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Research paper

An efficient scheme for solving a system of time- fractional order differential-algebraic equations by using fractional Laplace iteration method

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Abstract

In this article, we propose an efficient algorithm for solving system of time- fractional differential-algebraic equations by using a fractional Laplace iteration method. The scheme is tested for some examples and the results demonstrate reliability and accuracy of this method.

Keywords: *Riemann-Liouville Derivative; Analytic Solution; Fractional Laplace Iteration Method; Mittag-Leffller Functions; System of Time -Fractional Order Differential-Algebraic Equations.*

1. Introduction

The fractional order calculus is a generalization of the integer order calculus to a real or complex number. Fractional derivative have been extensively investigated to their broad applications in mathematics, physics and engineering [1-3]. A review of some applications of fractional calculus in continuum and statistical mechanics is given by Mainardi [4]. Recently, many important mathematical models can be expressed in term of system of differential-algebraic equations. The exact solution of most of the differential-algebraic equations (FDEs) cannot be found easily, and this has mandated the use of both analytical and numerical method. In recent years, many researchers have focused on the numerical solution of fractional differential –algebraic equations. Some numerical methods have been developed, such as implicit Runge-Kutta method [5], Padé approximation method [6-9], homotopy perturbation method [10-14], Adomian decomposition method [15-19], homotopy analysis method [20-21], variation iteration method [22-24], homotopy analysis transform method [25].
In 2013, Habibolla et al. [26] presented an alternative approach based on Laplace iterative method (LIM) for finding series solutions to linear and nonlinear systems of PDEs. The applied method gave rapidly convergent successive approximations.
In our article, we have use Fractional Laplace Iteration Method (FLIM) successfully to find the approximate analytical solution linear homogeneous, non-homogeneous FDAEs with time-and space fractional derivatives. These problems have not yet been solved by any researcher.
The rest of this paper is organized as follows. In section 2, we give the some necessary definitions and mathematical preliminaries of the fractional calculus theory. In section 3, we give analysis of the method used. In section 4, the proposed method is applied to several examples. Also a conclusion is given in the last section.

2. Preliminaries and notations

In this section, we give some basic definitions and properties of fractional calculus theory which will be used in this paper.

Definition 2.1: *[27] A real function f (t), t >0, is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0,\infty)$ and it is said to be in the space C_μ^m if and only if $f^{(m)} \in C_\mu$, $m \in N$.*

Definition 2.2: *The Riemann-Liouville fractional order integral of $\alpha > 0$ of function $f(t) \in C_\mu$, $\mu > -1$ is defined as [27]:*

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \tag{2.1}$$

$$J^0 f(t) = f(t)$$

Definition 2.3: *The fractional derivative of function $f(t)$ in Caputo sense is defined as [27]:*

$$_{RL}D^\alpha f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} f(\tau) d\tau, t > 0 \tag{2.2}$$

For $m-1 < \alpha \le m, m \in N, t > 0, f \in C_{-1}^m$.

Definition 2.4: *[28] The single parameter and the two parameters variants of the Mittag- Leffler function are denoted by $E_\alpha(t)$ and $E_{\alpha,\beta}(t)$, respectively, which are relevant for their connection with fractional calculus, and are defined as:*

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+1)}, \alpha > 0, t \in C \tag{2.3}$$

$$E_{\propto,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\propto k+\beta)}, \propto, \beta > 0, t \in C \quad (2.4)$$

For special choices of the values of the parameter $\propto, \beta$ we obtain well-known classical functions, e.g.:

$$E_1(t) = e^t, E_{\propto,1}(t) = E_{\propto}(t)$$

As we will see later, classical derivatives of the Mittag-Leffler function appear in solution of FDEs. Since the series (2.4) is uniformly convergent we may differentiate term by term and obtain

$$E^{(m)}{}_{\propto,\beta}(t) = \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{t^k}{\Gamma(\propto k+\alpha m+\beta)} \quad (2.5)$$

Lemma 2.5 [28]: *For $\alpha, \beta > 0$ and $s^\alpha > |a|$ we have the following inverse Laplace transforms formula*

$$\mathcal{L}^{-1}\left(\frac{s^{\alpha-\beta}}{s^\alpha+a}\right) = t^{\beta-1} E_{\propto,\beta}(-at^\alpha) \quad (2.6)$$

Definition 2.6: *The Laplace transform $\mathcal{L}[x(t)]$ of the Riemann-Liouville fractional derivative is given as [24]:*

$$\mathcal{L}\left[{}_{RL}D^\alpha f(t)\right] = s^\alpha \mathcal{L}[f(t)] - \sum_{k=0}^{m-1} s^k \left({}_{RL}D^{\alpha-k-1} f(t)\right)\Big|_{t=0} \quad (2.7)$$
$$m-1 < \alpha \le m.$$

Recall the Laplace transform of R-L derivative for $0 < \alpha \le 1$ is:

$$\mathcal{L}\left[{}_{RL}D^\alpha f(t)\right] = s^\alpha \mathcal{L}[f(t)] - \left({}_{RL}D^{\alpha-1} f(t)\right)\big|_{t=0} \quad (2.8)$$

Property 2.7: [27] The compositions of Riemann-Liouville derivative operators ${}_{RL}D^\alpha x(t)$ and ${}_{RL}D^\beta x(t)$ are as follows:

$${}_{RL}D^\alpha\left({}_{RL}D^\beta x(t)\right) = {}_{RL}D^{\alpha+\beta} x(t) - \sum_{j=0}^{n}\left({}_{RL}D^{\beta-j} x(t)\right)\big|_{t=0} \frac{t^{-\alpha-j}}{\Gamma(1-\propto-j)} \quad (2.9)$$

And

$${}_{RL}D^\beta\left({}_{RL}D^\alpha x(t)\right) = {}_{RL}D^{\alpha+\beta} x(t) - \sum_{j=0}^{m}\left({}_{RL}D^{\alpha-j} x(t)\right)\big|_{t=0} \frac{t^{-\beta-j}}{\Gamma(1-\beta-j)} \quad (2.10)$$

Where $m-1 < \alpha \le m, n-1 < \beta \le n$ and $m, n \in N$.

