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Research paper

Parametric inference for stochastic differential equations with random effects in the drift coefficient

Alsukaini Mohammed Sari*, Wang Xiang-Jun

School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, Hubei, 430074, P.R. China
**Corresponding author E-mail: mohammedsari@yahoo.com*

Abstract

In this paper we focus on estimating the parameters in the stochastic differential equations (SDE's) with drift coefficients depending linearly on a random variables $\emptyset_i$ and μ_i .The distributions of the random effects $\emptyset_i$ and μ_i are depends on unknown parameters from the continuous observations of the independent processes $(X_i(t), t \in [0, T_i], i = 1, \dots, n\,)$. When μ is an unknown parameter or restrict positive constant also studied in this paper. We propose the Gaussian distribution for the random effect $\emptyset_i$ and the exponential distribution for the random effect μ_i , we obtained an explicit formulas for the likelihood functions in each case and find the maximum likelihood estimators of the unknown parameters in the random effects and for the unknown parameter μ . Consistency and asymptotic normality are studied just when $\emptyset_i$ is normal random effect and μ is constant.

Keywords: *Stochastic Differential Equations; Maximum Likelihood Estimator; Linear Random Effects; Fisher Information Matrix; Asymptotic Normality; Consistency.*

1. Introduction

Stochastic differential equations play an important role in modeling various phenomena arising in fields as diverse as finance, physics, chemistry, engineering, biology, neuroscience and others, (Allen (2007)[1], Hindriks (2011) [2] Musiela and Rutkowski (2005) [3], Gugushvili, S. and P. Spreij [4] and Wong and Hajek (1985) [5]).

Parameters estimation in stochastic differential equations is a rapidly expanding area of research; (Nielsen, Madsen and Young(2000) [6], Sørensen (2004) [7]). Statistical estimation of parameters in the diffusion processes has been studied for a long time; Feigin [8] provided a useful historical overview of the early studies and introduced a general asymptotic theory of maximum likelihood estimation for continuous diffusion processes. In the recent years, the stochastic differential equations with random effects have been considered in various works (Overgaard et al. (2005) [9] Tornøe et al. (2005) [10] Ditlevsen and De Gaetano (2005)) and have been the subject of various applications such as pharmacokinetic/pharmacodynamics, neuronal modeling and modeling of electrical circuits (Delattre and Lavelle 2013 [11], Klim, Søren [12],Christoffer [10] ,Donnet and Samson 2013 [13], Picchini et al. 2010 [14], kampowsky and et al(1992)) [15].The problem of estimating parameters in SDE models is not straightforward, except in a simple cases. A natural approach would be likelihood inference, but the transition densities of the process are rarely known, and thus it is usually not possible to write the likelihood function explicitly. Many references proposed approximations for the unknown likelihood function, for general mixed SDEs an approximations of the likelihood have been proposed (Picchini and Ditlevsen, (2011) [16]), linearization (Beal and Sheiner (1982) [17])), approximate the transition density (Pedersen (1995) [18] Brandt and Santa-Clara (2002) [19] Nicolau (2002) [20], Hurn and Lindsay (1999) [21]), by solving numerically the Kolmogorov partial differential equations satisfied by the transition density (Lo [22] (1988)) or approximating the conditional transition density of the diffusion process given the random effects by a Hermit expansion‹ (Aït-Sahalia [23] (2002)). Delattre [24] studied the maximum likelihood estimator for random effects in more generally for fixed T and n tending to infinity and found an explicit expression for likelihood function and exact likelihood estimator by investigate the linear random effect in the drift (multiple case) together with a specific distribution for the random effect.

In this paper we consider the stochastic differential equation with (i- two random effects, ii- random effect and unknown parameter ,iii- random effect and constant) in drift coefficient and suppose that the diffusion coefficient without random effect. We study n real valued stochastic processes $(\,X_i(t), t \in [0, T_i], i = 1, \dots, n\,)$, with dynamics ruled by the following SDEs:

$$dX_i(t) = b(X_i(t), \emptyset_i, \mu_i)dt + \sigma\big(X_i(t)\big)dW_i(t), X_i(0) = x^i, \quad i = 1, \dots, n\,, \quad (1)$$

Where $W_1, \dots, W_n$ are n independent wiener processes. In the first case, $\emptyset_1, \dots, \emptyset_n$ and $\mu_1, \dots, \mu_n$ are n i. i. d. random variables taking values in ($\mathbb{R}$ and $\mathbb{R}^+$) respectively, $\emptyset_1, \dots, \emptyset_n$, $\mu_1, \dots, \mu_n$ and $W_1, \dots, W_n$ are independent and $x^i, i = 1, \dots, n$ are known real values. In The second case we suppose $\mu_i = \mu$ as an unknown parameter in $\mathbb{R}$ and the third case we suppose μ as a positive constant. The functions $b(x)$ and $\sigma(x)$ are known real valued functions. Each process $X_i(t)$ represents an individual, the variables $\emptyset_i$ and μ_i represents the random effects of individual i , the random variables $\emptyset_1, \dots, \emptyset_n$ have a common distribution $g(\varphi, \theta)d\upsilon(\varphi)$ on $\mathbb{R}$ and the random variables $\mu_1, \dots, \mu_n$ have a common distribution $h(\mu, \beta)du(\mu)$ on $\mathbb{R}^+$ where θ and β are an unknown parameters belonging to a set $\Theta \subset \mathbb{R}^p$ where υ and u are a dominating measures.

Our goal is to estimate $\psi=(\theta,\beta)$ from the continuous observations $(X_i(t), t\in[0,T_i], i=1,\dots,n)$ and estimate the unknown parameter μ .We focus on a special case of linear random effect in the drift coefficient in the model (1), i.e. $b(x,\emptyset_i,\mu_i)=\emptyset_i b(X_i(t))+\mu_i$, where b is a known real function and $\emptyset_i$ is a Gaussian and μ_i is an exponential in the first case , μ is unknown parameter in the second case and μ is constant in the third case. An explicit likelihood formula and the maximum likelihood estimators are obtained in the three cases; asymptotic properties are obtained in the third case only.

The structure of the paper is as follows. Section 2 contains the notation and assumptions. The general results of the estimation of the parameters are introduced in section 3. In section 4 we study the asymptotic properties (consistency and asymptotic normality) of the estimators. Conclusion is given in section 5.

2. Notations and assumptions

Consider n real valued stochastic processes $(X_i(t), t\geq 0)$, $i=1,\dots,n$ with dynamics ruled by (1). The processes $W_1,\dots,W_n$ and the random variables $\emptyset_1,\dots,\emptyset_n$ and $\mu_1,\dots,\mu_n$ are defined on a common probability space $(\Omega,\mathcal{F},\mathbb{P})$ .Consider the filtration $(\mathcal{F}_t, t\geq 0)$ defined by $\mathcal{F}_t=\sigma(\emptyset_i,\mu_i,W_i(s), s\leq t, i=1,\dots,n)$. As $\mathcal{F}_t=\sigma(\emptyset_i,\mu_i,W_i(s),s\leq t)\vee\mathcal{F}_t^i$, with $\mathcal{F}_t^i=\sigma(\emptyset_i,\emptyset_j,\mu_i,\mu_j,W_j(s), s\leq t, j\neq i)$ independent of W_i, each process W_i is a $(\mathcal{F}_t, t\geq 0)$-Brownian motion. Moreover, the random variables $\emptyset_i,\mu_i$ are $\mathcal{F}_0$ – measurable. We assume that:

H1

i) The function $b(x,\varphi,\mu)$ is C^1 on $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^+$ and such that:

$\exists K>0, \forall x\in\mathbb{R}, b^2(x,\varphi,\mu)\leq K(1+x^2+|\varphi|^2+|\mu|^2)$,

ii) The function $\sigma(x)$ is C^1 on $\mathbb{R}$ and

$\forall x\in\mathbb{R}, \sigma^2(x)\leq K(1+x^2)$.

From H1, the process $(X_i(t))$ is well define and $(\emptyset_i,\mu_i,X_i(t))$ adapted to filtration $(\mathcal{F}_t, t\geq 0)$.

