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On two diophantine equations $2^x + 3y^2 = 4^z$ and $2^x + 7y^2 = 4^z$

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Abstract

We find all solutions to the Diophantine equations $2^x + 3y^2 = 4^z$ and $2^x + 7y^2 = 4^z$. Also, we give solutions to $2^x + dy^2 = 4^z$ in non-negative integers for $d = (2^k - 1)/9$, where k is a natural number $\equiv 0 \pmod 6$.

Keywords: *Exponential Diophantine equation, integral solutions.*

1 Introduction

Different examples of Diophantine equations have been studied (*see* for instance [1, 2]). In [3], Acu studied some Diophantine equations of type $a^x + b^y = c^z$. Moreover, Acu [4] considered the equation $2^x + 5^y = z^2$. Cenberci and Senay [5] had studied the Diophantine equation $x^2 + B^2 = y^4$ and gave a conjecture, an analogue of Terai's conjecture. Furthermore, they proved in [5] that if $y \equiv 5 \pmod 8$ is a prime power then their conjecture holds. If B is a prime power, $y^2 = Y \equiv 1 \pmod 8$ then Terai's and their conjecture holds. Cenberci, Peker and Coskun [6] determined all solutions to the equation $x^a + y^b = z^c$, $(a, b, c) \in \{(2, 8, 6), (2, 6, 8), (8, 6, 2)\}$ in coprime integers x, y, z. Suvarnamani [7] studied the Diophantine equation $2^x + p^y = z^2$ where p is a prime number and x, y and z are non-negative integers. Rabago [8] have studied the two Diophantine equations $4^x - 7^y = 3z^2$ and $4^x - 19^y = 3z^2$, and in [9], Rabago considered the Diophantine equation $4^x - p^y = 3z^2$, p an odd prime $\equiv 3 \pmod 4$.

In this short note, we find all solutions to the Diophantine equations $2^x + 3y^2 = 4^z$ and $2^x + 7y^2 = 4^z$ in non-negative integers. Also, we give solutions to $2^x + dy^2 = 4^z$ for $d = (2^k - 1)/9$ in non-negative integers, where k a natural number $\equiv 0 \pmod 6$.

2 Main results

Theorem 2.1. *The solutions to the Diophantine equation $2^x + 3y^2 = 4^z$ in non-negative integers are given by*

$$(x, y, z) \in \{(2(n-1), 0, n-1) : n \in \mathbb{N}\} \cup \{(2(n-1), 2^{n-1}, n) : n \in \mathbb{N}\}.$$

Proof. We first consider the case when $z = 0$, obtaining $2^x + 3y^2 = 1$ in which we may deduce immediately that $x = 0$ and $y = 0$. For the case $x = 0$, we have $4^z - 3y^2 = 1$. This is true only when $z = 0$, $y = 0$ and $y = 1$, $z = 1$. On the other hand, if $y = 0$, we have $2^x = 2^{2z}$, or equivalently $x = 2z$. Now for the general case, $x, y, z > 0$, we have $2^{2z} - 2^x = 3y^2$. Then, $2^x(2^{2z-x} - 1) = 3y^2$. Hence, $2^x = y^2$ and $2^{2z-x} - 1 = 3$. The latter equation is true for $x = 2(z-1)$, and $2^x = y^2$ is satisfied for all $x = 2(n-1)$ and $y = 2^{n-1}$, where n is a natural number. Also, it follows that $z = n$. This proves the theorem. □

Theorem 2.2. *The solutions to the Diophantine equation $2^x + 7y^2 = 4^z$ in non-negative integers are given by*

$$(x, y, z) \in \{(2(n-1), 0, n-1) : n \in \mathbb{N}\} \cup \{(2(n-1), 3(2^{n-1}), n+2) : n \in \mathbb{N}\}.$$