3. Analysis of the FLIM

In this article, we consider the following non-homogenous, nonlinear system of fractional order differential-algebraic equations

$$\left.\begin{array}{c} {}_{RL}D^{\alpha_i}\left({}_{RL}D^{\beta_i} x_i(t)\right) + a_i x_i(t) = h_i(t, x_1, x_2, \dots, x_n) \\ 0 = g(t, x_1, x_2, \dots, x_n) \end{array}\right\} \quad (3.1)$$

With initial conditions

$$x_i(0) = b_i, x_i'(0) = c_i, \left({}_{RL}D^{\beta_i-1} x(t)\right)\Big|_{t=0} = d_i, \left({}_{RL}D^{\alpha_i+\beta_i-1} x(t)\right)\Big|_{t=0} = e_i \ i = 1, 2, \dots, n-1, x_n(0) = s_i.$$

Here $a_i, b_i, c_i, d_i, e_i, s_i$ are constants and ${}_{RL}D^{\alpha_i}\left({}_{RL}D^{\beta_i} x(t)\right)$ is the compositions of $\mathrm{Riemann-Liouville\ derivative\ operators}$ satisfying the relation $0 < \propto_i, \beta_i < 1$. Eq. (3.1) can be rewritten as:

$$\left.\begin{array}{c} L_1 x_1(t) + N_1(t, x_1, x_2, \dots, x_n) = f_1(t) \\ L_2 x_2(t) + N_2(t, x_1, x_2, \dots, x_n) = f_2(t) \\ \vdots \\ L_{n-1} x_{n-1}(t) + N_n(t, x_1, x_2, \dots, x_n) = f_{n-1}(t) \\ 0 = g(t, x_1, x_2, \dots, x_n) \end{array}\right\} \quad (3.2)$$

Where L_i is a linear operator, N_i a nonlinear operator and $f_i(t)$ is a nonhomogeneous item form i=1, 2… n-1. Eq. (3.2) can be rewritten down as a correction function in the following way:

$$L_i x_i(t) = f_i(t) - N_i(t, x_1, x_2, \dots, x_n) = R_i(t, x_1, x_2, \dots, x_n),$$
$$i=1, 2\dots n-1 \quad (3.3)$$

Therefore:

$$L_i x_i(t) = R_i(t, x_1, x_2, \dots, x_n), \ i=1, 2\dots n-1.$$

The Fractional Laplace Iteration Method assumed a series solution for x_i given by an infinite sum of components:

$$x_i(t) = \lim_{n\to\infty} x_i^p(t) = \lim_{n\to\infty} \sum_{j=0}^{p} v_i^j(t),$$
$$i = 1, 2, \dots, n-1 \quad (3.4)$$

In which x_i^n indicates the n-th approximation of x_i, where v_i^j is the j^{th} component of the solution of x_i and v_i^0 is the solution of $L_i x_i = 0$, along with the following initial conditions of the main problem:

$$v_i^1(t) = \varphi_i(v_i^0)$$

$$v_i^{k+1}(t) = \varphi_i\left(\sum_{j=0}^{k} v_i^j(t)\right) - \sum_{j=0}^{k} v_i^j(t), k \ge 1$$

In which $\varphi_i(v_i^k)$ is obtained as follows:

$$L_i \varphi_i\left(v_1^k, v_2^k, \dots, v_n^k\right) = R_i(t, x_1, x_2, \dots, x_n), \quad (3.5)$$

Using the homogenous initial conditions, supposing that L_i linear operator, therefore, taking Laplace transform to both sides of Eq. (3.5) in the usual way and using the homogenous initial conditions, the result can be obtained as following:

$$p_i(s). \Phi_i^k(s) = \mathcal{R}_i(v_i^k(s)), \quad (3.6)$$

Where $\mathcal{L}[\varphi_i(v_1^k, v_2^k, \dots, v_n^k)] = \Phi_i^k$, $p_i(s)$ is a fractional polynomial with the fractional degree of the highest derivative in Eq. (3.6) (The same as the highest order of the linear operator L_i). Thus,

$$\mathcal{L}[w] = \varpi, \psi_i(s) = \frac{1}{p_i(s)}, \mathcal{L}[u_i(t)] = \psi_i(s) \quad (3.7)$$

In Equations (3.5) and (3.6), the function $\mathcal{R}_i(v_i^k(s))$ and $R_i(t, x_1, x_2, \dots, x_n)$ are abbreviated as $\mathcal{R}_i$ and R_i respectively. Hence, Eq. (3.6) is rewritten as:

$$\Phi_i^k(s) = \mathcal{R}_i\left(\left(v_1^k, v_2^k, \dots, v_n^k\right)(s)\right) . \psi_i(s) \quad (3.8)$$

