]$$

$$=2-\left[\begin{array}{l}\frac{c}{H}-\frac{1}{2}\left(\frac{c^2}{H^2}\right)+\frac{1}{3}\left(\frac{c^3}{H^3}\right)-\frac{1}{4}\left(\frac{c^4}{H^4}\right)+\frac{1}{5}\left(\frac{c^5}{H^5}\right)-...\\ +2-\frac{c}{H}+\frac{2}{3}\left(\frac{c^2}{H^2}\right)-\frac{2}{4}\left(\frac{c^3}{H^3}\right)+\frac{2}{5}\left(\frac{c^4}{H^4}\right)-\frac{2}{6}\left(\frac{c^5}{H^5}\right)+...\end{array}\right]$$

$$=\frac{1}{2}\left(\frac{c^2}{H^2}\right)-\frac{1}{3}\left(\frac{c^3}{H^3}\right)+\frac{1}{4}\left(\frac{c^4}{H^4}\right)-\frac{1}{5}\left(\frac{c^5}{H^5}\right)+...$$
$$-\frac{2}{3}\left(\frac{c^2}{H^2}\right)+\frac{2}{4}\left(\frac{c^3}{H^3}\right)-\frac{2}{5}\left(\frac{c^4}{H^4}\right)+6\left(\frac{c^5}{H^5}\right)-...$$

$$=\left(\frac{1}{2}-\frac{2}{3}\right)\left(\frac{c^2}{H^2}\right)-\left(\frac{2}{4}-\frac{1}{3}\right)\left(\frac{c^3}{H^3}\right)+\left(\frac{1}{4}-\frac{2}{5}\right)\left(\frac{c^4}{H^4}\right)$$
$$+\left(\frac{2}{6}-\frac{1}{5}\right)\left(\frac{c^5}{H^5}\right)+...$$

$$=\frac{-1}{6}\left(\frac{c^2}{H^2}\right)+\frac{1}{6}\left(\frac{c^3}{H^3}\right)-\frac{3}{20}\left(\frac{c^4}{H^4}\right)+\frac{2}{15}\left(\frac{c^5}{H^5}\right)-...$$

$$=\frac{c^2}{6H^2}\left(1-\frac{c}{H}\right)+\frac{c^4}{H^4}\left(\frac{2c}{15H}-\frac{3}{20}\right)+...$$

$$=\frac{c^2}{6H^2}\left(1-\frac{c}{H}\right)+\frac{c^4}{60H^4}\left(\frac{8c}{H}-9\right)+... \qquad \text{(A.2)}$$

According to (A.1) and (A.2), we have

$$\hat{\theta}_{EBS\,2}-\hat{\theta}_{EBS\,1}=\hat{\theta}_{EBS\,3}-\hat{\theta}_{EBS\,2}>0$$

That is $\hat{\theta}_{EBS\,1}<\hat{\theta}_{EBS\,2}<\hat{\theta}_{EBS\,3}$

ii) From (A.1) and (A.2), we get

$$\lim_{H\to\infty}\left(\hat{\theta}_{EBS\,2}-\hat{\theta}_{EBS\,1}\right)=\lim_{H\to\infty}\left(\hat{\theta}_{EBS\,3}-\hat{\theta}_{EBS\,2}\right)$$
$$=\lim_{H\to\infty}\left[\frac{c^2}{6H^2}\left(1-\frac{c}{H}\right)+\frac{c^4}{60H^4}\left(\frac{8c}{H}-9\right)+...\right]=0$$

That is $\lim_{H\to\infty}\hat{\theta}_{EBS\,1}=\lim_{H\to\infty}\hat{\theta}_{EBS\,2}=\lim_{H\to\infty}\hat{\theta}_{EBS\,3}$

Thus, the proof is complete

Proof of Proposition 2

i) From (3-10), (3-11) and (3-12), we get

$$\hat{\theta}_{EBL\,2}-\hat{\theta}_{EBL\,1}=\hat{\theta}_{EBL\,3}-\hat{\theta}_{EBL\,2}$$
$$=\left(\frac{2r+1}{2sc}\right)\left[\begin{array}{l}-\left[\left(\frac{(H+s)^2}{c}+(H+s)\right)\ln\left(1+\frac{c}{H+s}\right)\right]\\ +\left(\frac{H^2}{c}+H\right)\ln\left(1+\frac{c}{H}\right)+s\end{array}\right] \qquad \text{(A.3)}$$

For $-1<x<1$, we have:

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+...=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}.$$

