On the application of Fubini's theorem in the integration of functions of two variables in a measure space

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Abstract

We consider the integration of functions of two variables in a measure space. Some definitions, theorems and proves relating to measurable functions and measure space were considered by using Fubini's theorem. Application on the improvement of the Jensen's inequality with respect to the probability measure space is treated.

Keywords: *Fubini's Theorem, Hermite-Hadamard inequality, Jensen's inequality, Measure, Product measure.*

1 Introduction

In Mathematical analysis, integration of functions is of great importance. In the 1850's Bernhard Riemann adopted a new and different viewpoint of the calculus of integration by I. Newton and G. Leibniz. He separated the concept of integration from its companion, differentiation, and examined the motivating summation and limit process of finding areas by itself. He broadened the scope by considering all functions on an interval for which this process of integration could be defined: the class of 'integrable' functions. The viewpoint of Riemann led others to invent other integration theories, the most significant being Lebesgue's thoery of integration.

Measure theory was developed in successive stages during the late 19th- early 20th century by Emile Borel, Henri Lebesgue, Johann Radon among others. In measure integration theory, specifying a measure allows one to define integrals on spaces more general and it gives more theorem than its predecessor, the Riemann integral. Also, in classical analysis the problem of reducing double(or multiple) integrals to iterated integrals plays an important role. In integration theory, the key result along this line is the Fubini's theorem.

Fubini's theorem, named after Guido Fubini, is a result which gives conditions under which it is possible to compute a double integral using iterated integrals. As a consequence it allows the order of integration to be changed in iterated integrals.

In 1992 Dragomir and Ionescu [6] studied some aspects of convex functions and some interesting inequalities were obtained. Mitrinovic, Pecaric and Fink in [12] discussed classical and new inequalities in analysis like the Holder's, Minkowski's, Jensen's, Bernoullli's and Steffensen's inequalities in which they proved these inequalities with some generalizations. Dragomir and Ionescu [8] later in 1994 proved useful inequality which counterparts the Jensen's inequality for continuous functions. Rooin[14] proved some application on discrete function.

2 Preliminary notes

We briefly give some basic definitions of the concepts of measure, integration and product space which serves as background to this work.

Definiton 2.1.0[11] A system of set is a set whose elements are sets. A system of set I is called a *semiring* if

   (i) I contains the empty set.

   (ii) $A \cap B \in I$ whenever $A \in I, B \in I$

   (iii) if $A \in I$ and $A_1 \subset A, A = \cup_{k=1}^n A_k$ such that $A_K \cap A_l = \phi$ k=l where A_1 is the first term.

Definiton 2.1.1[11] A set function is a function whose domain of definition is a class of sets. An extended real valued set function m defined on class of sets say I_m with the following properties:

   (i) the domain of definition I_m of m is a semiring.

   (ii) m is real and non-negative.

   (iii) m is additive i.e if $P = \cup_{k=1}^n p_k$ then $m(P) = \sum_{k=1}^n m(p_k)$ where $p_k's$ are pairwise disjoint rectangles is called a measure.

Definition 2.1.2 Let X be a set and $\mathcal{S}$ be a σ-algebra defined on X and a measure μ defined on X, then the pair $(X, \mathcal{S})$ is called a measurable space and the triple $(X, \mathcal{S}, \mu)$ is called a measure space. e.g the probability measure space, complex measure space, projection-valued measure space. A measure space is called σ-finite if $X = \cup_n X_n$ where $m(X_n) < \infty$.

Definition 2.1.3[11] Let f be a simple function then by the Lebesgue integral of f over the set A we mean

$$\int_A f(x)dx = \sum_n y_n \mu(A_n) \quad A_n = \{x : x \in A, f(x) = y_n\} \tag{1}$$

provided this series above is absolutely convergent. If the integral exist then we say f is integrable with repect to the measure μ on the set A.

Definition 2.1.4[11] A measurable function f is said to be integrable on a set A if $\exists$ a sequence $\{f_n\}$ of integrable simple functions converging uniformly to f on A. The limit $\lim_{n\to\infty} \int_A f_n(x)dx$ is called the lebesgue integral of f over set A.

$$\lim_{n\to\infty} \int_A f_n(x)dx = \int_A f(x)dx \tag{2}$$

Definition 2.1.5[15] Let X and Y be two sets, by product sets of X and Y we mean the set of all ordered pairs (x, y) where $x \in X$ and $y \in Y$ and denoted by $X \times Y$. If $A \subset X$ and $B \subset Y$ then $A \times B \subset X \times Y$, we call set $A \times B$ a rectangle in $X \times Y$.

Definition 2.1.6[15] If $E \subset X \times Y$ such that $x \in X$ $y \in Y$ we define

$$E_x = \{y : (x, y) \in E\}, E^y = \{x : (x, y) \in E\} \tag{3}$$

and $E_x \subset Y$; $E^y \subset X$. Then all E_x the x-section and E^y the y-section.

