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Positive solutions for boundary value problems with fractional order

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Abstract

In this paper we investigate the existence of at least one, two positive solutions by using the Krasnoselskii fixed-point theorem in cones for nonlinear boundary value problem with fractional order.

Keywords: *Differential Equation, Caputo fractional derivative, fractional integral, existence, positive solutions, Krasnoselskii fixed-point theorem, nonlinear boundary value problem.*

1 Introduction

In this paper, we are concerned with the existence of at least one, two positive solutions of the boundary value problem for fractional differential equation of the form

$$^{c}D^{\alpha}y(t)+\varphi(t)f(t,y(t))=0, \quad \text{a.e} \ \ t\in J=[0,1], \quad 0<\alpha\leq 1, \tag{1}$$

$$ay(0)+by(1)=c, \tag{2}$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative, $f: J\times \mathbb{R}\rightarrow[0,+\infty)$ is a continuous function, a,b,c are real constants with $a+b\neq 0$, and $\varphi : [0,1]\rightarrow \mathbb{R}$ is a given function.

With the development of fractional calculus and its applications [26, 27, 28, 30, 32] in mathematics, technology, biology, chemical process etc., increasing attention has been paid to the study of fractional differential equations including the existence of solutions to fractional differential equations; see the books by Abbas *et al.* [1], Baleanu *et al.* [7], Lakshmikantham *et al.* [23], and the papers [3, 5, 8, 25, 31, 33, 34, 35, 37, 38, 39], the stability analysis of fractional differential equations [14, 22, 29], and so on. As a fundamental issue of the theory of fractional differential equations, the existence of (positive) solutions for kinds of boundary-value problems (BVPs) of fractional differential equations has been studied recently by many scholars, and lots of excellent results have been obtained for both two-point BVPs and nonlocal BVPs by means of fixed point index theory [5] fixed point theorems [3, 35, 39] mixed monotone method upper and lower solutions technique [31], seen also Benchohra *et al.* [9, 10, 12] and so on. The use of cone theoretic techniques in the study of solutions to boundary value problems has a rich and diverse history. Some authors have used fixed point theorems to show the existence of positive solutions to boundary value problems for ordinary differential equations, difference equations, and dynamic equations on time scales, see for example [2, 15, 19, 20, 21, 24, 36] and references therein. In other papers, [20, 21], authors have use fixed point theory to show the existence of solutions to singular boundary value problems. Still other papers have used cone theoretic techniques to compare the smallest eigenvalues of two operators, see [4, 6]. The books by Agarwal *et al.* [2] and Guo and Lakshmikantham [18] are excellent resources for the use of fixed point theory in the study of existence of solutions to boundary value problems.

This paper is organized as follows. In Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following sections. In Section 3, we shall provide sufficient conditions ensuring the existence of at least one, two positive solutions to for problem $(1)-(2)$ via an application of the the

Krasnoselskii fixed-point theorem in cones [13]. Finally in Section 4 we give an example to illustrate the theory presented in the previous sections.

2 Preliminary notes

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

$L^\infty(J, \mathbb{R})$ denotes the Banach space of measurable and essentially bounded functions with norm

$$\|y\|_{L^\infty} = \inf\{d > 0 : |y(t)| \leq d, \text{ a.e. } t \in J\}.$$

Definition 2.1 *([16]-[17]). The fractional (arbitrary) order integral of the function $h \in L^1([a, b], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = h(t) * \varphi_\alpha(t)$, where $\varphi_\alpha(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \leq 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2.2 *([16]-[17]). For a function h given on the interval $[a, b]$, the αth Riemann-Liouville fractional-order derivative of h, $\alpha \in (0, 1)$, is defined by*

$$\begin{aligned}
(D_{a+}^\alpha h)(t) &= \frac{d^\alpha h(t)}{dt^\alpha} \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} h(s)ds \\
&= \frac{d}{dt} I_a^{1-\alpha} h(t).
\end{aligned}$$

Definition 2.3 *For a function h given on the interval $[a, b]$, the Caputo fractional-order derivative of h, $\alpha \in (0, 1)$, is defined by*

$$({}^cD_{a+}^\alpha h)(t) = \big(D_{a+}^\alpha [h(x) - h(a)]\big)(t).$$

Theorem 2.4 *Let $\mathcal{K}$ be a cone in a Banach space $\mathcal{B}$. Assume Ω_1 and Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. If $N : \mathcal{K} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{K}$ is a completely continuous operator such that either*