Proof. It can be shown easily that $(x, y, z) = (2(n-1), 0, n-1)$ is a solution for all natural number n. Now, if $x, y, z > 0$, we have $2^{2z} - 2^x = 7y^2$. Then, $2^x(2^{2z-x} - 1) = 7y^2$. Hence, x is even so $7y^2 \equiv 4^z - 2^x \equiv 0 \pmod 3$. It follows that y is divisible by three, *i.e.* $y = 3k$ for some $k \in \mathbb{N}$. Letting $y = 3k$, we obtain $2^x(2^{2z-x} - 1) = 63k^2$, implying $2^x = k^2$ and $2^{2z-x} - 1 = 63$. The solution to $2^x = k^2$ is then given by $x = 2(n-1)$ and $k = 2^{n-1}$. For $2^{2z-x} - 1 = 63$ we have the solution $2z - x = 6$ or $x = 2(z-3)$. Furthermore, we see that $2(n-1) = 2(z-3)$, that is $z = n+2$. This completes the proof of the theorem. □

Lemma 2.3. *If k is a natural number and $k \equiv 0 \pmod 6$, then $2^k - 1 \equiv 0 \pmod 9$.*

Proof. Let $k \equiv 0 \pmod 6$ hence $k = 6m$, $m \in \mathbb{N}$. For $m = 1$, we have $2^6 - 1 = 64 - 1 = 63$ which is divisible by 9. Suppose $2^k - 1 \equiv 0 \pmod 9$. Then, $2^k - 1 = 9l$, where l a natural number. So, for $m > 1$, we have $2^{k+1} - 1 = 2^{6m+1} - 1 = 64^{m+1} - 1$. It follows that $2^{k+1} - 1 = 64(64^m - 1) + 63 = 64(9l) + 63 = 9(64l + 63)$. Thus, $2^{k+1} - 1 \equiv 0 \pmod 9$. By the principle of mathematical induction, conclusion follows. □

In Theorem 2.2, it is interesting to note that $7 = (2^6 - 1)/9$. This observation provides us a motivation to generalize the given theorem. Our generalization is stated in the following result.

Theorem 2.4. *Let $d = (2^k - 1)/9$, where k is a natural number such that $k \equiv 0 \pmod 6$. Then the solutions to the Diophantine equation $2^x + dy^2 = 4^z$ in non-negative integers are given by*

$$(x, y, z) \in \{(2(n-1), 0, n-1) : n \in \mathbb{N}\} \cup \{(2(n-1), 3(2^{n-1}), n-1+k/2 : n \in \mathbb{N}\}.$$

Proof. The proof is very similar to Theorem 2.2. It is clear that $(x, y, z) = (2(n-1), 0, n-1)$ are solutions to $2^x = 4^z$. Now, for positive integers x, y and z, we have $2^x(2^{2z-x} - 1) = dy^2$. It follows that x is even so $dy^2 \equiv 4^z - 2^x \equiv 0 \pmod 3$. This implies that y is divisible by 3. Letting $y = 3m$, m a natural number, we have $2^x(2^{2z-x} - 1) = (2^k - 1)m^2$. That is, $2^x = m^2$ and $2^{2z-x} - 1 = 2^k - 1$. Thus, $x = 2(n-1)$ and $m = 2^{n-1}$ which implies that $y = 3(2^{n-1})$. Furthermore, we see that $2z - x = k = 6l$, for some $l \in \mathbb{N}$. Therefore, $x = 2(z - 3l) = 2(z - k/2)$. Here we conclude that $2(n-1) = 2(z - k/2)$ or $z = n - 1 + k/2$. The theorem is proved. □

3 Conclusion

In the paper, we have found all solutions to the Diophantine equation $2^x + 3y^2 = 4^z$ in non-negative integers. The solutions are given by $(x, y, z) \in \{(2(n-1), 0, n-1) : n \in \mathbb{N}\} \cup \{(2(n-1), 2^{n-1}, n) : n \in \mathbb{N}\}$. Also, we have shown that for $d = (2^k - 1)/9$ and natural number $k \equiv 0 \pmod 6$, the solutions to the Diophantine equation $2^x + dy^2 = 4^z$ in non-negative integers are $(x, y, z) \in \{(2(n-1), 0, n-1) : n \in \mathbb{N}\} \cup \{(2(n-1), 3(2^{n-1}), n-1+k/2 : n \in \mathbb{N}\}$.

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