Definition 2.1.7[15] Let (X, S, μ) and (Y, T, λ) be measure spaces and if $Q \in S \times T$ we define

$$(\mu \times \lambda)(Q) = \int_X \lambda(Q_x)d\mu(x) = \int_Y \mu(Q^y)d\lambda(y) \tag{4}$$

$(\mu \times \lambda)$ is called the product of the measures μ and λ.

Definition 2.1.8[2] Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \lambda)$ be two probability measure spaces, by a weight function defined on $(X \times Y)$ we imply a product-measurable mapping $w : X \times Y \to [0, \infty)$ such that

$$\int_X w(x, y)d\mu = 1$$
$$\int_Y w(x, y)d\lambda = 1 \tag{5}$$

3   The Fubinni's theorem

Let f be a measurable function.

Theorem 3.1 If $E \in S \times T$, then $E_x \in T$ and $E^y \in S$ for every $x \in X$ and $y \in Y$. This theorem shows that E_x and E^y are measurable sets.

Theorem 3.2 Let (X, S, μ) and (Y, T, λ) be σ-measure spaces. Supppose $Q \in S \times T$ and if

$$\varphi(x) = \lambda(Q_x), \psi(y) = \mu(Q^y) \tag{6}$$

for every $x \in X$ and $y \in Y$, then φ is $S - measurable, \psi$ is $T - measurable$ and

$$\int_X \varphi d\mu = \int_Y \psi d\lambda \tag{7}$$

where μ and λ are positive measures on S and T respectively, $X = \cup_n X_n$ such that $\mu(X_n) < \infty$ and $Y = \cup_m Y_m$ such that $\lambda(Y_m) < \infty$. since by theorem 3.1

$$\lambda(Q_x) = \int_Y \chi_Q(x,y)d\lambda(y) \quad (x \in X) \tag{8}$$

$$\mu(Q^y) = \int_X \chi_Q(x,y)d\mu(x) \quad (y \in Y)$$

with (6);(7) can be expressed as

$$\int_X d\mu(x) \int_Y \chi_Q(x,y)d\lambda(y) = \int_Y d\lambda(y) \int_X \chi_Q(x,y)d\mu(x)$$

which is called an iterated integral.

Theorem 3.3 The Fubini's theorem[15]: Let (X, S, μ) and (Y, T, λ) be a $\sigma - finite$ measure spaces and let f be a $(S \times T) - measurable$ function on $X \times Y$

(a) If $0 \leq f \leq \infty$ and if

$$\varphi(x) = \int_Y f_x d\lambda, \psi(y) = \int_X f^y d\mu \quad (x \in X, y \in Y) \tag{9}$$

then φ is $S - measurable$, ψ is $T - measurable$, and

$$\int_X \varphi d\mu = \int_{X \times Y} f d(\mu \times \lambda) = \int_Y \psi d\lambda \tag{10}$$

(b) If f is complex and if

$$\varphi^*(x) = \int_Y |f|_x d\lambda \quad and \quad \int_X \varphi^* d\mu \leq \infty \tag{11}$$

then $f \in L'(\mu \times \lambda)$

(c) If $f \in L'(\mu \times \lambda)$, then $f_x \in L'(\lambda)$ for all $x \in X$, $f^y \in L'(\mu)$ for almost all $y \in Y$, the functions φ and ψ defined by (7) a.e, are $L'(\mu)$ and $L'(\lambda)$, respectively and (10) holds.

Proof:

lemma 1: Let $\{f_n(x)\}$ be a sequence of non-negative monotonic increasing functions on a set E and sequence that the sequence converges to $f(x)$ then

$$\lim_{n \to \infty} \int_E f_n(x)dx = \int_E f(x)dx$$

For (a) since f_x and f^y are $T-$ and $S - measurable$ the definition in equation (10) holds

$$\varphi(x) = \int_Y f_x d\lambda, \quad \psi(y) = \int_X f^y d\mu \quad x \in X, y \in Y$$

Suppose $Q \in S \times T$ and $f = \chi_Q$ by definition 2.1.7, equation (10) is exactly (7). Hence (a) holds $\forall$ non-negative simple $(S \times T)$-measurable function s. There is a sequence of such function s_n such that $0 \leq s_1 \leq s_2 \leq \ldots\ldots$ and $s_n(x,y) \to f(x,y)$ at every of $(x,y) \in X \times Y$. But

$$\int_X \varphi(x)d\mu = \int_{X \times Y} f d(\mu \times \lambda)$$

$$\Rightarrow \quad \int_X \varphi_n(x)d\mu = \int_{X \times Y} s_n d(\mu \times \lambda) \quad (n = 1, 2, \ldots\ldots) \tag{12}$$