(i) $\|Ny\| \leq \|y\|$ for all $y \in \mathcal{K} \cap \partial\Omega_1$, and $\|Ny\| \geq \|y\|$ for all $y \in \mathcal{K} \cap \partial\Omega_2$, or

(ii) $\|Ny\| \geq \|y\|$ for all $y \in \mathcal{K} \cap \partial\Omega_1$ and $\|Ny\| \leq \|y\|$ for all $y \in \mathcal{K} \cap \partial\Omega_2$.

Then N has a fixed point in $\mathcal{K} \cap (\overline{\Omega_2} \backslash \Omega_1)$.

3 Main results

Let us start by defining what we mean by positive a solution of the problem $(1) - (2)$.

Definition 3.1 *A function $y \in C^1([0, 1], \mathbb{R})$ is said to be a solution of (1)–(2) if y satisfies the equation ${}^cD^\alpha y(t) + \varphi(t) f(t, y(t)) = 0$ on J, $y(t) \geq 0$ a.e $t \in J$ and the condition $ay(0) + by(1) = c$.*

For the existence of solutions for the problem $(1) - (2)$, we need the following auxiliary lemma:

Lemma 3.2 *[38] Let $\alpha > 0$, then the differential equation*

$${}^cD^\alpha h(t) = 0$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}, c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

Lemma 3.3 *[38] Let $\alpha > 0$, then*

$$I^{\alpha c}D^{\alpha}h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1}t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

Lemma 3.4 *[11] Let $0 < \alpha \leq 1$ and let $h : [0,1] \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$\begin{aligned} y(t) &= \frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)ds - c\right] \\ &- \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds \end{aligned}$$

if and only if y is a solution of the fractional BVP

$$^cD^{\alpha}y(t) + h(t) = 0, \ \ t \in [0,1],$$

$$ay(0) + by(1) = c.$$

Let us now introduce additional conditions that will be used to show our existence result.

(H_1) There exist functions $\psi_1 : [0,\infty) \to [0,\infty)$ continuous, nondecreasing and $q_1 \in L^{\infty}(J, \mathbb{R}^+)$ such that
$$f(t,y) \leq q_1(t)\psi_1(y), \ \text{for each } t \in J, \ y \geq 0.$$

(H_2) There exist functions $\psi_2 : [0,\infty) \to [0,\infty)$ continuous, nondecreasing and $q_2 \in L^{\infty}(J, \mathbb{R}^+)$ such that
$$f(t,y) \geq q_2(t)\psi_2(y), \ \text{for each } t \in J, \ y \geq 0.$$

(H_3) There exists a constant $r > 0$ such that

$$\frac{\frac{b}{a+b}M\psi_2(r)\|q_2\|_{\infty}}{\Gamma(\alpha+1)} - \frac{\|\varphi\|_{\infty}\psi_1(r)\|q_1\|_{\infty}}{\Gamma(\alpha+1)} - \frac{c}{a+b} \geq r.$$

(H_4) There exists a constant $R > 0$ such that

$$\frac{\|\varphi\|_{\infty}\|q_1\|_{\infty}\psi_1(R)}{\Gamma(\alpha+1)}\left(1 + \frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|} \leq R.$$

(H_5) There exists a function $\varphi : [0,1] \to \mathbb{R}$ with

(1) $\varphi \in L^{\infty}[0,1]$.

(2) There exists $M > 0$ such that $\varphi(t) \geq M$ a.e. $t \in [0,1]$.

Theorem 3.5 *Suppose that hypotheses $(H_1) - (H_5)$ are satisfied. Then the boundary value problem (1)-(2) has at least one positive solution.*

Define the cone $\mathcal{K} \subset C([0,1], \mathbb{R})$ by :

$$\mathcal{K} = \{y \in C([0,1], \mathbb{R}), y(t) \geq 0, \ t \in [0,1]\},$$

and the operator $N : \mathcal{K} \to \mathcal{K}$ by :

$$\begin{aligned} Ny(t) &= \frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\varphi(s)f(s,y(s))ds - c\right] \\ &- \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi(s)f(s,y(s))ds. \end{aligned}$$

Note that fixed points of N are solutions of $(1)-(2)$. In order to use Theorem 2.4 we must show that $N:\mathcal{K}\to\mathcal{K}$ is completely continuous. The proof will be given in several steps.