The n processes $(\emptyset_i,\mu_i,X_i(t), i=1,\dots,n)$ are independent. For all φ, μ and all $x^i\in\mathbb{R}$, the stochastic differential equation

$$dX_i^{\varphi,\mu}(t)=b\left(X_i^{\varphi,\mu}(t),\varphi,\mu\right)dt+\sigma\left(X_i^{\varphi,\mu}(t)\right)dW_i(t),$$

$$X_i^{\varphi,\mu}(0)=x^i. \qquad (2)$$

Admits a unique strong solution process $(X_i^{\varphi,\mu}(t), t\geq 0)$ adapted to filtration $(\mathcal{F}_t, t\geq 0)$. We deduce that the conditional distribution of X_i given $\emptyset_i=\varphi$ and $\mu_i=\mu$ identical to the distribution o f $X_i^{\varphi,\mu}$.

3. A general results of parameters estimation

3.1. When φ and μ are random effects

3.1.1. Exact likelihood

We introduce the distribution $Q_{\varphi,\mu}^{x^i,T_i}$ of $(X_i^{\varphi,\mu}(t), t\in[0,T_i])$.

Let $P_\psi^i=g(\varphi,\theta)dv(\varphi)\otimes h(\mu,\beta)du(\mu)\otimes Q_\varphi^{x^i,T_i}$ denote the joint distribution of $(\emptyset_i,\mu_i,X_i(t))$ and let Q_ψ^i denote the marginal distribution of $(X_i(t), t\in[0,T_i])$. Let us consider the following assumption:

H2 For $i=1,\dots,n$ and for all $\varphi,\mu,\varphi',\mu'$,

$$Q_{\varphi,\mu}^{x^i,T_i}\left(\int_0^{T_i}\frac{b^2(X_i^{\varphi,\mu}(t),\varphi',\mu')}{\sigma^2\left(X_i^{\varphi,\mu}(t)\right)}\,dt<+\infty\right)=1.$$

Under H1-H2, the derivative of the distribution $Q_{\varphi,\mu}^{x^i,T_i}$ with respect to derivative of $Q^i=Q_{\varphi_0,\mu_0}^{x^i,T_i}$ has the density:

$$\frac{dQ_{\varphi,\mu}^{x^i,T_i}}{dQ^i}(X_i)=L_{T_i}(X_i,\varphi,\mu)$$

$$=\exp\left(\int_0^{T_i}\frac{b(X_i(s),\varphi,\mu)}{\sigma^2(X_i(s))}dX_i(s)-\frac{1}{2}\int_0^{T_i}\frac{b^2(X_i(s),\varphi,\mu)}{\sigma^2(X_i(s))}\,ds\right) \qquad (3)$$

(for more details see Liptser and Shiryaev [25]).

The density is depending on the statistics:

$$A_i=\int_0^{T_i}\frac{b(X_i(s))}{\sigma^2(X_i(s))}\,dX_i(s)\ , B_i=\int_0^{T_i}\frac{b^2(X_i(s))}{\sigma^2(X_i(s))}\,ds\ ,$$

$$C_i=\int_0^{T_i}\frac{1}{\sigma^2(X_i(s))}\,dX_i(s),\ D_i=\int_0^{T_i}\frac{1}{\sigma^2(X_i(s))}\,ds,$$

$$E_i=\int_0^{T_i}\frac{b(X_i(s))}{\sigma^2(X_i(s))}\,ds.$$

By independent of individuals, $P_\psi=\otimes_{i=1}^n P_\psi^i$ is the distribution of $(\emptyset_i,\mu_i,X_i(.))$, $i=1,\dots,n$ and $Q_\psi=\otimes_{i=1}^n Q_\psi^i$ is the distribution of the sample $(X_i(t), t\in[0,T_i], i=1,\dots,n)$.

We can compute the density of Q_ψ w.r.t. $Q=\otimes_{i=1}^n Q^i$ as follow:

$$\gamma_i(X_i,\psi)=\frac{dQ_\psi^i}{dQ^i}(X_i)$$

$$=\int_{\mathbb{R}^+}\int_{\mathbb{R}} L_{T_i}(X_i,\varphi,\mu)g(\varphi,\theta)h(\mu,\beta)dv(\varphi)du(\mu)$$

And the exact likelihood of whole sample $(X_i(t), t\in[0,T_i], i=1,\dots,n)$ is

$$\xi_n(\psi)=\prod_{i=1}^n\gamma_i(X_i,\psi).$$

3.1.2. The distributions of the random effects

Consider model (1) with linear random effects in the drift coefficient $b(x,\varphi,\mu)=\varphi b(x)+\mu$ where $\varphi\in\mathbb{R}, \mu\in\mathbb{R}^+$ and $b(.), \sigma(.)$ are known functions. We assume that:

$$\int_0^{T_i}\frac{b^2(X_i(s))}{\sigma^2(X_i(s))}\,ds<\infty\ , Q_{\varphi,\mu}^{x^i,T_i}-a.s,$$

for all φ,μ and for $i=1,\dots,n$; $T_i=T, x^i=x$, so that $(X_i(t), t\in[0,T], i=1,\dots,n)$ are $i.i.d.$ We will use the well define statistics as follow:

$$A_i=\int_0^T\frac{b(X_i(s))}{\sigma^2(X_i(s))}\,dX_i(s)\ ,\ B_i=\int_0^T\frac{b^2(X_i(s))}{\sigma^2(X_i(s))}\,ds$$

$$C_i=\int_0^T\frac{1}{\sigma^2(X_i(s))}\,dX_i(s), D_i=\int_0^T\frac{1}{\sigma^2(X_i(s))}\,ds,$$

$$E_i=\int_0^T\frac{b(X_i(s))}{\sigma^2(X_i(s))}\,ds \qquad (4)$$

So that the density $\gamma_i(X_i,\psi)$ is given by:

$$\gamma_i(X_i,\psi)==\int_{\mathbb{R}^+}\int_{\mathbb{R}} exp\left(\varphi A_i-\frac{1}{2}\varphi^2B_i\right)exp\left(\left(\mu C_i-\frac{1}{2}\mu^2D_i\right)\right)exp(-\varphi\,\mu E_i)\,g(\varphi,\theta)h(\mu,\beta)dv(\varphi)du(\mu) \qquad (5)$$

For a general distributions, $g(\varphi,\theta)dv(\varphi)$ for the random effect $\emptyset_i$ and $h(\mu,\beta)du(\mu)$ for the random effectμ, it is not possible find an explicit expression for $\gamma_i(X_i,\psi)$ above, therefor we propose a specific distributions, Gaussian (λ,ω^2) for the random effect φ and an exponential (β) for the random effect, which will give an explicit likelihood and then find the maximum likelihood estimators of the unknown parameters. In the next proposition an evident expression for $\gamma_i(X_i,\psi)$ is obtained when the above distributions of the random effects is with unknown parameter $\psi=(\lambda,\omega^2,\beta)\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+$ .The true value is denoted by $\psi_0=(\lambda_0,{\omega^2}_0,\beta_0)$.

Proposition3.1 suppose that $g(\varphi,\theta)dv(\varphi)=\mathcal{N}(\lambda,\omega^2)$, and $h(\mu,\beta)du(\mu)=exp(\beta)$ then:

$$\gamma_i(X_i,\psi)=\frac{\beta\sqrt{2\pi}}{\sqrt{D_i(1+\omega^2B_i)-\omega^2{E_i}^2}}exp\left[\frac{\omega^2\,{A_i}^2+2\lambda\,A_i-\lambda^2B_i}{2(1+\omega^2B_i)}\right]\times$$

$$exp\left[\frac{[(1+\omega^2B_i)(\,C_i-\beta)-\,A_iE_i\omega^2-\lambda E_i]^2}{2(1+\omega^2B_i)[D_i(1+\omega^2B_i)-\omega^2{E_i}^2]}\right].$$

Proof: From (5) we compute the joint density of $(\emptyset_i,\mu_i,X_i)$:

$$=\int_{\mathbb{R}^+}\int_{\mathbb{R}}\,exp\left(\varphi A_i-\frac{1}{2}\varphi^2B_i\right)exp\left(\left(\,\mu C_i-\frac{1}{2}\mu^2D_i\right)\right)exp(-\varphi\mu E_i)\frac{1}{\sqrt{2\pi\omega^2}}exp\left(-\frac{1}{2\omega^2}(\varphi-\lambda)^2\right)\beta exp(-\beta\mu)dv(\varphi)du(\mu)$$