Assuming $x=\frac{c}{H}$ when $0<c<H$, $0<\frac{c}{H}<1$, we obtain

$$\left[\left(\frac{H^2}{c}+H\right)\ln\left(1+\frac{c}{H}\right)\right]-\left[\left(\frac{(H+s)^2}{c}+(H+s)\right)\ln\left(1+\frac{c}{H+s}\right)\right]+s=$$

$$s+\left(\frac{H^2}{c}+H\right)\left[\frac{c}{H}-\frac{c^2}{2H^2}+\frac{c^3}{3H^3}-\frac{c^4}{4H^4}+\frac{c^5}{5H^5}-...\right]$$

$$-\left(\frac{(H+s)^2}{c}+(H+s)\right)\left[\begin{array}{l}\frac{c}{H+s}-\frac{c^2}{2(H+s)^2}+\frac{c^3}{3(H+s)^3}\\ -\frac{c^4}{4(H+s)^4}+\frac{c^5}{5(H+s)^5}-...\end{array}\right]$$

$$=s+\left[H-\frac{c}{2}+\frac{c^2}{3H}-\frac{c^3}{4H^2}+\frac{c^4}{5H^3}-...\right]$$
$$+\left[c-\frac{c^2}{2H}+\frac{c^3}{3H^2}-\frac{c^4}{4H^3}+\frac{c^5}{5H^4}-...\right]$$

$$-\left[(H+s)-\frac{c}{2}+\frac{c^2}{3(H+s)}-\frac{c^3}{4(H+s)^2}+\frac{c^4}{5(H+s)^3}-...\right]$$

$$+\left[c-\frac{c^2}{2(H+s)}+\frac{c^3}{3(H+s)^2}-\frac{c^4}{4(H+s)^3}+\frac{c^5}{5(H+s)^4}-...\right]$$

$$=s+H-\frac{c}{2}+\frac{c^2}{3H}-\frac{c^3}{4H^2}+\frac{c^4}{5D^3}-...$$
$$+c-\frac{c^2}{2H}+\frac{c^3}{3H^2}-\frac{c^4}{4H^3}+\frac{c^5}{5H^4}-...$$

$$-H-s+\frac{c}{2}-\frac{c^2}{3(H+s)}+\frac{c^3}{4(H+s)^2}-\frac{c^4}{5(H+s)^3}+...$$

$$-c+\frac{c^2}{2(H+s)}-\frac{c^3}{3(H+s)^2}+\frac{c^4}{4(H+s)^3}-\frac{c^5}{5(H+s)^4}$$

$$=\left(\frac{1}{3}-\frac{1}{2}\right)\frac{c^2}{H}-\left(\frac{1}{4}-\frac{1}{3}\right)\frac{c^3}{H^2}+\left(\frac{1}{5}-\frac{1}{4}\right)\frac{c^4}{H^3}-...$$

$$-\left(\frac{1}{3}-\frac{1}{2}\right)\frac{c^2}{(H+s)}+\left(\frac{1}{4}-\frac{1}{3}\right)\frac{c^3}{(H+s)^2}$$
$$-\left(\frac{1}{5}-\frac{1}{4}\right)\frac{c^4}{(H+s)^3}+...$$

$$=\frac{-c^2}{6H}+\frac{c^3}{12H^2}-\frac{c^4}{20H^3}+...$$
$$+\frac{c^2}{6(H+s)}-\frac{c^3}{12(H+s)^2}+\frac{c^4}{20(H+s)^3}-...$$

$$=\frac{c^2}{2(H+s)}\left[\frac{1}{3}-\frac{c}{6(H+s)}+\frac{c^2}{10(H+s)^2}-...\right]$$
$$-\frac{c^2}{2H}\left[\frac{1}{3}+\frac{c}{6H}-\frac{c^2}{10H^2}-...\right] \qquad \text{(A.4)}$$

According to (A.3) and (A.4), we have

$$\hat{\theta}_{EBL2}-\hat{\theta}_{EBL1}=\hat{\theta}_{EBL3}-\hat{\theta}_{EBL2}>0$$

That is $\hat{\theta}_{EBL1}<\hat{\theta}_{EBL2}<\hat{\theta}_{EBL3}$

ii) From (A.3) and (A.4), we get

$$\lim_{H\to\infty}\left(\hat{\theta}_{EBL2}-\hat{\theta}_{EBL1}\right)=\lim_{H\to\infty}\left(\hat{\theta}_{EBL3}-\hat{\theta}_{EBL2}\right)$$

$$=\lim_{H\to\infty}\left\{\frac{c^2}{2(H+s)}\left[\frac{1}{3}-\frac{c}{6(H+s)}+\frac{c^2}{10(H+s)^2}-...\right]\right\}$$

$$-\lim_{H\to\infty}\left\{\frac{c^2}{2H}\left[\frac{1}{3}+\frac{c}{6H}-\frac{c^2}{10H^2}-...\right]\right\}=0$$