Step 1: *N is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n\to y$ in $\mathcal{K}$. Then for each $t\in[0,1]$.

$$\begin{aligned}
& |N(y_n)(t)-N(y)(t)| \\
\leq\ & \frac{|b|}{\Gamma(\alpha)|a+b|}\int_0^1(1-s)^{\alpha-1}\varphi(s)|f(s,y_n(s))-f(s,y(s))|ds \\
+\ & \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\varphi(s)|f(s,y_n(s))-f(s,y(s))|ds \\
\leq\ & \frac{|b|\|\varphi\|_\infty}{\Gamma(\alpha)|a+b|}\int_0^1(1-s)^{\alpha-1}\sup_{s\in[0,1]}|f(s,y_n(s))-f(s,y(s))|ds \\
+\ & \frac{\|\varphi\|_\infty}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\sup_{s\in[0,1]}|f(s,y_n(s))-f(s,y(s))|ds \\
\leq\ & \frac{\|\varphi\|_\infty\left(1+\frac{|b|}{|a+b|}\right)\|f(\cdot,y_n(\cdot))-f(\cdot,y(\cdot))\|_\infty}{\alpha\Gamma(\alpha)} \\
+\ & \frac{\|\varphi\|_\infty\|f(\cdot,y_n(\cdot))-f(\cdot,y(\cdot))\|_\infty}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds \\
\leq\ & \frac{\|\varphi\|_\infty\left(1+\frac{|b|}{|a+b|}\right)\|f(\cdot,y_n(\cdot))-f(\cdot,y(\cdot))\|_\infty}{\alpha\Gamma(\alpha)}.
\end{aligned}$$

Since f is a continuous function, we have

$$\|N(y_n)-N(y)\|_\infty\leq\frac{\|\varphi\|_\infty\left(1+\frac{|b|}{|a+b|}\right)\|f(\cdot,y_n(\cdot))-f(\cdot,y(\cdot))\|_\infty}{\Gamma(\alpha+1)}\to 0\quad as\ n\to\infty.$$

Step 2: *N maps bounded sets into bounded sets in $\mathcal{K}$.*

Indeed, it is enough to show that for any $\eta^*>0$, there exists a positive constant ℓ such that for each $y\in B_{\eta^*}=\{y\in\mathcal{K}:\|y\|_\infty\leq\eta^*\}$, we have $\|N(y)\|_\infty\leq\ell$.
By $(H1),(H_5)$ we have for each $t\in[0,1]$,

$$\begin{aligned}
|N(y)(t)| &\leq \frac{|b|}{\Gamma(\alpha)|a+b|}\int_0^1(1-s)^{\alpha-1}|\varphi(s)f(s,y(s))|+\frac{|c|}{|a+b|} \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\varphi(s)f(s,y(s))ds \\
&\leq \frac{|b|\|\varphi\|_\infty\psi_1(\|y\|)\|q_1\|_\infty}{\Gamma(\alpha)|a+b|}\int_0^1(1-s)^{\alpha-1}ds+\frac{|c|}{|a+b|} \\
&+ \frac{\|\varphi\|_\infty\psi_1(\|y\|)\|q_1\|_\infty}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds \\
&\leq \frac{\|\varphi\|_\infty\psi(\|\eta^*\|)\|q_1\|_\infty|b|}{\alpha\Gamma(\alpha)|a+b|}+\frac{|c|}{|a+b|}+\frac{\|\varphi\|_\infty\psi(\|\eta^*\|)\|q_1\|_\infty}{\alpha\Gamma(\alpha)}.
\end{aligned}$$

Thus

$$\|N(y)\|_\infty\leq\frac{\|\varphi\|_\infty\psi(\|\eta^*\|)\|q_1\|_\infty}{\Gamma(\alpha+1)}\left[1+\frac{|b|}{|a+b|}\right]+\frac{|c|}{|a+b|}:=\ell.$$