Now, by applying the inverse Laplace Transform to both side of Eq. (3.8) and using the convolution Theorem, the following relation can be presented:

$$\emptyset_i\left(v_1^k, v_2^k, \dots, v_n^k\right) = \int_0^t R_i\left(\left(v_1^k, v_2^k, \dots, v_n^k\right)(\tau)\right) . u_i(t-\tau)\, d\tau \quad (3.9)$$

Therefore

$$x_i^{p+1}(t) = \sum_{j=0}^{p+1} v_i^j(t) = x_i^0(t) + \int_0^t R_i\left(x_i^p(\tau)\right) . u_i(t-\tau) d\tau, \quad (3.10)$$
$$i = 1, 2, \dots, n-1$$

After identifying the initial approximation of x_i^0, the remaining approximations $x_i^p, p > 0$ can be determined so that each term can be determined by previous terms and the approximation of iteration formula can be entirely evaluated.
Consequently, the exact solution may be obtained by:

$$x_i = \lim_{p\to\infty} x_i^p(t) = \lim_{p\to\infty} \sum_{j=0}^{p} v_i^j(t), \quad (3.11)$$
$$i = 1, 2, \dots, n-1.$$

Which is the Fractional Laplace Iteration method.

4. Numerical experiments

In this part, we introduce some applications on FLIM to solve differential- algebraic equations with time- fractional derivatives:

Example 1: consider the following non-homogenous system of linear space-time fractional order differential –algebraic equations:

$$\left.\begin{aligned} {}_{RL}D^{\alpha}\left({}_{RL}D^{\beta}x(t)\right)+x(t)-y(t)=-\sin t,\\ x(t)+y(t)=\sin t+\cos t\ ,t\in[o,1],0<\alpha,\beta<1 \end{aligned}\right\} \quad (4.1)$$

Subject to initial conditions x (0) = 1,y(0)=0, $\acute{x}(0)=0$,

$$\left({}_{RL}D^{\beta-1}x(t)\right)\Big|_{t=0}=a,\left({}_{RL}D^{\alpha+\beta-1}x(t)\right)\Big|_{t=0}=b$$

For the special case $\alpha=\beta=1$, we have analytical solution $x(t)=\cos t\,,y(t)=\sin t.$

Solution:
From the Eq. (4.1), optimal selection auxiliary linear operator the equation is represented as follows:

$$Lx(t)\colon {}_{RL}D^{\alpha}\left({}_{RL}D^{\beta}x(t)\right)+x(t)$$

Therefore $\emptyset(v_1^k,v_2^k)$ is defined as:

$$\emptyset\left(v_1^k,v_2^k\right)=\int_0^t u(t-\tau)[\,v_2^k(\tau)-\sin\tau]d\tau \quad (4.2)$$

Then, using Eq. (4.1), the fractional Laplace iteration method in t-direction for the calculation of the approximate solution of Eq. (4.2) can be readily obtained as:

$$\left.\begin{aligned} x_{n+1}(t)=x_0(t)+\int_0^t u(t-\tau)[\,y_n(\tau)-\sin\tau]d\tau,\\ y_n(t)=x_n(t)+\sin t-\cos t \end{aligned}\right\} \quad (4.3)$$

<u>Case 1:</u> $\alpha+\beta<1\ (if\ \alpha=\frac{1}{5},\beta=\frac{1}{4})$

$$Lx(t)\colon {}_{RL}D^{\frac{1}{5}}\left({}_{RL}D^{\frac{1}{4}}x(t)\right)+x(t)$$

$$\Rightarrow p(s)=s^{\frac{9}{20}}+1$$

$$\Rightarrow \psi(s)=\frac{1}{p(s)}=\frac{1}{s^{\frac{9}{20}}+1}$$

$$\Rightarrow u(t)=\mathcal{L}^{-1}[\psi(s)]=t^{\frac{-11}{20}}E_{\frac{9}{20},\frac{9}{20}}(-t^{\frac{9}{20}})$$

Where the initial approximation must be satisfied by the following equations:

$$Lx(t)=0,x(0)=1,\left({}_{RL}D^{\beta-1}x(t)\right)\Big|_{t=0}=\left({}_{RL}D^{\frac{-3}{4}}x(t)\right)\Big|_{t=0}=0,$$

$$\left({}_{RL}D^{\alpha+\beta-1}x(t)\right)\Big|_{t=0}=\left({}_{RL}D^{\frac{-11}{20}}x(t)\right)\Big|_{t=0}=0,\quad\Rightarrow$$

$$x_0(t)=E_{\frac{9}{20},1}\,(-t^{\frac{9}{20}})$$

$$y_0(t)=x_0(t)+\sin t-\cos t$$

$$=E_{\frac{9}{20},1}\left(-t^{\frac{9}{20}}\right)+\sin t-\cos t$$

Accordingly, by Eq. (4.3) the higher order approximation of the exact solution can be obtained as follows:

$$x_1(t)=\sum_{k=0}^{\infty}\frac{\left(-t^{\frac{9}{20}}\right)^k}{\Gamma\left(\frac{9}{20}\,k+1\right)}+$$