That is $\lim_{H\to\infty}\hat{\theta}_{EBL1}=\lim_{H\to\infty}\hat{\theta}_{EBL2}=\lim_{H\to\infty}\hat{\theta}_{EBL3}$

Thus, the proof is complete

Proof of Proposition 3

i) From (3-13), (3-14) and (3-15), we obtain

$$\hat{\theta}_{EBD1}-\hat{\theta}_{EBD2}=\hat{\theta}_{EBD2}-\hat{\theta}_{EBD3}=\left(\frac{2r+3}{2c}\right)\left[\left(1+\frac{2H}{c}\right)\ln\left(1+\frac{c}{H}\right)-2\right] \quad \text{(A.5)}$$

Substituting from (A.2) in (A.5), we get

$$\left(1+\frac{2H}{c}\right)\ln\left(1+\frac{c}{H}\right)-2=\frac{c^2}{6H^2}\left(1-\frac{c}{H}\right)-\frac{c^4}{60H^4}\left(9-\frac{8c}{H}\right)+...$$

$$=\frac{c^2}{6H^2}\left(1-\frac{c}{H}\right)-\frac{c^4}{60H^4}\left(9-\frac{8c}{H}\right)+... \quad \text{(A.6)}$$

According to (A.5) and (A.6), we have

$$\hat{\theta}_{EBD1}-\hat{\theta}_{EBD2}=\hat{\theta}_{EBD2}-\hat{\theta}_{EBD3}>0$$

That is $\hat{\theta}_{EBD3}<\hat{\theta}_{EBD2}<\hat{\theta}_{EBD1}$

ii) From (A.5) and (A.6), we get

$$\lim_{H\to\infty}\left(\hat{\theta}_{EBD1}-\hat{\theta}_{EBD2}\right)=\lim_{H\to\infty}\left(\hat{\theta}_{EBD2}-\hat{\theta}_{EBD3}\right)=\lim_{H\to\infty}\left[\frac{c^2}{6H^2}\left(1-\frac{c}{H}\right)-\frac{c^4}{60H^4}\left(9-\frac{8c}{H}\right)+...\right]=0$$

That is $\lim_{H\to\infty}\hat{\theta}_{EBD1}=\lim_{H\to\infty}\hat{\theta}_{EBD2}=\lim_{H\to\infty}\hat{\theta}_{EBD3}$

Thus, the proof is complete

Proof of Proposition 4

i) From (3-16), (3-17) and (3-18) that

$$\hat{\theta}_{EBQ2}-\hat{\theta}_{EBQ1}=\hat{\theta}_{EBQ3}-\hat{\theta}_{EBQ2}=\left(\frac{2r-3}{2c}\right)\left[\left(1+\frac{2H}{c}\right)\ln\left(1+\frac{c}{H}\right)-2\right] \quad \text{(A.7)}$$

Substituting from (A.2) in (A.7), we get

$$\left(1+\frac{2H}{c}\right)\ln\left(1+\frac{c}{H}\right)-2=\frac{c^2}{6H^2}\left(1-\frac{c}{H}\right)-\frac{c^4}{60H^4}\left(9-\frac{8c}{H}\right)+...$$

$$=\frac{c^2}{6H^2}\left(1-\frac{c}{H}\right)-\frac{c^4}{60H^4}\left(9-\frac{8c}{H}\right)+... \quad \text{(A.8)}$$

According to (A.7) and (A.8), we have

$$\hat{\theta}_{EBQ2}-\hat{\theta}_{EBQ1}=\hat{\theta}_{EBQ3}-\hat{\theta}_{EBQ2}>0$$

That is $\hat{\theta}_{EBQ3}<\hat{\theta}_{EBQ2}<\hat{\theta}_{EBQ1}$

ii) From (A.7) and (A.8), we get

$$\lim_{H\to\infty}\left(\hat{\theta}_{EBQ2}-\hat{\theta}_{EBQ1}\right)=\lim_{H\to\infty}\left(\hat{\theta}_{EBQ3}-\hat{\theta}_{EBQ2}\right)=\lim_{H\to\infty}\left[\frac{c^2}{6H^2}\left(1-\frac{c}{H}\right)-\frac{c^4}{60H^4}\left(9-\frac{8c}{H}\right)+...\right]=0$$

That is $\lim_{H\to\infty}\hat{\theta}_{EBQ1}=\lim_{H\to\infty}\hat{\theta}_{EBQ2}=\lim_{H\to\infty}\hat{\theta}_{EBQ3}$

Thus, the proof is complete

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