We progress the first part of the exponent as follow:

$$(\,\varphi A_i-\frac{1}{2}\varphi^2B_i)-\frac{1}{2\omega^2}(\varphi-\lambda)^2-\,\varphi\mu E_i$$

$$=\,\varphi A_i-\frac{1}{2}\varphi^2B_i-\frac{1}{2\omega^2}\varphi^2+\frac{1}{\omega^2}\varphi\lambda-\frac{1}{2\omega^2}\lambda^2-\varphi\mu E_i$$

$$=\left(-\frac{1}{2}B_i-\frac{1}{2\omega^2}\right)\varphi^2+\left(\,A_i-\mu E_i+\frac{1}{\omega^2}\lambda\right)\varphi-\frac{1}{2\omega^2}\lambda^2$$

$$=\,-\frac{1}{2}\left[(B_i+\omega^{-2})\left(\varphi^2-2\frac{A_i-\mu E_i+\omega^{-2}\lambda}{B_i+\omega^{-2}}\varphi\right)-\,\frac{1}{2}\omega^{-2}\lambda^2\right]$$

$$=-\frac{1}{2(B_i+\omega^{-2})^{-1}}\left(\varphi-\frac{A_i-\mu E_i+\omega^{-2}\lambda}{B_i+\omega^{-2}}\right)^2$$

$$+\frac{(\,A_i-\mu E_i+\omega^{-2}\lambda)^2}{2(B_i+\omega^{-2})}-\frac{1}{2}\omega^{-2}\lambda^2$$

By rearrange the first integral, the first part is normal depend on the random effect φ with mean,

$$m_i=\frac{\lambda+\omega^2(\,A_i-E_i)}{1+\omega^2B_i},$$

And variance,

$$\sigma_i^2=\frac{\omega^2}{1+\omega^2B_i}.$$

By substituting in (5), the result of the first integral is:

$$exp\left[\frac{(\,A_i-\mu E_i+\omega^{-2}\lambda)^2}{2(B_i+\omega^{-2})}-\frac{1}{2}\omega^{-2}\lambda^2\right]\times\frac{\beta}{\sqrt{1+\omega^2B_i}},$$

Using this result in the second integral and developing the exponent as follow:

$$\frac{(\,A_i-\mu E_i+\omega^{-2}\lambda)^2}{2(B_i+\omega^{-2})}-\frac{1}{2}\omega^{-2}\lambda^2-\beta\mu+\,\mu C_i-\frac{1}{2}\mu^2D_i$$

$$=\,\left(-\frac{1}{2}D_i+\frac{1}{2}\frac{E_i^2\omega^2}{1+\omega^2B_i}\right)\mu^2$$

$$+\left(\,C_i-\beta-\frac{2\,A_iE_i\omega^2+2\lambda E_i}{2(1+\omega^2B_i)}\right)\mu+\frac{{A_i}^2\omega^2+2\lambda\,A_i-B_i\lambda^2}{2(1+\omega^2B_i)}$$

$$=\frac{-1}{2\alpha_i^2}(\mu-\varrho_i)^2+\frac{{\varrho_i}^2}{2\alpha_i^2}+\,\frac{{A_i}^2\omega^2+2\lambda\,A_i-B_i\lambda^2}{2(1+\omega^2B_i)}$$

Where,

$$\varrho_i=\frac{(1+\omega^2B_i)\,(\,C_i-\beta)-A_iE_i\omega^2-\lambda E_i}{D_i(1+\omega^2B_i)-E_i^2\omega^2},$$

And

$$\alpha_i^2=\left(\frac{(1+\omega^2B_i)D_i-E_i^2\omega^2}{1+\omega^2B_i}\right)^{-1}.$$

Now, by splitting the result into two parts that are independent and dependent on the random effect μ respectively.

Then the second integral gives us $\gamma_i(X_i,\psi)$.

3.1.3. Estimators of parameters of the random effects

We use the maximum likelihood approach to estimate $\psi=(\lambda,\omega^2,\beta)$, the likelihood function is written as:

$$\xi_n(\psi)\,=\prod_{i=1}^n\,\gamma_i(X_i,\psi).$$

The logarithm of likelihood function is,

$$\mathcal{L}_n(\psi)=log\xi_n(\psi)$$

$$=log(2\pi)^{\frac{n}{2}}+log\beta^n-\sum_{i=1}^n log\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)^{\frac{1}{2}}$$

$$+\sum_{i=1}^n\frac{{A_i}^2\omega^2+2\lambda\,A_i-B_i\lambda^2}{2(1+\omega^2B_i)}$$

$$+\sum_{i=1}^n\frac{1}{2}\frac{((1+\omega^2B_i)\,(\,C_i-\beta)-A_iE_i\omega^2-\lambda E_i)^2}{(1+\omega^2B_i)\left(D_i(1+\omega^2B_i)-{E_i}^2\omega^2\right)} \qquad (6)$$

With score function

$$S_n(\psi)=\left(\frac{\partial}{\partial\lambda}\mathcal{L}_n(\psi)\;\frac{\partial}{\partial\omega^2}\mathcal{L}_n(\psi)\;\frac{\partial}{\partial\beta}\mathcal{L}_n(\psi)\right)'$$

Where

$$\frac{\partial}{\partial\lambda}\mathcal{L}_n(\psi)=\sum_{i=1}^n\frac{2A_i-2\lambda B_i}{2(1+\omega^2B_i)}$$

$$-\sum_{i=1}^n\frac{((1+\omega^2B_i)\,(\,C_i-\beta)-A_iE_i\omega^2-\lambda E_i)\times E_i}{(1+\omega^2B_i)\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)}$$

$$=\sum_{i=1}^n\frac{A_i}{1+\omega^2B_i}-\lambda\sum_{i=1}^n\frac{B_i}{1+\omega^2B_i}$$

$$+\lambda\sum_{i=1}^n\frac{E_i^2}{(1+\omega^2B_i)\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)}$$

$$-\sum_{i=1}^n\frac{E_i(1+\omega^2B_i)(\,C_i-\beta)}{(1+\omega^2B_i)\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)}$$

$$+\sum_{i=1}^n\frac{A_iE_i^2\omega^2}{(1+\omega^2B_i)\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)}$$

$$=\lambda\left(\sum_{i=1}^n\frac{E_i^2}{(1+\omega^2B_i)\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)}-\frac{B_i}{1+\omega^2B_i}\right)$$

$$+\sum_{i=1}^n\left[\frac{A_i}{1+\omega^2B_i}+\frac{A_iE_i^2\omega^2}{(1+\omega^2B_i)\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)}-\frac{E_i(\,C_i-\beta)}{\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)}\right].$$

$$\frac{\partial}{\partial\beta}\mathcal{L}_n(\psi)=$$

$$\frac{n}{\beta}-\sum_{i=1}^{n}\frac{\left((1+\omega^2B_i)(C_i-\beta)-A_iE_i\omega^2-\lambda E_i\right)(1+\omega^2B_i)}{(1+\omega^2B_i)\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)}$$

$$=\frac{n}{\beta}-\sum_{i=1}^{n}\frac{(1+\omega^2B_i)(C_i-\beta)}{D_i(1+\omega^2B_i)-E_i^2\omega^2}$$

$$+\sum_{i=1}^{n}\frac{E_i(A_i\omega^2-\lambda)}{D_i(1+\omega^2B_i)-E_i^2\omega^2}$$

$$=\sum_{i=1}^{n}\frac{(1+\omega^2B_i)}{D_i(1+\omega^2B_i)-E_i^2\omega^2}\beta^2$$

$$-\sum_{i=1}^{n}\frac{\left(C_i(1+\omega^2B_i)-E_i(A_i\omega^2-\lambda)\right)}{D_i(1+\omega^2B_i)-E_i^2\omega^2}\beta+n.$$

$$\frac{\partial}{\partial\omega^2}\mathcal{L}_n(\psi)=-\frac{1}{2}\sum_{i=1}^{n}\frac{D_iB_i-E_i^2}{D_i(1+\omega^2B_i)-E_i^2\omega^2}$$

$$+\sum_{i=1}^{n}\frac{2A_i^2(1+\omega^2B_i)-2B_i(\omega^2A_i^2+2\lambda A_i-\lambda^2B_i)}{\left(2(1+\omega^2B_i)\right)^2}$$

$$-\frac{1}{2}\sum_{i=1}^{n}\frac{\left((1+\omega^2B_i)(C_i-\beta)-E_i(A_i\omega^2+\lambda)\right)^2}{(1+\omega^2B_i)^2\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)^2}\times$$

$$\left[(1+\omega^2B_i)\left(D_iB_i-E_i^2\right)+B_i\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)\right]$$

$$+\sum_{i=1}^{n}\frac{[B_i(C_i-\beta)-A_iE_i][(1+\omega^2B_i)(C_i-\beta)-E_i(A_i\omega^2+\lambda)]}{(1+\omega^2B_iD_i)\left((1+\omega^2B_i)-E_i^2\omega^2\right)}.$$