$$\int_0^{\tau}\left[\left((t-\tau)^{\frac{-11}{20}}\sum_{k=0}^{\infty}\frac{\left(-(t-\tau)^{\frac{9}{20}}\right)^k}{\Gamma\left(\frac{9}{20}k+\frac{9}{20}\right)}\right)\left(\sum_{k=0}^{\infty}\frac{\left(-\tau^{\frac{9}{20}}\right)^k}{\Gamma\left(\frac{9}{20}\,k+1\right)}-\cos\tau\right)\right]d\tau$$

$$y_1(t)=\sum_{k=0}^{\infty}\frac{\left(-t^{\frac{9}{20}}\right)^k}{\Gamma\left(\frac{9}{20}\,k+1\right)}+$$

$$\int_0^{\tau}\left[\left((t-\tau)^{\frac{-11}{20}}\sum_{k=0}^{\infty}\frac{\left(-(t-\tau)^{\frac{9}{20}}\right)^k}{\Gamma\left(\frac{9}{20}k+\frac{9}{20}\right)}\right)\left(\sum_{k=0}^{\infty}\frac{\left(-\tau^{\frac{9}{20}}\right)^k}{\Gamma\left(\frac{9}{20}\,k+1\right)}-\cos\tau\right)\right]d\tau+\sin t-\cos t$$

<u>Case 2:</u> $\alpha+\beta=1\ (if\ \alpha=\frac{3}{10},\beta=\frac{7}{10})$

$$Lx(t)\colon {}_{RL}D^{\frac{3}{10}}\left({}_{RL}D^{\frac{7}{10}}x(t)\right)+x(t)$$

$$\Rightarrow p(s)=s+1$$

$$\Rightarrow \psi(s)=\frac{1}{p(s)}=\frac{1}{s+1}$$

$$\Rightarrow u(t)=\mathcal{L}^{-1}[\psi(s)]=e^{-t}$$

Where the initial approximation must be satisfied by the following equations:

$$Lx(t)=0,x(0)=1,\left({}_{RL}D^{\beta-1}x(t)\right)\Big|_{t=0}=\left({}_{RL}D^{\frac{-3}{10}}x(t)\right)\Big|_{t=0}=0$$

$$\Rightarrow x_0(t)=E_{1,1}\,(-t)=e^{-t}$$

$$y_0(t)=x_0(t)+\sin t-\cos t$$

$$=e^{-t}+\sin t-\cos t$$

Accordingly, by Eq. (4.3) the higher order approximation of the exact solution can be obtained as follows:

$$x_1(t)=\sum_{k=0}^{\infty}\frac{(-t)^k}{\Gamma(k+1)}+\int_0^{\tau}\left[\left(\sum_{k=0}^{\infty}\frac{(\tau-t)^k}{\Gamma(k+1)}\right)\left(\sum_{k=0}^{\infty}\frac{(-\tau)^k}{\Gamma(k+1)}-\cos\tau\right)\right]d\tau$$

$$y_1(t)=\sum_{k=0}^{\infty}\frac{(-t)^k}{\Gamma(k+1)}+\int_0^{\tau}\left[\left(\sum_{k=0}^{\infty}\frac{(\tau-t)^k}{\Gamma(k+1)}\right)\left(\sum_{k=0}^{\infty}\frac{(-\tau)^k}{\Gamma(k+1)}-\cos\tau\right)\right]d\tau+\sin t-\cos t$$

<u>Case 3:</u> $\alpha+\beta>1\ (if\ \alpha=\frac{1}{2},\beta=\frac{2}{3})$

$$Lx(t)\colon {}_{RL}D^{\frac{1}{2}}\left({}_{RL}D^{\frac{2}{3}}x(t)\right)+x(t)$$

$$\Rightarrow p(s)=s^{\frac{7}{6}}+1$$

$$\Rightarrow \psi(s)=\frac{1}{p(s)}=\frac{1}{s^{\frac{7}{6}}+1}$$

$$\Rightarrow u(t)=\mathcal{L}^{-1}[\psi(s)]=t^{\frac{1}{6}}E_{\frac{7}{6},\frac{7}{6}}(-t^{\frac{7}{6}})$$

Where the initial approximation must be satisfied by the following equations:

$Lx(t)=0, x(0)=1, \acute{x}(0)=0, \left({}_{RL}D^{\beta-1}x(t)\right)\Big|_{t=0} = \left({}_{RL}D^{\frac{-1}{3}}x(t)\right)\Big|_{t=0}=0, \left({}_{RL}D^{\alpha+\beta-1}x(t)\right)\Big|_{t=0} = \left({}_{RL}D^{\frac{1}{6}}x(t)\right)\Big|_{t=0} = 0, \Longrightarrow x_0(t)=E_{\frac{7}{6},1}\left(-t^{\frac{7}{6}}\right)$

$$y_0(t)=x_0(t)+\sin t-\cos t$$

$$=E_{\frac{7}{6},1}\left(-t^{\frac{7}{6}}\right)+\sin t-\cos t$$

Accordingly, by Eq. (4.3) the higher order approximation of the exact solution can be obtained as follows:

$$x_1(t)=\sum_{k=0}^{\infty}\frac{\left(-t^{\frac{7}{6}}\right)^k}{\Gamma\left(\frac{7}{6}k+1\right)}+$$

$$\int_0^{\tau}\left[\left((t-\tau)^{\frac{1}{6}}\sum_{k=0}^{\infty}\frac{\left(-(t-\tau)^{\frac{7}{6}}\right)^k}{\Gamma\left(\frac{7}{6}k+\frac{7}{6}\right)}\right)\left(\sum_{k=0}^{\infty}\frac{\left(-\tau^{\frac{7}{6}}\right)^k}{\Gamma\left(\frac{7}{6}k+1\right)}-\cos\tau\right)\right]d\tau$$