Step 3: *N maps bounded sets into equicontinuous sets of $\mathcal{K}$.*

Let $t_1, t_2 \in (0,1]$, $t_1 < t_2$, B_{η^*} be a bounded set of $\mathcal{K}$ as in Step 2, and let $y \in B_{\eta^*}$. Then, by $(H1), (H_5)$, we have

$$\begin{aligned}
|N(y)(t_2) - N(y)(t_1)| &= \Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\varphi(s)f(s,y(s))ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_1-s)^{\alpha-1}\varphi(s)f(s,y(s))ds\Big| \\
&\le \frac{\|\varphi\|_\infty\psi_1(\|y\|)\|q_1\|_\infty}{\Gamma(\alpha)}\int_0^{t_2}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]ds \\
&+ \frac{\|\varphi\|_\infty\psi_1(\|y\|)\|q_1\|_\infty}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_1-s)^{\alpha-1}ds \\
&\le \frac{\|\varphi\|_\infty\psi_1(\|y\|)\|q_1\|_\infty}{\Gamma(\alpha+1)}[(t_1-t_2)^\alpha + t_2^\alpha - t_1^\alpha] \\
&+ \frac{\|\varphi\|_\infty\psi_1(\|y\|)\|q_1\|_\infty}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha \\
&\le \frac{\|\varphi\|_\infty\psi_1(\|\eta^*\|)\|q_1\|_\infty}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha \\
&+ \frac{\|\varphi\|_\infty\psi_1(\|\eta^*\|)\|q_1\|_\infty}{\Gamma(\alpha+1)}(t_1^\alpha - t_2^\alpha).
\end{aligned}$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $N : \mathcal{K} \longrightarrow \mathcal{K}$ is continuous and completely continuous.

Define the set

$$\Omega_2 = \{y \in \mathcal{K} : \|y\|_\infty < R\}.$$

Let $y \in \mathcal{K} \cap \partial\Omega_2$. From $(H_1), (H_4)$ and (H_5), we have

$$\begin{aligned}
\|Ny\|_\infty &= \max_{0\le t\le 1}\Big|\frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\varphi(s)f(s,y(s))ds - c\right] \\
&- \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\varphi(s)f(s,y(s))ds\Big| \\
&\le \max_{0\le t\le 1}\left(\left|\int_0^t(t-s)^{\alpha-1}\varphi(s)f(s,y(s))ds\right| + \left|\frac{b}{a+b}\int_0^1(1-s)^{\alpha-1}\varphi(s)f(s,y(s))ds\right| + \frac{|c|}{|a+b|}\right) \\
&\le \|\varphi\|_\infty\|q_1\|_\infty\psi_1(R)\left(\frac{1}{\alpha\Gamma(\alpha)} + \frac{|b|}{\alpha\Gamma(\alpha)|a+b|}\right) + \frac{|c|}{|a+b|}. \\
&\le \|\varphi\|_\infty\|q_1\|_\infty\psi_1(R)\left(\frac{1}{\Gamma(\alpha+1)} + \frac{|b|}{\Gamma(\alpha+1)|a+b|}\right) + \frac{|c|}{|a+b|}. \\
&\le R \\
&= \|y\|_\infty,
\end{aligned}$$

that is

$$\|Ny\|_\infty \le \|y\|_\infty \quad y \in \mathcal{K} \cap \partial\Omega_2. \tag{3}$$

Define the set

$$\Omega_1 = \{y \in \mathcal{K} : \|y\|_\infty < r\}.$$

Let $y \in \mathcal{K} \cap \partial\Omega_1$. From (H_1), (H_2), (H_3) and (H_5), for $t \in [0,1]$ we have

$$\begin{aligned}
Ny(t) &= \frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\varphi(s)f(s,y(s))ds - c\right] \\
&- \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\varphi(s)f(s,y(s))ds
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{\psi_2(\|y\|)q_2(t)Mb}{(a+b)\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}ds-\frac{c}{a+b}\\
&- \|\varphi\|_\infty\psi_1(\|y\|)\|q_1\|_\infty\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds\\
&\geq \frac{\frac{b}{a+b}M\psi_2(r)\|q_2\|_\infty}{\Gamma(\alpha+1)}-\frac{\|\varphi\|_\infty\psi_1(r)\|q_1\|_\infty}{\Gamma(\alpha+1)}-\frac{c}{a+b}\\
&\geq r\\
&= \|y\|_\infty,
\end{aligned}$$

which implies that

$$\|Ny\|_\infty \geq \|y\|_\infty \quad y\in\mathcal{K}\cap\partial\Omega_1. \tag{4}$$