We study the estimators defined by the estimating function

$$S_n(\hat{\psi}_n)=0\ ,$$

Hence, the MLEs of $\psi_0=(\lambda_0,\omega_0^2,\beta_0)$ are given by the system:

$$\hat{\lambda}_n=$$

$$\frac{\sum_{i=1}^{n}\left[\frac{E_i(C_i-\hat{\beta}_n)\left(1+\widehat{\omega_n^2}B_i\right)-A_iE_i^2\widehat{\omega_n^2}-\left(D_i\left(1+\widehat{\omega_n^2}B_i\right)-E_i^2\widehat{\omega_n^2}\right)A_i}{\left(1+\widehat{\omega_n^2}B_i\right)\left(D_i\left(1+\widehat{\omega_n^2}B_i\right)-E_i^2\widehat{\omega_n^2}\right)}\right]}{\sum_{i=1}^{n}\left[\frac{E_i^2}{\left(1+\widehat{\omega_n^2}B_i\right)\left(D_i\left(1+\widehat{\omega_n^2}B_i\right)-E_i^2\widehat{\omega_n^2}\right)}-\frac{B_i}{1+\widehat{\omega_n^2}B_i}\right]},$$

$$\hat{\beta}_n=\frac{H_i\pm\sqrt{H_i^2-4nK_i}}{2K_i},$$

Where,

$$H_i=\sum_{i=1}^{n}\frac{C_i\left(1+\widehat{\omega_n^2}B_i\right)-E_i\left(A_i\widehat{\omega_n^2}-\hat{\lambda}_n\right)}{D_i\left(1+\widehat{\omega_n^2}B_i\right)-{E_i}^2\widehat{\omega_n^2}},$$

And

$$K_i=\sum_{i=1}^{n}\frac{\left(1+\widehat{\omega_n^2}B_i\right)}{D_i\left(1+\widehat{\omega_n^2}B_i\right)-E_i^2\widehat{\omega_n^2}}.$$

$$\sum_{i=1}^{n}\frac{D_iB_i-E_i^2}{D_i\left(1+\widehat{\omega_n^2}B_i\right)-E_i^2\widehat{\omega_n^2}}=$$

$$+\sum_{i=1}^{n}\frac{\left(\left(1+\widehat{\omega_n^2}B_i\right)(C_i-\hat{\beta}_n)-E_i\left(A_i\widehat{\omega_n^2}+\hat{\lambda}_n\right)\right)^2}{\left(1+\widehat{\omega_n^2}B_i\right)^2\left(D_i\left(1+\widehat{\omega_n^2}B_i\right)-E_i^2\widehat{\omega_n^2}\right)^2}$$

$$\times\left[\left(1+2\widehat{\omega_n^2}B_i\right)\left(D_iB_i-E_i^2\right)+D_iE_i\right]$$

$$-\sum_{i=1}^{n}\frac{\left[2\left[B_i(C_i-\hat{\beta}_n)-A_iE_i\right]\left[\left(1+\widehat{\omega_n^2}B_i\right)(C_i-\hat{\beta}_n)-E_i\left(A_i\widehat{\omega_n^2}+\hat{\lambda}_n\right)\right]\right]}{\left(1+\widehat{\omega_n^2}B_i\right)\left(D_i\left(1+\widehat{\omega_n^2}B_i\right)-E_i^2\widehat{\omega_n^2}\right)}$$

$$-\sum_{i=1}^{n}\frac{A_i^2-2\hat{\lambda}_nA_iB_i+\hat{\lambda}_n^2B_i^2}{2\left(1+\widehat{\omega_n^2}B_i\right)^2}.$$

For the fisher information matrix

$$I(\psi)=$$

$$-\begin{pmatrix} E_\psi\left(\frac{\partial^2}{\partial\lambda^2}\mathcal{L}_n(\psi)\right) & E_\psi\left(\frac{\partial^2}{\partial\lambda\partial\omega^2}\mathcal{L}_n(\psi)\right) & E_\psi\left(\frac{\partial^2}{\partial\lambda\partial\beta}\mathcal{L}_n(\psi)\right)\\ E_\psi\left(\frac{\partial^2}{\partial\omega^2\partial\lambda}\mathcal{L}_n(\psi)\right) & E_\psi\left(\frac{\partial^2}{\partial\omega^2\partial\omega^2}\mathcal{L}_n(\psi)\right) & E_\psi\left(\frac{\partial^2}{\partial\omega^2\partial\beta}\mathcal{L}_n(\psi)\right)\\ E_\psi\left(\frac{\partial^2}{\partial\beta\partial\lambda}\mathcal{L}_n(\psi)\right) & E_\psi\left(\frac{\partial^2}{\partial\beta\partial\omega^2}\mathcal{L}_n(\psi)\right) & E_\psi\left(\frac{\partial^2}{\partial\beta^2}\mathcal{L}_n(\psi)\right)\end{pmatrix}, \quad (7)$$

We get:

$$\frac{\partial^2}{\partial\lambda^2}\mathcal{L}_n(\psi)=\sum_{i=1}^{n}\frac{E_i^2}{(1+\omega^2B_i)\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)}\quad-\frac{B_i}{1+\omega^2B_i} \quad (8)$$

$$\frac{\partial^2}{\partial\lambda\partial\beta}\mathcal{L}_n(\psi)=\ \sum_{i=1}^{n}\frac{E_i}{\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)}$$

$$=\frac{\partial^2}{\partial\beta\partial\lambda}\mathcal{L}_n(\psi) \quad (9)$$

$$\frac{\partial^2}{\partial\lambda\partial\omega^2}\mathcal{L}_n(\psi)=\sum_{i=1}^{n}\frac{-(A_i-\lambda B_i)\mathrm{B}_i}{(1+\omega^2B_i)^2}$$

$$-\sum_{i=1}^{n}\frac{B_iE_i(C_i-\beta)-A_iE_i^2}{(1+\omega^2B_i)\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)}$$

$$+\sum_{i=1}^{n}\frac{[(1+\omega^2B_i)(C_i-\beta)E_i-E_i^2(C_i-\beta)]}{(1+\omega^2B_i)^2\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)^2}$$

$$\times\left[2D_iB_i(1+\omega^2B_i)-E_i^2(1+2\omega^2B_i)\right].$$

After some elementary algebra we get:

$$\frac{\partial^2}{\partial\lambda\partial\omega^2}\mathcal{L}_n(\psi)=\frac{\partial^2}{\partial\omega^2\partial\lambda}\mathcal{L}_n(\psi) \quad (10)$$

$$\frac{\partial^2}{\partial\beta^2}\mathcal{L}_n(\psi)=2\beta K_i-H_i \quad (11)$$

$$\frac{\partial^2}{\partial\omega^2\partial\beta}\mathcal{L}_n(\psi)=$$

$$\sum_{i=1}^{n}\frac{-(1+\omega^2B_i)[B_i(C_i-\beta)-A_iE_i]}{(1+\omega^2B_i)\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)}$$

$$-\sum_{i=1}^{n}\frac{[(1+\omega^2B_i)(C_i-\beta)-E_i(A_i\omega^2+\lambda)]B_i}{(1+\omega^2B_i)\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)}$$

$$+\sum_{i=1}^{n}\frac{[(1+\omega^2B_i)(C_i-\beta)-E_i(A_i\omega^2+\lambda)]}{(1+\omega^2B_i)\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)^2}$$

$$\times\left[(1+\omega^2B_i)\left(D_iB_i-E_i^2\right)+B_i\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)\right] \quad (12)$$

After some elementary algebra we get:

$$\frac{\partial^2}{\partial\omega^2\partial\beta}\mathcal{L}_n(\psi)=\frac{\partial^2}{\partial\beta\partial\omega^2}\mathcal{L}_n(\psi) \quad (13)$$

$$\frac{\partial^2}{\partial\omega^2\partial\omega^2}\mathcal{L}_n(\psi)=\frac{1}{2}\sum_{i=1}^{n}\frac{\left(D_iB_i-E_i^2\right)^2}{\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)^2}$$

$$-\sum_{i=1}^{n}\frac{B_i^2\left(\omega^2A_i^2+2\lambda A_i-\lambda^2B_i\right)}{(1+\omega^2B_i)^3}$$

$$+\sum_{i=1}^{n}\frac{[B_i(C_i-\beta)-A_iE_i]B_i(C_i-\beta)}{(1+\omega^2B_i)\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)}$$

$$-\sum_{i=1}^{n}\frac{[(1+\omega^2B_i)(C_i-\beta)-E_i(A_i\omega^2+\lambda)]^2B_i\left(D_iB_i-E_i^2\right)}{(1+\omega^2B_i)^2\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)^2}$$

$$\times\left[2D_iB_i(1+\omega^2B_i)-E_i^2(1-2\omega^2B_i)\right]$$

$$\times\left[\frac{(D_iB_i-E_i^2)(1+\omega^2B_i)+\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)(D_iB_i-E_i^2)B_i}{(1+\omega^2B_i)^3\left(D_i(1+\omega^2B_i)-E_i^2\omega^2\right)^3}\right]. \quad (14)$$

3.2. When φ is random effect and μ is a parameter

We introduce the distribution $Q_{\varphi,\mu}^{x^i,T_i}$ of$(X_i^{\varphi,\mu}(t), t\in[0,T_i])$, where μ is unknown pararmeter.