$$y_1(t)=\sum_{k=0}^{\infty}\frac{\left(-t^{\frac{7}{6}}\right)^k}{\Gamma\left(\frac{7}{6}\ k+1\right)}+$$

$$\int_0^{\tau}\left[\left((t-\tau)^{\frac{1}{6}}\sum_{k=0}^{\infty}\frac{\left(-(t-\tau)^{\frac{7}{6}}\right)^k}{\Gamma\left(\frac{7}{6}k+\frac{7}{6}\right)}\right)\left(\sum_{k=0}^{\infty}\frac{\left(-\tau^{\frac{7}{6}}\right)^k}{\Gamma\left(\frac{7}{6}k+1\right)}-\cos\tau\right)\right]d\tau+$$

$$\sin t-\cos t$$

Table 1: Numerical Results of the Solution in Example 1

T	x_1 $\alpha=\frac{1}{5},\beta=\frac{1}{4}$	x_1 $\alpha=\frac{3}{10},\beta=\frac{7}{10}$	x_1 $\alpha=\frac{1}{2},\beta=\frac{2}{3}$	Exact solution
0	1	1	1	1
0.1	0.622	0.9	0.939	0.995
0.2	0.507	0.802	0.868	0.98
0.3	0.421	0.708	0.797	0.955
0.4	0.36	0.613	0.729	0.921
0.5	0.313	0.534	0.666	0.878
0.6	0.276	0.45	0.611	0.825
0.7	0.249	0.387	0.563	0.765
0.8	0.231	0.325	0.523	0.697
0.9	0.221	0.27	0.492	0.622
1	0.22	0.224	0.471	0.54

Table 1 shows the approximate solutions for Eq. (4.3) obtained for different values of α and β using our method. The results are in good agreement with the results of the exact solutions.

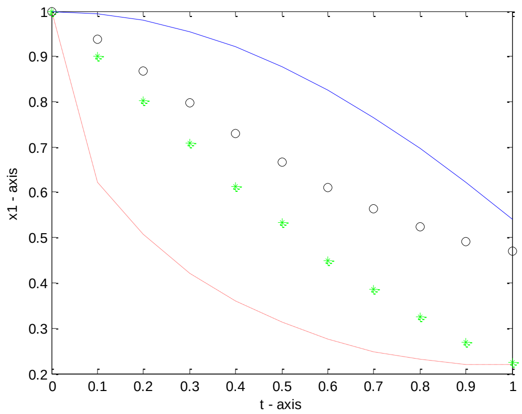

Fig. 1: The red dotted line (…) is the approximate solution when $\alpha=\frac{1}{5},\beta=\frac{1}{4}(\alpha+\beta<1)$, the green asterisks (***) is the approximate solution when $\alpha=\frac{3}{10},\beta=\frac{7}{10}(\alpha+\beta=1)$, the black circles (ooo) is the approximate solution when $\alpha=\frac{1}{2},\beta=\frac{2}{3}(\alpha+\beta>1)$, and the Continuous Line (---) is the Exact Solution.

Example 2: consider the following homogenous system of linear space-time fractional order differential –algebraic equations:

$$\left.\begin{array}{c} {}_{RL}D^{\alpha}\left({}_{RL}D^{\beta}x(t)\right)+x(t)-2t\acute{y}(t)+2\acute{x}(t)+2t\acute{z}(t)- \\ (4+2t)y(t)+(2t^2+6t)z(t)=0, \\ {}_{RL}D^{\alpha}\left({}_{RL}D^{\beta}y(t)\right)-\acute{z}(t)-y(t)+2tz(t)=0, \\ z(t)-\sin t=0\ ,t\in[o,1],0<\alpha,\beta<1 \end{array}\right\} \quad (4.4)$$

Subject to initial conditions x (0) =y (0) =1, z (0) =0, $\acute{x}(0)=0$,

$\acute{y}(0)=1\ ,\left({}_{RL}D^{\alpha+\beta-1}x(t)\right)\Big|_{t=0}=a,$

$\left({}_{RL}D^{\beta-1}x(t)\right)\Big|_{t=0}=b, \left(D^{\alpha+\beta-1}y(t)\right)\Big|_{t=0}= c, \left({}_{RL}D^{\beta-1}y(t)\right)\Big|_{t=0}=d.$

For the special case $\alpha=\beta=1$, we have analytical solution

$x(t)=e^{-t}+te^{t}, y(t)=e^{t}+t\sin t,$ and z (t)= $\sin t$

Solution:
From the Eq. (4.4), optimal selection auxiliary linear operator the equation is represented as follows:

$$L_1x(t){:}\ {}_{RL}D^{\alpha}\left({}_{RL}D^{\beta}x(t)\right)+x(t)$$

$$L_2y(t){:}\ {}_{RL}D^{\alpha}\left({}_{RL}D^{\beta}x(t)\right)-y(t)$$

Therefore $\emptyset_i(v_1^k,v_2^k,v_3^k)$, i=1, 2; are defined as:

$$\left.\begin{array}{c} \emptyset_1(v_1^k,v_2^k,v_3^k)=\int_0^t u_1(t-\tau)[\,2\tau\left(\acute{v_2^k}(\tau)\right)-2\left(\acute{v_1^k}(\tau)\right) \\ -2\tau\left(\acute{v_3^k}(\tau)\right)+(4+2\tau)v_2^k(\tau)-(2\tau^2+6\tau)v_3^k(\tau)]d\tau \\ \emptyset_2(v_1^k,v_2^k,v_3^k)=\int_0^t u_2(t-\tau)[\left(\acute{v_3^k}(\tau)\right)-2\tau v_3^k(\tau)]d\tau \end{array}\right\} \quad (4.5)$$