Since $0\in\overline{\Omega}_1\subset\Omega_2$ and inequalities (3) and (4) hold, then by part (ii) of Theorem2.4, there exists at least one fixed point of N in $\mathcal{K}\cap(\overline{\Omega}_2\backslash\Omega_1)$, and then (1)-(2) has at least one positive solution and the proof is complete.

Next, we deal with the existence of at least two distinct positive solutions to problem (1)-(2). For $y\in\mathcal{K}$, we denote

$$f_\infty=\lim_{y\to\infty}\inf_{t\in[0,1]}\frac{f(t,y)}{y},$$

$$f_0=\lim_{y\to 0}\inf_{t\in[0,1]}\frac{f(t,y)}{y}.$$

Let us now introduce additional conditions that will be used after.

(H_6) There exist constant $p>s>0$ such that $f(t,y)\leq s\Lambda$ for $(t,y)\in[0,1]\times[0,p]$, where

$$\Lambda=\left(\|\varphi\|_\infty\frac{|\frac{b}{a+b}|+1}{\Gamma(\alpha+1)}\right)^{-1},$$

and

$$s+\frac{|c|}{|a+b|}\leq p.$$

(H_7)

$$f_0=\infty,\ f_\infty=\infty.$$

Theorem 3.6 *Assume that the conditions (H_1), (H_5) (H_6) and (H_7) hold, then the problem* $(1)-(2)$ *has at least two distinct positive solutions y_1, $y_2\in\mathcal{K}$.*

From (H_7) In view of $f_0=\infty$, there exists K_1, $0<K_1<p$ such that

$$f(t,y)\geq m\|y\| \quad \text{for } (t,y)\in[0,1]\times(0,K_1], \tag{5}$$

where m is given by

$$\left(\frac{Mm\frac{b}{a+b}-\|\varphi\|_\infty\|q_1\|_\infty\psi_1(K_1)}{\Gamma(\alpha+1)K_1}-\frac{c}{(a+b)K_1}\right)\geq 1. \tag{6}$$

Take

$$\Omega_{K_1}=\{y\in\mathcal{K}:\|y\|<K_1\}.$$

If $y\in\Omega_{K_1}$ with $\|y\|=K_1$, it means that

$$\max_{t\in[0,1]}y(t)\leq\|y\|=K_1 \text{ for } t\in[0,1].$$

It follows from (5), (6) and $(H_1),(H_5)$ that

$$\begin{aligned}
Ny(t) &\geq \min_{t\in[0,1]}\left(\frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\varphi(s)f(s,y(s))ds-c\right]\right.\\
&- \left.\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\varphi(s)f(s,y(s))ds\right)\\
&\geq \min_{t\in[0,1]}\left(\frac{1}{a+b}\frac{b}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}Mm\|y\|ds-\frac{c}{a+b}\right.\\
&- \left.\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|\varphi\|_\infty\psi_1(y(s))q_1(s)ds\right)\\
&\geq \min_{t\in[0,1]}\left(\frac{1}{a+b}\frac{b}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}Mm\|y\|ds-\frac{c}{a+b}\right.\\
&- \left.\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|\varphi\|_\infty\psi_1(y(s))q_1(s)ds\right)\\
&\geq \min_{t\in[0,1]}\left(Mm\|y\|\frac{b}{(a+b)\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}ds-\frac{c}{a+b}\right.\\
&- \left.\frac{\|\varphi\|_\infty\psi_1(\|y\|)\|q_1\|_\infty}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds\right)\\
&\geq \left(\frac{Mm\frac{b}{a+b}-\|\varphi\|_\infty\|q_1\|_\infty\psi_1(K_1)}{\Gamma(\alpha+1)K_1}-\frac{c}{(a+b)K_1}\right)\|y\|_\infty\\
&\geq \|y\|_\infty.
\end{aligned}$$

which implies that

$$\|Ny\|_\infty \geq \|y\|_\infty \quad \text{for} y\in\mathcal{K}\cap\partial\Omega_{K_1}. \tag{7}$$