Let $P_\psi^i = g(\varphi,\theta)dv(\varphi)\otimes Q_\varphi^{x^i,T_i}$ denote the joint distribution of $(\emptyset_i, X_i(t))$ and let Q_ψ^i denote the marginal distribution of $(X_i(t), t\in[0,T_i])$. Let us consider the following assumption:

H3 For $i=1,\dots,n$ and for all $,\varphi'$,

$$Q_\varphi^{x^i,T_i}\left(\int_0^{T_i}\frac{b^2\left(X_i^{\varphi,\mu}(t),\varphi'\right)}{\sigma^2\left(X_i^{\varphi,\mu}(t)\right)}\,dt<+\infty\right)=1.$$

By the same approach of part one, we can derivative the joint density and estimators of the parameters (except there is no distribution of μ and the random effect φ is normal) as follow:

$$\gamma_i(X_i,\psi) = \int_{\mathbb{R}} exp\left(\varphi A_i-\frac{1}{2}\varphi^2B_i\right)exp\left(\mu C_i-\frac{1}{2}\mu^2D_i\right)exp(-\varphi\,\mu E_i)\,g(\varphi,\theta)dv(\varphi)$$

Hence,

$$\gamma_i(X_i,\psi)=\frac{1}{\sqrt{1+\omega^2B_i}}exp\left[\frac{-B_i}{2(1+\omega^2B_i)}\left(-\frac{(A_i-\mu E_i)}{B_i}\right)^2\right]$$

$$\times exp\left(\frac{(A_i-\mu E_i)^2}{2B_i}\right)\times exp\left(\mu C_i-\frac{1}{2}\mu^2D_i\right).$$

And

$$\mathcal{L}_n(\psi)=-\frac{1}{2}\sum_{i=1}^n log(1+\omega^2B_i)$$

$$-\sum_{i=1}^n\frac{-B_i}{2(1+\omega^2B_i)}\left(\lambda-\frac{(A_i-\mu E_i)}{B_i}\right)^2$$

$$+\sum_{i=1}^n\frac{(A_i-\mu E_i)^2}{2B_i}+\sum_{i=1}^n\left(\mu C_i-\frac{1}{2}\mu^2D_i\right) \quad (15)$$

The estimators are given by the system:

$$\hat\lambda_n=\sum_{i=1}^n\frac{A_i-\hat\mu_nE_i}{1+\widehat{\omega_n^2}B_i}\Big/\sum_{i=1}^n\frac{B_i}{1+\widehat{\omega_n^2}B_i},$$

$$\sum_{i=1}^n\left(\frac{A_i-\hat\mu_nE_i}{1+\widehat{\omega_n^2}B_i}-\hat\lambda_n\frac{B_i}{1+\widehat{\omega_n^2}B_i}\right)^2=\sum_{i=1}^n\frac{B_i}{1+\widehat{\omega_n^2}B_i},$$

$$\hat\mu_n=\frac{\sum_{i=1}^n\left[\frac{E_i\hat\lambda_nB_i-A_iE_i}{B_i\left(1+\widehat{\omega_n^2}B_i\right)}-\frac{A_iE_i}{B_i^2}+C_i\right]}{\sum_{i=1}^n\left[\frac{D_i^2}{B_i\left(1+\widehat{\omega_n^2}B_i\right)}-\frac{D_i^2}{B_i^2}+D_i\right]}.$$

The fisher information matrix is,

$$I(\psi)=$$

$$-\begin{pmatrix} E_\psi\left(\frac{\partial^2}{\partial\lambda^2}\mathcal{L}_n(\psi)\right) & E_\psi\left(\frac{\partial^2}{\partial\lambda\partial\omega^2}\mathcal{L}_n(\psi)\right) & E_\psi\left(\frac{\partial^2}{\partial\lambda\partial\mu}\mathcal{L}_n(\psi)\right)\\ E_\psi\left(\frac{\partial^2}{\partial\omega^2\partial\lambda}\mathcal{L}_n(\psi)\right) & E_\psi\left(\frac{\partial^2}{\partial\omega^2\partial\omega^2}\mathcal{L}_n(\psi)\right) & E_\psi\left(\frac{\partial^2}{\partial\omega^2\partial\mu}\mathcal{L}_n(\psi)\right)\\ E_\psi\left(\frac{\partial^2}{\partial\mu\partial\lambda}\mathcal{L}_n(\psi)\right) & E_\psi\left(\frac{\partial^2}{\partial\mu\partial\omega^2}\mathcal{L}_n(\psi)\right) & E_\psi\left(\frac{\partial^2}{\partial\mu^2}\mathcal{L}_n(\psi)\right)\end{pmatrix}, \quad (16)$$

Where,

$$\frac{\partial^2}{\partial\lambda^2}\mathcal{L}_n(\psi)=\sum_{i=1}^n\frac{B_i}{1+\omega^2B_i}$$

$$\frac{\partial^2}{\partial\lambda\partial\omega^2}\mathcal{L}_n(\psi)=\sum_{i=1}^n\left[\frac{-B_i(A_i-\mu E_i)}{(1+\omega^2B_i)^2}+\lambda\frac{B_i^2}{(1+\omega^2B_i)^2}\right]$$

$$=\frac{\partial^2}{\partial\omega^2\partial\lambda}\mathcal{L}_n(\psi)$$

$$\frac{\partial^2}{\partial\lambda\partial\mu}\mathcal{L}_n(\psi)=\sum_{i=1}^n\frac{-E_i}{1+\omega^2B_i}=\frac{\partial^2}{\partial\mu\partial\lambda}\mathcal{L}_n(\psi)$$

$$\frac{\partial^2}{\partial\omega^2\partial\omega^2}\mathcal{L}_n(\psi)=\sum_{i=1}^n\left[\left(\frac{-B_i(A_i-\mu E_i)}{(1+\omega^2B_i)^2}+\lambda\frac{B_i^2}{(1+\omega^2B_i)^2}\right)+\frac{B_i^2}{(1+\omega^2B_i)^2}\right]$$

$$\frac{\partial^2}{\partial\omega^2\partial\mu}\mathcal{L}_n(\psi)$$

$$=\sum_{i=1}^n\left[\left(\frac{A_i-\mu E_i}{1+\omega^2B_i}+\lambda\frac{B_i}{1+\omega^2B_i}\right)\frac{-E_i}{1+\omega^2B_i}\right]$$

$$=\frac{\partial^2}{\partial\mu\partial\omega^2}\mathcal{L}_n(\psi)$$

$$\frac{\partial^2}{\partial\mu^2}\mathcal{L}_n(\psi)=\sum_{i=1}^n\left[\frac{E_i^2}{B_i^2}-\frac{E_i^2}{(1+\omega^2B_i)B_i}-D_i\right].$$

3.3. When φ is random effect with normal distribution and μ is a constant (μ > 0).

From equation (15), we have the estimators for the parameters λ and ω^2 by the system:

$$\hat\lambda_n=\sum_{i=1}^n\frac{A_i-\mu E_i}{1+\widehat{\omega_n^2}B_i}\Big/\sum_{i=1}^n\frac{B_i}{1+\widehat{\omega_n^2}B_i},$$

$$\sum_{i=1}^n\left(\frac{A_i-\mu E_i}{1+\widehat{\omega_n^2}B_i}-\hat\lambda_n\frac{B_i}{1+\widehat{\omega_n^2}B_i}\right)^2=\sum_{i=1}^n\frac{B_i}{1+\widehat{\omega_n^2}B_i}$$