Then, using Eq. (4.4), the fractional Laplace iteration method in t-direction for the calculation of the approximate solution of Eq. (4.5) can be readily obtained as:

$$\left.\begin{array}{c} x_{n+1}(t)=x_0(t)+\int_0^t u_1(t-\tau)[\,2\tau(\acute{y_n}(\tau))- \\ 2(\acute{x_n}(\tau))-2\tau(\acute{z_n}(\tau))+(4+2\tau)(\acute{y_n}(\tau)) \\ -(2\tau^2+6\tau)z_n(\tau)]d\tau, \\ y_{n+1}(t)=z_0(t)+\int_0^t u_2(t-\tau)[\,(\acute{z_n}(\tau))-2\tau z_n(\tau)]d\tau, \\ z_n=\sin t \end{array}\right\} \quad (4.6)$$

<u>**Case 1:**</u> $\alpha+\beta<1\ (if\ \alpha=\frac{1}{2},\beta=\frac{1}{3})$

$$L_1x(t){:}\ {}_{RL}D^{\frac{1}{2}}\left({}_{RL}D^{\frac{1}{3}}x(t)\right)+x(t)$$

$$\Longrightarrow p_1(s)=s^{\frac{5}{6}}+1$$

$$\Longrightarrow \psi_1(s)=\frac{1}{p_1(s)}=\frac{1}{s^{\frac{5}{6}}+1}$$

$$\Longrightarrow u_1(t)=\mathcal{L}^{-1}[\psi_1(s)]=t^{\frac{-1}{6}}E_{\frac{5}{6},\frac{5}{6}}\left(-t^{\frac{5}{6}}\right)$$

And

$$L_2z(t){:}\ {}_{RL}D^{\frac{1}{2}}\left({}_{RL}D^{\frac{1}{3}}y(t)\right)-y(t)$$

$$\Longrightarrow p_2(s)=s^{\frac{5}{6}}-1$$

$\Rightarrow \psi_2(s) = \frac{1}{p_2(s)} = \frac{1}{s^{\frac{5}{6}}-1}$

$\Rightarrow u_2(t) = \mathcal{L}^{-1}[\psi_2(s)] = t^{\frac{-1}{6}} E_{\frac{5}{6},\frac{5}{6}}(t^{\frac{5}{6}})$

Where the initial approximation must be satisfied by the following equations:

$L_1x(t) = 0, x(0) = 1, \left({}_{RL}D^{\alpha+\beta-1}x(t)\right)\Big|_{t=0} = \left({}_{RL}D^{\frac{-1}{6}}x(t)\right)\Big|_{t=0} = 0, \left({}_{RL}D^{\beta-1}x(t)\right)\Big|_{t=0} = \left({}_{RL}D^{\frac{-2}{3}}x(t)\right)\Big|_{t=0} = 0 \;\Rightarrow x_0(t) = E_{\frac{5}{6},1}(-t^{\frac{5}{6}})$

$L_2y(t) = 0, y(0) = 1, \left(D^{\alpha+\beta-1}y(t)\right)\Big|_{t=0} = \left({}_{RL}D^{\frac{-1}{6}}y(t)\right)\Big|_{t=0} = 0, \left({}_{RL}D^{\beta-1}y(t)\right)\big|_{t=0} = \left({}_{RL}D^{\frac{-2}{3}}y(t)\right)\Big|_{t=0} = 0, y_0(t) = E_{\frac{5}{6},1}(t^{\frac{5}{6}})$

$z_0(t) = \sin t$

Accordingly, by Eq. (4.6) the higher order approximation of the exact solution can be obtained as follows:

$$x_1(t) = \sum_{k=0}^{\infty} \frac{\left(-t^{\frac{5}{6}}\right)^k}{\Gamma\left(\frac{5}{6}k+1\right)} + \int_0^\tau \left[\left((t-\tau)^{\frac{-1}{6}} \sum_{k=0}^{\infty} \frac{\left(-(t-\tau)^{\frac{5}{6}}\right)^k}{\Gamma\left(\frac{5}{6}k+\frac{5}{6}\right)}\right)\left(2\tau \sum_{k=0}^{\infty} \frac{(k+1)!}{k!} \frac{\left(\tau^{\frac{5}{6}}\right)^k}{\Gamma\left(\frac{5}{6}k+\frac{11}{6}\right)} - 2\sum_{k=0}^{\infty} \frac{(k+1)!}{k!} \frac{\left(-\tau^{\frac{5}{6}}\right)^k}{\Gamma\left(\frac{5}{6}k+\frac{11}{6}\right)} - 2\tau\cos\tau + (4+2\tau)\sum_{k=0}^{\infty} \frac{\left(\tau^{\frac{5}{6}}\right)^k}{\Gamma\left(\frac{5}{6}k+1\right)} - (2\tau^2+6\tau)\sin\tau\right)\right] d\tau$$

$$y_1(t) = \sum_{k=0}^{\infty} \frac{\left(t^{\frac{5}{6}}\right)^k}{\Gamma\left(\frac{5}{6}k+1\right)} + \int_0^\tau \left[\left((t-\tau)^{\frac{-1}{6}} \sum_{k=0}^{\infty} \frac{\left((t-\tau)^{\frac{5}{6}}\right)^k}{\Gamma\left(\frac{5}{6}k+\frac{5}{6}\right)}\right)(\cos\tau - 2\tau\sin\tau)\right] d\tau$$