Let

$$\Omega_p=\{y\in\mathcal{K}:\|y\|_\infty<p\}.$$

For $y\in\mathcal{K}\cap\partial\Omega_p$ and from $(H_5),(H_6)$ we have:

$$\begin{aligned}
|Ny(t)| &\leq \max_{t\in[0,1]}\left|\frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\varphi(s)f(s,y(s))ds-c\right]\right.\\
&- \left.\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\varphi(s)f(s,y(s))ds\right|\\
&\leq \max_{t\in[0,1]}\left(\frac{\|\varphi\|_\infty s\Lambda|b|}{\Gamma(\alpha)|a+b|}\int_0^1(1-s)^{\alpha-1}ds+\frac{\|\varphi\|_\infty s\Lambda}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds+\frac{|c|}{|a+b|}\right)\\
&\leq s\Lambda\left(\|\varphi\|_\infty\frac{|\frac{b}{a+b}|+1}{\Gamma(\alpha+1)}\right)+\frac{|c|}{|a+b|}\\
&\leq s+\frac{|c|}{|a+b|}\\
&\leq p\\
&= \|y\|_\infty.
\end{aligned}$$

And then

$$\|Ny\|_\infty \leq \|y\|_\infty, \qquad y\in\mathcal{K}\cap\partial\Omega_p. \tag{8}$$

Since $0\in\Omega_{K_1},(\overline{\Omega}_{K_1}\subset\Omega_p$. In view to (7), (8) and according to Theorem 2.4, (ii) problem (1)-(2) has a positive solution y_1 in $\mathcal{K}\cap(\overline{\Omega}_p\backslash\Omega_{K_1})$.

It follows from (H_7), $f_\infty=\infty$ that there exists $K_2>4p$ such that

$$f(t,y)\geq k\|y\|_\infty, \tag{9}$$

where $t \in [0,1]$ and $\|y\|_\infty \geq K_2$. Moreover, k satisfies :

$$\frac{\frac{Mkb}{a+b} - \|\varphi\|_\infty \psi_1(K_2)\|q_1\|_\infty}{\Gamma(\alpha+1)} - \frac{c}{(a+b)K_2} \geq 1 \tag{10}$$

Let

$$\Omega_{K_2} = \{y \in \mathcal{K} : \|y\|_\infty < K_2\}$$

,

and

$$\Omega_{4p} = \{y \in C([0,1], \mathbb{R}) : \|y\|_\infty < 4p\}$$

,

then we see that $\overline{\Omega}_{4p} \subset \Omega_{K_2}$. Then we obtain

$$\|Ny\|_\infty \leq \|y\|_\infty, \quad \mathcal{K} \cap \partial\Omega_{4p}, \tag{11}$$

holds by using (H_6),

For any $y \in \mathcal{K} \cap \partial\Omega_{K_2}$, we have $\|y\|_\infty = K_2$.
Then according to equation $(9), (10)$ and from $(H_1), (H_5)$ we deduce that :

$$\begin{aligned}
Ny(t) &\geq \min_{t\in[0,1]} \left(\frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \varphi(s) f(s,y(s)) ds - c \right] \right. \\
&- \left. \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) f(s,y(s)) ds \right) \\
&\geq \min_{t\in[0,1]} \left(\frac{b}{(a+b)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} Mk\|y\| ds - \frac{c}{a+b} \right. \\
&- \left. \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\varphi\|_\infty \psi_1(y(s)) q_1(s) ds \right) \\
&\leq \min_{t\in[0,1]} \left(\frac{Mbk\|y\|}{(a+b)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds - \frac{c}{a+b} \right. \\
&- \left. \|\varphi\|_\infty \psi_1(\|y\|)\|q_1\|_\infty \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \right) \\
&\geq \left(\frac{\frac{Mkb}{a+b} - \|\varphi\|_\infty \psi_1(K_2)\|q_1\|_\infty}{\Gamma(\alpha+1)} - \frac{c}{(a+b)K_2} \right) \|y\| \\
&\geq \|y\|_\infty.
\end{aligned}$$

and then

$$\|Ny\|_\infty \geq \|y\|_\infty \quad \mathcal{K} \cap \partial\Omega_{K_2}. \tag{12}$$

From (11), (12) and according to Theorem 2.4 we deduce that problem (1)-(2) has at least one positive solution y_2 in $\mathcal{K} \cap (\overline{\Omega}_{K_2} \backslash \Omega_{4p})$ with

$$4p \leq \|y_2\|_\infty \text{ and } \|y_2\|_\infty \leq K_2.$$

It is easily seen that y_1 and y_2 are distinct.