By suppose that $F_i=A_i-\mu E_i$,

$$\hat\lambda_n=\sum_{i=1}^n\frac{F_i}{1+\widehat{\omega_n^2}B_i}\Big/\sum_{i=1}^n\frac{B_i}{1+\widehat{\omega_n^2}B_i},$$

$$\sum_{i=1}^n\left(\frac{F_i}{1+\widehat{\omega_n^2}B_i}-\hat\lambda_n\frac{B_i}{1+\widehat{\omega_n^2}B_i}\right)^2=\sum_{i=1}^n\frac{B_i}{1+\widehat{\omega_n^2}B_i}$$

In order to studying the maximum likelihood estimator of $\psi=(\lambda,\omega^2)$, we consider properties of the following random variables:

$$\mathcal{J}_i(\psi)=\frac{F_i-\lambda B_i}{1+\omega^2B_i}\ ,\mathcal{S}_i(\omega^2)=\frac{B_i}{1+\omega^2B_i}. \quad (17)$$

The score function is

$$\frac{\partial}{\partial\lambda}\mathcal{L}_n(\psi)=\sum_{i=1}^n\mathcal{J}_i(\psi),$$

$$\frac{\partial}{\partial\omega^2}\mathcal{L}_n(\psi)=\frac{1}{2}\sum_{i=1}^n\left(\mathcal{J}_i^2(\psi)-\mathcal{S}_i(\omega^2)\right) \quad (18)$$

We need the following lemma to study an important moments of $\mathcal{J}_i(\psi),\mathcal{S}_i(\omega^2)$.

Lemma 1: *For all* $\psi=(\lambda,\omega^2)\in\mathbb{R}\times\mathbb{R}^+$ *and all* $h\in\mathbb{R}$,

$$E_\psi(exp(h\frac{F_1}{1+\omega^2B_1}))<+\infty.$$

Proof: From (17), we set $\mathcal{J}_1(\psi)=\mathcal{J}_1$ and $\mathcal{S}_1(\omega^2)=\mathcal{S}_1$. Let $l(X_1,\psi)=log\gamma_1(X_1,\psi)$and set $\psi(h)=(\lambda+h,\omega^2)$, then:

$$l(X_1,\psi(h))=log\gamma_1(X_1,\psi)$$

$= -\frac{1}{2}log(1+\omega^2 B_i)$

$-\frac{B_i}{2(1+\omega^2 B_i)}\left(\lambda - \frac{(A_i-\mu E_i)}{B_i}\right)^2 + \frac{(A_i-\mu E_i)^2}{2B_i}$

$+\left(\mu C_i - \frac{1}{2}\mu^2 D_i\right)$

$= l(X_1,\psi) + h\frac{F_i-\lambda B_1}{1+\omega^2 B_1} - \frac{F_1^2}{2}\mathcal{S}_1 + \left(\mu C_1 - \frac{1}{2}\mu^2 D_1\right)$

$= l(X_1,\psi) + h\mathcal{J}_1 - \frac{h^2}{2}\mathcal{S}_1 + \left(\mu C_1 - \frac{1}{2}\mu^2 D_1\right).$

The first and second derivative w.r.t λ are given as follow,

$\frac{\partial}{\partial\lambda} l(X_1,\psi) = -\frac{B_i}{(1+\omega^2 B_1)}\left(\lambda - \frac{F_i}{B_1}\right) = \mathcal{J}_1$

, and

$\frac{\partial^2}{\partial\lambda^2} l(X_1,\psi) = -\mathcal{S}_1.$

Hence,

$\gamma_1(X_1,\psi)exp(h\mathcal{J}_1) = l\big(X_1,\psi(h)\big)exp\left(\frac{h^2}{2}\mathcal{S}_1\right),$

And since $\mathcal{S}_1 \le \frac{1}{\omega^2}$,

$E_\psi exp(h\mathcal{J}_1) = E_{\psi(h)} exp\left(\frac{h^2}{2}\mathcal{S}_1\right)$

$\le exp\left(\frac{h^2}{2\omega^2}\right)$

$< +\infty$,

Then,

$E_\psi(exp(h\frac{F_1}{1+\omega^2 B_1})) \le E_\psi exp(h\mathcal{J}_1)exp\left(\frac{(h+\lambda)^2}{4\omega^2}\right)$

$< +\infty.$

Proposition 3.2 For all $\psi = (\lambda,\omega^2) \in \mathbb{R}\times\mathbb{R}^+$, the following relations hold:

$E_\psi\big(\mathcal{J}_1(\psi)\big) = 0, E_\psi\left(\mathcal{J}_1^2(\psi)\right) = E_\psi\big(\mathcal{S}_1(\omega^2)\big),$

$E_\psi\left(\mathcal{J}_1^3(\psi)\right) = 3E_\psi\big(\mathcal{J}_1(\psi)\mathcal{S}_1(\omega^2)\big),$

$E_\psi\left(\mathcal{J}_1^2(\psi) - \mathcal{S}_1(\omega^2)\right)^2$

$= 4E_\psi\left(\mathcal{J}_1^2(\psi)\mathcal{S}_1(\omega^2)\right) - 2E_\psi\left(\mathcal{S}_1^2(\omega^2)\right).$

Proof: We set $\mathcal{J}_1(\psi) = \mathcal{J}_1$ and $\mathcal{S}_1(\omega^2) = \mathcal{S}_1$. Let $\psi = (\lambda,\omega^2)$ and $\tau = (0,\omega^2)$ and from the relation:

$\gamma_i(X_i,\psi) = \frac{dQ_\psi^i}{dQ^i}(X_i),$

We set

$g_1(\psi) = \frac{\gamma_1(X_1,\psi)}{\gamma_1(X_1,\tau)} = \frac{dQ_\psi^1}{dQ^1}\Big/\frac{dQ_\tau^1}{dQ^1} = \frac{dQ_\psi^1}{dQ_\tau^1}$

$= exp\left(\lambda\frac{F_1}{1+\omega^2 B_1} - \frac{\lambda^2 B_1}{2(1+\omega^2 B_1)}\right),$

So that

$\int_{C_T} g_1(\psi)\, dQ_\tau^1 = 1.$

Provided that we can swap derivation with respect to λ and integration with respect to Q_τ^1 ,

$$\int_{C_T} \frac{\partial^j g_1}{\partial\lambda^j}(\psi)\, dQ_\tau^1 = 0, \tag{19}$$

Hold for $j \ge 1$, where C_T denote the space of all real continuous functions $(x(t), t \in [0,T])$ defined on $[0,T]$.
For = 1,2,3,4 , the moments relations hold from (19) as follow:

$\frac{\partial g_1}{\partial\lambda}(\psi) = \mathcal{J}_1 g_1(\psi), \frac{\partial^2 g_1}{\partial\lambda^2}(\psi) = (\mathcal{J}_1^2 - \mathcal{S}_1)g_1(\psi),$

$\frac{\partial^3 g_1}{\partial\lambda^3}(\psi) = (\mathcal{J}_1^3 - 3\mathcal{J}_1\mathcal{S}_1)g_1(\psi),$

$\frac{\partial^4 g_1}{\partial\lambda^4}(\psi) = (\mathcal{J}_1^4 - 6\mathcal{J}_1^2\mathcal{S}_1 + 3\mathcal{S}_1^2)g_1(\psi).$

Then,

$\int_{C_T}\frac{\partial g_1}{\partial\lambda}(\psi)\, dQ_\tau^1 = \int_{C_T}\mathcal{J}_1 g_1(\psi)dQ_\tau^1 = 0$ yield , $E_\psi\big(\mathcal{J}_1(\psi)\big) = 0,$

$\frac{\partial^2 g_1}{\partial\lambda^2}(\psi)dQ_\tau^1 = \int_{C_T}(\mathcal{J}_1^2 - \mathcal{S}_1)g_1(\psi)dQ_\tau^1$ yields , $E_\psi\left(\mathcal{J}_1^2(\psi)\right) = E_\psi(\mathcal{S}_1),$