$z_1(t) = \sin t$

Case 2: $\alpha + \beta = 1$ $(if\ \alpha = \frac{2}{5}, \beta = \frac{6}{10})$

$L_1x(t)\colon {}_{RL}D^{\frac{2}{3}}\left({}_{RL}D^{\frac{6}{10}}x(t)\right) + x(t)$

$\Rightarrow p_1(s) = s + 1$

$\Rightarrow \psi_1(s) = \frac{1}{p_1(s)} = \frac{1}{s+1}$

$\Rightarrow u_1(t) = \mathcal{L}^{-1}[\psi_1(s)] = e^{-t}$

And

$L_2y(t)\colon {}_{RL}D^{\frac{2}{3}}\left({}_{RL}D^{\frac{6}{10}}y(t)\right) - y(t)$

$\Rightarrow p_2(s) = s - 1$

$\Rightarrow \psi_2(s) = \frac{1}{p_2(s)} = \frac{1}{s-1}$

$\Rightarrow u_2(t) = \mathcal{L}^{-1}[\psi_2(s)] = e^{t}$

Where the initial approximation must be satisfied by the following equations:

$L_1x(t) = 0, x(0) = 1, \left({}_{RL}D^{\beta-1}x(t)\right)\Big|_{t=0} = \left({}_{RL}D^{\frac{-4}{10}}x(t)\right)\Big|_{t=0} = 0 \Rightarrow x_0(t) = e^{-t}$

$L_2y(t) = 0, y(0) = 1, \left({}_{RL}D^{\beta-1}y(t)\right)\Big|_{t=0} = \left({}_{RL}D^{\frac{-4}{10}}y(t)\right)\Big|_{t=0} = 0 \Rightarrow y_0(t) = e^{t}$

$z_0(t) = \sin t$

Accordingly, by Eq. (4.6) the higher order approximation of the exact solution can be obtained as follows:

$$x_1(t) = e^{-t} + \int_0^\tau \left[\left(\sum_{k=0}^{\infty} \frac{(\tau-t)^k}{\Gamma(k+1)}\right)\left(2\tau^2 \sum_{k=0}^{\infty} \frac{(\tau)^k}{\Gamma(k+1)} - 2\tau \sum_{k=0}^{\infty} \frac{(-\tau)^k}{\Gamma(k+1)} - 2\tau\cos\tau + (4+2\tau)\sum_{k=0}^{\infty} \frac{(\tau)^k}{\Gamma(k+1)} - (2\tau^2+6\tau)\sin\tau\right)\right] d\tau$$

$$y_1(t) = e^{t} + \int_0^\tau \left[\left(\sum_{k=0}^{\infty} \frac{(t-\tau)^k}{\Gamma(k+1)}\right)(\cos\tau - 2\tau\sin\tau)\right] d\tau$$

$z_1(t) = \sin t$

Case 3: $\alpha + \beta > 1$ $(if\ \alpha = \frac{7}{12}, \beta = \frac{3}{4})$

$L_1x(t)\colon {}_{RL}D^{\frac{7}{12}}\left({}_{RL}D^{\frac{3}{4}}x(t)\right) + x(t)$

$\Rightarrow p_1(s) = s^{\frac{4}{3}} + 1$

$\Rightarrow \psi_1(s) = \frac{1}{p_1(s)} = \frac{1}{s^{\frac{4}{3}}+1}$

$\Rightarrow u_1(t) = \mathcal{L}^{-1}[\psi_1(s)] = t^{\frac{1}{3}} E_{\frac{4}{3},\frac{4}{3}}(-t^{\frac{4}{3}})$

And

$L_2y(t)\colon {}_{RL}D^{\frac{7}{12}}\left({}_{RL}D^{\frac{3}{4}}y(t)\right) - y(t)$

$\Rightarrow p_1(s) = s^{\frac{4}{3}} - 1$

$\Rightarrow \psi_2(s) = \frac{1}{p_2(s)} = \frac{1}{s^{\frac{4}{3}}-1}$

$\Rightarrow u_2(t) = \mathcal{L}^{-1}[\psi_2(s)] = t^{\frac{1}{3}} E_{\frac{4}{3},\frac{4}{3}}(t^{\frac{4}{3}})$

Where the initial approximation must be satisfied by the following equations:

$L_1x(t) = 0, x(0) = 1, \acute{x}(t) = 0, \left({}_{RL}D^{\alpha+\beta-1}x(t)\right)\Big|_{t=0} = \left({}_{RL}D^{\frac{1}{3}}x(t)\right)\Big|_{t=0} = 0, \left({}_{RL}D^{\beta-1}x(t)\right)\Big|_{t=0} = \left({}_{RL}D^{\frac{-1}{4}}x(t)\right)\Big|_{t=0} = 0$