4 Example

We give an example to illustrate the usefulness of our main results. Let us consider the following fractional boundary value problem,

$$^cD^{\frac{1}{2}}y(t)+\frac{e^{t-2}}{2}\frac{2\sqrt{\pi}e^t|y(t)|}{(9+e^t)(1+|y(t)|)}=0,\quad t\in J:=[0,1],\quad \alpha=\frac{1}{2}, \tag{13}$$

$$y(0)+y(1)=0. \tag{14}$$

Set

$$f(t,u)=\frac{2\sqrt{\pi}e^tu}{(1+e^t)(1+u)},\quad (t,u)\in J\times[0,\infty),$$
$$\varphi(t)=1,\quad t\in[0,1],$$
$$\psi_1(u)=\frac{\sqrt{\pi}u}{(1+u)},\quad u\in[0,\infty),$$
$$\psi_2(u)=\frac{\sqrt{\pi}u}{(1+u)},\quad u\in[0,\infty).$$
$$q_1(t)=e^t,\quad q_2(t)=\frac{2}{1+e^t},\quad t\in[0,1].$$

We have

$$f(t,u)\le q_1(t)\psi_1(u),\quad (t,u)\in[0,1]\times[0,\infty),$$
$$f(t,u)\ge q_2(t)\psi_2(u),\quad (t,u)\in[0,1]\times[0,\infty).$$

Conditions (H_1), (H_2) hold. Since $a=1,\ b=1,\ c=0$, we have :

$$\frac{\frac{b}{a+b}M\psi_2(r)\|q_2\|_\infty}{\frac{\sqrt{\pi}}{2}}-\frac{\|\varphi\|_\infty\psi_1(r)\|q_1\|_\infty}{\frac{\sqrt{\pi}}{2}}\ge r. \tag{15}$$

Since $\varphi(s)=\frac{1}{2}(e^{s-2}),\ \forall s\in[0,1],\quad M=1,\ \|\varphi\|_\infty=\frac{1}{2},\ \|q_1\|_\infty=e,\ \|q_2\|_\infty=1$
Condition (H_3) which is (15) becomes:

$$2(4-e)\frac{r}{r+1}\ge r. \tag{16}$$

Choose $r=1$, (16) holds and then (H_3) holds.
Define

$$\Omega_1=\{y\in\mathcal{K}:\|y\|_\infty<1\}.$$

Condition (H_4)

$$\frac{\|\varphi\|_\infty\|q_1\|_\infty\psi_1(R)}{\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right)+\frac{|c|}{|a+b|}\le R.$$

Becomes :

$$\|\varphi\|_\infty\|q_1\|_\infty\psi_1(R)\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{2\Gamma(\alpha+1)}\right)\le R. \tag{17}$$

Since $\|\varphi\|_\infty=1,\ \|q_1\|_\infty=e,\ \psi_1(R)=\frac{5\sqrt{\pi}R}{R+1},\ \Gamma(\alpha+1)=\Gamma(\frac{3}{2})=\frac{\sqrt{\pi}}{2}$, we have :

$$15e\frac{R}{R+1}\le R. \tag{18}$$

Choose $R=15e$, (18) holds and then (H_5) which is (17) holds. define

$$\Omega_2=\{y\in\mathcal{K}:\|y\|_\infty<15e\}.$$

Since $0\in\Omega_1,(\overline{\Omega}_1\subset\Omega_2$. It follows from Theorem 2.4, that problem $(13)-(14)$ has a positive solution y in $\mathcal{K}\cap(\overline{\Omega}_2\backslash\Omega_1)$.

5 Conclusion

In this paper we have considered a boundary value problem with fractional order, for which we give sufficient conditions for existence of at least one positive, two positive solutions by using Krasnoselskii fixed-point theorem.

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