$\frac{\partial^3 g_1}{\partial\lambda^3}(\psi)dQ_\tau^1 = \int_{C_T}(\mathcal{J}_1^3 - 3\mathcal{J}_1\mathcal{S}_1)g_1(\psi)dQ_\tau^1$ yield, $E_\psi\left(\mathcal{J}_1^3(\psi)\right) = 3E_\psi\big(\mathcal{J}_1(\psi)\mathcal{S}_1(\omega^2)\big),$

And, $\frac{\partial^4 g_1}{\partial\lambda^4}(\psi)dQ_\tau^1 = \int_{C_T}(\mathcal{J}_1^4 - 6\mathcal{J}_1^2\mathcal{S}_1 + 3\mathcal{S}_1^2)g_1(\psi)dQ_\tau^1$ yield,

$E_\psi\left(\mathcal{J}_1^2(\psi) - \mathcal{S}_1(\omega^2)\right)^2 = 4E_\psi\left(\mathcal{J}_1^2(\psi)\mathcal{S}_1(\omega^2)\right)$

$-2E_\psi\left(\mathcal{S}_1^2(\omega^2)\right).$

For justify the swap of derivation and integration. Let us fix $\bar{\lambda}$ and $\epsilon > 0$. For $\lambda \in [\bar{\lambda}-\epsilon, \bar{\lambda}+\epsilon]$, we have the bound

$\left|\frac{\partial g_1}{\partial\lambda}(\psi)\right| \le \left(\left|\frac{F_1}{1+\omega^2 B_1}\right| + \frac{C}{\omega^2}\right)\left(exp((\bar{\lambda}-\epsilon)\frac{F_1}{1+\omega^2 B_1}) + exp((\bar{\lambda}+\epsilon)\frac{F_1}{1+\omega^2 B_1})\right)$

Where $C = |\bar{\lambda}-\epsilon| + |\bar{\lambda}+\epsilon|$.

The upper bound is independent of λ and integrable w.r.t. Q_τ^1 by lemma 1, by the same way we investigate the other derivatives.□

Remark 1: *From (17) we get,* $\frac{\partial\mathcal{J}_i}{\partial\lambda} = -\mathcal{S}_i, \frac{\partial\mathcal{J}_i}{\partial\omega^2} = -\mathcal{J}_i\mathcal{S}_i$ *and* $\frac{\partial\mathcal{S}_i}{\partial\omega^2} = -\mathcal{S}_i^2(\omega^2)$.

Depending on the law of large number, CLT, lemma 1, remark1 and the result of proposition3.2,
The random vector

$v = \frac{1}{\sqrt{n}}\left(\frac{\partial}{\partial\lambda}\mathcal{L}_n(\psi)\ \frac{\partial}{\partial\omega^2}\mathcal{L}_n(\psi)\right)' = \frac{1}{\sqrt{n}}\left(\sum_{i=1}^n \mathcal{J}_i(\psi)\ \frac{1}{2}\sum_{i=1}^n\left(\mathcal{J}_i^2(\psi) - \mathcal{S}_i(\omega^2)\right)\right)'$

Converge in distribution to $\mathcal{N}_2\big(0, I(\psi)\big)$ for all, under Q_ψ , as n goes to infinity.

The matrix

$$-\frac{1}{n}\begin{pmatrix} \frac{\partial^2}{\partial\lambda^2}\mathcal{L}_n(\psi) & \frac{\partial^2}{\partial\lambda\partial\omega^2}\mathcal{L}_n(\psi) \\ \frac{\partial^2}{\partial\omega^2\partial\lambda}\mathcal{L}_n(\psi) & \frac{\partial^2}{\partial\omega^2\partial\omega^2}\mathcal{L}_n(\psi) \end{pmatrix}$$

Converge in probability to $I(\psi)$, where

$$\frac{\partial^2}{\partial\lambda^2}\mathcal{L}_n(\psi) = -\sum_{i=1}^{n}\mathcal{S}_i(\omega^2),$$

$$\frac{\partial^2}{\partial\lambda\partial\omega^2}\mathcal{L}_n(\psi) = -\sum_{i=1}^{n}\mathcal{J}_i(\psi)\mathcal{S}_i(\omega^2) \quad (20)$$

$$\frac{\partial^2}{\partial\omega^2\partial\omega^2}\mathcal{L}_n(\psi)$$

$$-\frac{1}{2}\sum_{i=1}^{n}\left(2\mathcal{J}_1^2(\psi)\mathcal{S}_1(\omega^2) - \mathcal{S}_1^2(\omega^2)\right) \quad (21)$$

And

$$I(\psi) = \begin{pmatrix} E_\psi\left(\mathcal{S}_1(\omega^2)\right) & E_\psi\left(\mathcal{J}_1(\psi)\mathcal{S}_1(\omega^2)\right) \\ E_\psi\left(\mathcal{J}_1(\psi)\mathcal{S}_1(\omega^2)\right) & E_\psi\left(\mathcal{J}_1^2(\psi)\mathcal{S}_1(\omega^2)\right) - \frac{1}{2}E_\psi\left(\mathcal{S}_1^2(\omega^2)\right) \end{pmatrix},$$

Is the covariance matrix of the vector

$$\left(\mathcal{J}_i(\psi)\ \frac{1}{2}\left(\mathcal{J}_i^2(\psi) - \mathcal{S}_i(\omega^2)\right)\right)'.$$

4. Asymptotic properties of the estimators

In this section we focus on the consistency and asymptotic normality when the random effect φ is normal and μ is constant.
We need the following assumptions in order to prove the properties of estimators:
H4 the parameter set Θ is a compact subset of $\mathbb{R}\times\mathbb{R}^+$.
H5 the true value ψ_0 belongs to $int(\Theta)$.
H6 $I(\psi_0)$ is invertible.

4.1. Strong consistency

Proposition4.1 Under H1-H3, $Q_{\psi_0}^1 = Q_\psi^1$ implies that, $\theta_0 = \theta$. Hence, $\psi \to K(Q_{\psi_0}^1, Q_\psi^1)$ admits a unique minimum at $\psi_0 = \psi$, where $K(Q_{\psi_0}^1, Q_\psi^1)$ is the kullback information of $Q_{\psi_0}^1$w.r.t Q_ψ^1.

Proof: By analogue way of proof of proposition7 in [24].

Proposition4.2 Under H1-H3 and H4 and under Q_{ψ_0}, $\hat{\psi}_n$ converges in probability to ψ_0 , where $\hat{\psi}_n$ is the maximum likelihood estimator defined as any solution of $\mathcal{L}_n(\hat{\psi}_n) = \sup_{\psi\in\Theta}\mathcal{L}_n(\psi)$.

Proof: Since $\frac{1}{n}\left(\mathcal{L}_n(\psi_0) - \mathcal{L}_n(\psi)\right)$ converges in probability to $K(Q_{\psi_0}^1, Q_\psi^1)$, the likelihood $-\frac{1}{n}\mathcal{L}_n(\psi)$ is a contrast process with $\psi \to K(Q_{\psi_0}^1, Q_\psi^1)$,by the standard proof of consistency (van der vaart 2000 [26]).
Now we will proof the continuity of $-\frac{1}{n}\mathcal{L}_n(\psi)$:
Define

$$w_n(\delta) = \sup_{\|\psi-\psi'\|\le\delta,\psi,\psi'\in\Theta}|\mathcal{L}_n(\psi) - \mathcal{L}_n(\psi')|/n$$

Let $w_n(\delta) \le \delta\sup_{\psi\in\Theta}\|\nabla\mathcal{L}_n(\psi)/n\|$ and bounded the score function (18).by H4, we have

$$\Theta \subset [\underline{\lambda}, \overline{\lambda}] \times \left[\underline{\omega^2}, \overline{\omega^2}\right] \text{ With } \underline{\lambda} < \overline{\lambda},\ 0 < \underline{\omega^2} < \overline{\omega^2}.$$

We have

$$\zeta_i(\psi) = \frac{F_i}{1+\omega_0^2 B_i}\left(1 + \frac{(\omega_0^2-\omega^2)B_i}{1+\omega^2 B_i}\right) - \lambda\frac{B_i}{1+\omega^2 B_i}.$$

Then,

$$\sup_{\psi\in\Theta}|\zeta_i(\psi)| \le \left|\frac{F_i}{1+\omega_0^2 B_i}\right|\left(2 + \frac{\omega_0^2}{\underline{\omega^2}}\right) + \frac{|\overline{\lambda}|}{\underline{\omega^2}}. \quad (22)$$

There is a constant K such that

$$E_{\psi_0}w_n(\delta) \le K\delta E_{\psi_0}\left(\left|\frac{F_1}{1+\omega_0^2 B_1}\right| + \left(\frac{F_1}{1+\omega_0^2 B_1}\right)^2\right).$$

Hence, the consistency of $\hat{\psi}_n$ hold.