$\Rightarrow x_0(t) = E_{\frac{4}{3},1}(-t^{\frac{4}{3}})$

$L_2y(t) = 0, y(0) = 1,\ \acute{y}(t) = 0, \left({}_{RL}D^{\alpha+\beta-1}y(t)\right)\Big|_{t=0} = \left({}_{RL}D^{\frac{1}{3}}y(t)\right)\Big|_{t=0} = 0, \left({}_{RL}D^{\beta-1}y(t)\right)\Big|_{t=0} = \left({}_{RL}D^{\frac{-1}{4}}y(t)\right)\Big|_{t=0} = 0$

$\Rightarrow y_0(t) = E_{\frac{4}{3},1}(t^{\frac{4}{3}})$

$z_0(t) = \sin t$

Accordingly, by Eq. (4.6) the higher order approximation of the exact solution can be obtained as follows:

$$x_1(t) = \sum_{k=0}^{\infty} \frac{\left(-t^{\frac{4}{3}}\right)^k}{\Gamma\left(\frac{4}{3}k+1\right)} +$$

$$\int_0^{\tau}\left[\left((t-\tau)^{\frac{1}{3}}\sum_{k=0}^{\infty}\frac{\left(-(t-\tau)^{\frac{4}{3}}\right)^k}{\Gamma\left(\frac{4}{3}k+\frac{4}{3}\right)}\right)\left(2\tau\sum_{k=0}^{\infty}\frac{(k+1)!}{k!}\frac{\left(\tau^{\frac{4}{3}}\right)^k}{\Gamma\left(\frac{4}{3}k+\frac{7}{3}\right)}-\right.\right.$$
$$2\sum_{k=0}^{\infty}\frac{(k+1)!}{k!}\frac{\left(-\tau^{\frac{4}{3}}\right)^k}{\Gamma\left(\frac{4}{3}k+\frac{7}{3}\right)}-2\tau\cos\tau+(4+2\tau)\sum_{k=0}^{\infty}\frac{\left(\tau^{\frac{4}{3}}\right)^k}{\Gamma\left(\frac{4}{3}k+1\right)}-$$
$$\left.\left.(2\tau^2+6\tau)\sin\tau\right)\right]d\tau$$

$$y_1(t)=\sum_{k=0}^{\infty}\frac{\left(t^{\frac{4}{3}}\right)^k}{\Gamma\left(\frac{4}{3}k+1\right)}+\int_0^{\tau}\left[\left((t-\tau)^{\frac{1}{6}}\sum_{k=0}^{\infty}\frac{\left((t-\tau)^{\frac{4}{3}}\right)^k}{\Gamma\left(\frac{4}{3}k+\frac{4}{3}\right)}\right)(\cos\tau-\right.$$
$$\left.2\tau\sin\tau)\right]d\tau$$

$$z_1(t)=\sin t$$

Table 2: Numerical Results of the Solution in Example 2

T	x_1 $\alpha=\frac{1}{2},\beta=\frac{1}{3}$	x_1 $\alpha=\frac{2}{5},\beta=\frac{6}{10}$	x_1 $\alpha=\frac{7}{12},\beta=\frac{3}{4}$	Exact solution
0	1	1	1	1
0.1	1.366	1.313	1.094	1.015
0.2	1.755	1.647	1.242	1.063
0.3	1.796	1.75	1.391	1.146
0.4	1.983	1.9	1.555	1.267
0.5	2.125	1.982	1.711	1.431
0.6	2.22	1.988	1.846	1.642
0.7	2.457	2.212	1.949	1.906
0.8	2.761	2.549	2.341	2.23
0.9	3.117	2.903	2.674	2.62
1	3.852	3.561	3.196	3.086

Table 2 shows the approximate solutions for Eq. (4.6) obtained for different values of α and β using our method. The results are in good agreement with the results of the exact solutions.

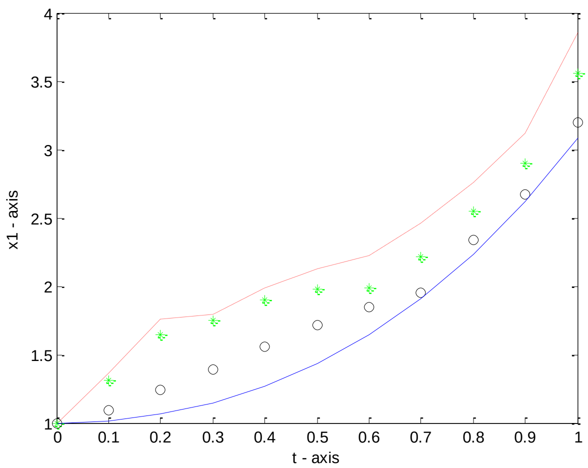

Fig. 2: The red dotted line (…) is the approximate solution when $\alpha=\frac{1}{2},\beta=\frac{1}{3}(\alpha+\beta<1)$, the green asterisks (***) is the approximate solution when $\alpha=\frac{2}{5},\beta=\frac{6}{10}(\alpha+\beta=1)$, the black circles (ooo) is the approximate solution when $\alpha=\frac{7}{12},\beta=\frac{3}{4}(\alpha+\beta>1)$, the continuous line (---) is the exact solution.

5. Conclusions

The fundamental goal of this work has been to propose an efficient algorithm for the solution of non-homogeneous system of time-fractional differential-algebraic equations. The goal has been achieved by using the fractional Laplace iteration method (FLIM).The results show that fractional Laplace iteration method is powerful and efficient techniques in finding approximate solutions for a system of time-fractional linear differential-algebraic equations Mathcad has been used for computations in this paper.

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