4.2. Asymptotic normality

Proposition4.3 Assume H1-H3, the function

$\psi \to K(Q_{\psi_0}^1, Q_\psi^1)$ is continuous on $\mathbb{R}\times\mathbb{R}^+$.

Proof: From (15), let $\mathcal{L}_1(\psi) = \gamma_1(X_1, \psi)$, then

$$E_{\psi_0}\left(\log\frac{\gamma_1(X_1,\psi_0)}{\gamma_1(X_1,\psi)}\right) = K\left(Q_{\psi_0}^1, Q_\psi^1\right)$$

$$= E_{\psi_0}\left(\mathcal{L}_1(\psi_0) - \mathcal{L}_1(\psi)\right).$$

We obtain

$$\mathcal{L}_1(\psi_0) - \mathcal{L}_1(\psi) = \frac{1}{2}\log\left(\frac{1+\omega^2 B_1}{1+\omega_0^2 B_1}\right)$$

$$+\frac{1}{2}\frac{(\omega_0^2-\omega^2)F_1^2}{(1+\omega^2 B_1)(1+\omega_0^2 B_1)} + \frac{\lambda^2 B_1}{2(1+\omega^2 B_1)}$$

$$-\frac{\lambda F_1}{1+\omega^2 B_1} - \left(\frac{\lambda_0^2 B_1}{2(1+\omega_0^2 B_1)} - \frac{\lambda_0 F_1}{1+\omega_0^2 B_1}\right)$$

Now, we show that this random variable has finite expectation.
Let us consider the upper bound:

$$0 < \frac{1+\omega^2 B_1}{1+\omega_0^2 B_1} < 1 + \frac{\omega^2}{\omega_0^2}.$$

From the relation $(x) = x - 1 - \log x$, which is non-negative and define on $\mathbb{R}^+$, we get the lower bound:

$$f\left(\frac{1+\omega_0^2 B_1}{1+\omega^2 B_1}\right) = \frac{1+\omega_0^2 B_1}{1+\omega^2 B_1} - 1 - \log\left(\frac{1+\omega_0^2 B_1}{1+\omega^2 B_1}\right),$$

$$\log(F_1) = f\left(\frac{1+\omega_0^2 B_1}{1+\omega^2 B_1}\right) + (\omega^2 - \omega_0^2)\frac{B_1}{1+\omega^2 B_1}$$

$$\ge (\omega^2 - \omega_0^2)\frac{B_1}{1+\omega^2 B_1}.$$

So that

$$\left|\log\left(\frac{1+\omega^2 B_1}{1+\omega_0^2 B_1}\right)\right| \le \log\left(1 + \frac{\omega^2}{\omega_0^2}\right) + \frac{|\omega^2-\omega_0^2|}{\omega^2}.$$

And

$$0 < \frac{F_1^2}{(1+\omega^2 B_1)(1+\omega_0^2 B_1)} = \left(\frac{F_1}{1+\omega_0^2 B_1}\right)^2\frac{1+\omega_0^2 B_1}{1+\omega^2 B_1}$$

$$\le \left(\frac{F_1}{1+\omega_0^2 B_1}\right)^2\left(1 + \frac{\omega^2}{\omega_0^2}\right)$$

Has finite expectation due to lemma1.

Also,

$$\frac{F_1}{1+\omega^2 B_1} = \frac{F_1}{1+\omega_0^2 B_1}\left(1 + (\omega_0^2 - \omega^2)\frac{B_1}{1+\omega^2 B_1}\right).$$

Then

$$\left|\frac{F_1}{1+\omega^2 B_1}\right| \le \left|\frac{F_1}{1+\omega_0^2 B_1}\right|\left(1 + \frac{|\omega_0^2-\omega^2|}{\omega^2}\right).$$

Has finite expectation under E_{ψ_0} by lemma1.

From above and for all $= (\lambda, \omega^2) \in [\underline{\lambda}, \overline{\lambda}] \times [\underline{\omega^2}, \overline{\omega^2}] \subset \mathbb{R} \times \mathbb{R}^+$, the term $|\mathcal{L}_1(\psi_0) - \mathcal{L}_1(\psi)|$ has finite E_{ψ_0} -expectation which means $K(Q_{\psi_0}^1, Q_\psi^1)$ is continuous. Hence the result.

Proposition4.4: Under H1-H3 and H4-H6, as n tends to infinity, the maximum likelihood estimator satisfies $\sqrt{n}(\hat{\psi}_n - \psi_0) \to^D \mathcal{N}_2(0, I^{-1}(\psi_0))$.

Proof: Let $\hat{\psi}_{n,i}, \psi_{0,i}$be the components of $\hat{\psi}_n, \psi_0$, assume that

$$A_n(\psi) = -\frac{1}{n}\mathcal{L}_n(\psi)$$

And

$$A'_{n,i} = -\frac{1}{n}\frac{\partial}{\partial\psi_i}\mathcal{L}_n(\psi) \text{ And } A''_{n,ij} = -\frac{1}{n}\frac{\partial^2}{\partial\psi_i\partial\psi_i}\mathcal{L}_n(\psi).$$

From the proof of consistency and assumption (5), $Q_{\psi_0}(\hat{\psi}_n \in int(\Theta)) \to 1$.

By using Taylor formula, we get:

$$0 = A'_{n,i}(\hat{\psi}_n)$$

$$= A'_{n,i}(\psi_0) + \sum_{j=1,2}(\hat{\psi}_{n,j} - \psi_{0,j})(A''_{n,ij}(\psi_0) + R_n),$$

Where

$$R_n = \int_0^1 \left(A''_{n,ij}(\psi_0 + s(\hat{\psi}_n - \psi_0)) - A''_{n,ij}(\psi_0)\right) ds.$$

We must prove that R_n goes to zero in probability. Using (20)-(21), compute the derivatives as follow:

$$\frac{1}{n}\frac{\partial^3}{\partial\lambda^3}\mathcal{L}_n(\psi) = 0,$$

$$\frac{1}{n}\frac{\partial^3}{\partial\omega^2\partial\omega^2\partial\omega^2}\mathcal{L}_n(\psi) = \frac{1}{n}\sum_{i=1}^n\left[3\mathcal{J}_i^2(\psi)\mathcal{S}_1^2(\omega^2) - \mathcal{S}_1^3(\omega^2)\right],$$

$$\frac{1}{n}\frac{\partial^3}{\partial^2\lambda\partial\omega^2}\mathcal{L}_n(\psi) = \frac{1}{n}\sum_{i=1}^n \mathcal{S}_i^2(\omega^2),$$

$$\frac{1}{n}\frac{\partial^3}{\partial\lambda\partial\omega^2\partial\omega^2}\mathcal{L}_n(\psi) = \frac{2}{n}\sum_{i=1}^n \mathcal{J}_i(\psi)\mathcal{S}_1^2(\omega^2).$$

From (22), for a constant K, we obtain

$$|R_n| \le K|\hat{\psi}_n - \psi_0|\frac{1}{n}\sum_{i=1}^n\left(1 + \left(\left|\frac{F_1}{1+\omega_0^2 B_1}\right| + \left(\frac{F_1}{1+\omega_0^2 B_1}\right)^2\right)\right),$$

By using the proof of Proposition 4.2. , R_n tends to zero. This means the result.

5. Conclusion

We depend on SDE with random effects model framework and consider the linearity assumption in the drift function given by:

i) $b(x, \emptyset_i, \mu_i) = \emptyset_i b(X_i(t)) + \mu_i$ where $\emptyset_i$ are supposed to be Gaussian random variables with mean λ and variance ω^2 , and μ_i to be exponential random variables with parameterβ

ii) $b(x, \emptyset_i, \mu_i) = \emptyset_i b(X_i(t)) + \mu$ where $\emptyset_i$ are an(λ, ω^2) and μ is unknown parameter,

iii) $b(x, \emptyset_i, \mu_i) = \emptyset_i b(X_i(t)) + \mu$ Where $\emptyset_i$ are Gaussian (λ, ω^2) and μ is a constant. A closed form expression of the likelihoods of the parameters of the $i.i.d$ random effects and the maximum likelihood estimator are obtained. We proved consistency and asymptotic normality of the estimators in the third case only.

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