by monotone convergence theorem [9] $\varphi_n(x) \to \varphi(x)$ for every $x \in X$ as $x \to \infty$. Applying this theorem on equation (12) we have

$$\lim_{n\to\infty} \int_X \varphi_n d\mu = \lim_{n\to\infty} \int_{X\times Y} s_n d(\mu \times \lambda)$$

$$\Rightarrow \int_X \varphi d\mu = \int_{X\times Y} f(x,y) d(\mu \times \lambda)$$

hence the 1st half of the inequality and associating ψ with s_n as

$$\int_Y \psi d\lambda = \int_{X\times Y} f d(\mu \times \lambda)$$

we have

$$\int_Y \psi_n d\lambda = \int_{X\times Y} s_n d(\mu \times \lambda)$$

by monotone convergence theorem we have

$$\int_Y \psi d\lambda = \int_{X\times Y} f(x,y) d(\mu \times \lambda)$$

hence (a) is proved and for (b) by (a) let $f = |f|$ then

$$\varphi(x) = \int_Y |f|_x d\lambda \qquad \psi(y) = \int_X |f|^y d\mu \qquad (13)$$

since f is measurable so do $|f|$ [18] hence

$$\varphi^*(x) = \int_Y |f|_x d\lambda \quad and \int_X \varphi^* d\mu < \infty$$

$f \in L'(\mu \times \lambda)$

For (c) we prove for real $L'(\mu \times \lambda)$ then the complex case follows. If f is real by (a) we have f^+ and f^-.let φ_1 and φ_2 correspond to f^+ and f^- as

$$\varphi(x) = \int_X f_x d\lambda$$

$$\Rightarrow \varphi_1(x) = \int_X f_x^+ d\lambda \qquad \varphi_2(x) = \int_X f_x^- d\lambda$$

since $f \in L'(\mu \times \lambda)$ and $f^+ \le |f|$ [18] and (a) holds for f^+ then $\varphi_1 \in L'(\mu)$ also $f^- \le |f|$ and $\varphi_2 \in L'(\mu)$ since

$$f_x = (f^+)_x - (f^-)_x \qquad (14)$$

$f_x(x) \in L'(\mu) \forall x$ for which $\varphi_1(x) < \infty$ and $\varphi_2(x) < \infty$ since $\varphi_1, \varphi_2 \in L'(\mu)$ which occurs for almost all x, and at such x, $\varphi(x) = \varphi_1(x) - \varphi_2(x)$ hence $\varphi \in L'(\mu)$

$$\varphi_1(x) = \int_Y f^+ d\lambda, \qquad \varphi_2(x) = \int_Y f^- d\lambda \qquad (x \in X)$$

$$\begin{aligned} \varphi_1(x) - \varphi_2(x) &= \int_Y f^+ d\lambda - \int_Y f^- d\lambda \\ \varphi(x) &= \int_Y (f^+ - f^-) d\lambda \\ &= \int_Y f d\lambda \end{aligned}$$

hence φ as defined in (7) a.e are in $L'(\mu)$, also for f real

$$\psi_1(y) = \int_X (f^+)^y d\mu \qquad \psi_2(y) = \int_X (f^-)^y d\mu$$

since $f \in L'(\mu \times \lambda)$ and $f^+ \leq |f|$ and $f^- \leq |f|$ then (a) holds for f^+ and f^- and $\psi_1 \in L'(\lambda)$, $\psi_2 \in L'(\lambda)$ since

$$f^y = (f^+)^y - (f^-)^y \tag{15}$$

we have $f^y \in L'(\lambda)$ $\forall y$ for which $\psi_1 < \infty$ and $\psi_2 < \infty$. since $\psi_1, \psi_2 \in L'(\lambda)$ which happens for almost all y and at such y, $\psi(y) = \psi_1(y) - \psi_2(y)$. Hence $\psi \in L'(\lambda)$. Now we have

$$\psi_1(y) = \int_Y (f^+)^y d\mu, \quad \psi_2(y) = \int_Y (f^-)^y d\mu \quad y \in Y$$

$$
\begin{aligned}
\psi_1(y) - \psi_2(y) &= \int_Y (f^+)^y d\mu - \int_Y (f^-)^y d\mu \\
\psi(y) &= \int_Y [(f^+)^y - (f^-)^y] d\mu \quad by \quad (15) \\
&= \int_Y f^y d\mu
\end{aligned}
$$

Fubini's theorem asserts that if f is $(\mathcal{S} \times \mathcal{T})$-measurable and if the iterated integral is finite then the two iterated integrals of f are finite and equal.

4   Application

In this section, we will consider some applications as shown in the following examples

Example 4.1 We prove the Marcinkiewiez's theorem. Let F be a closed subset of a bounded open interval (a,b) and let $\delta(x) = \delta(x; F)$ be the corresponding distance function then given $\lambda > 0$, the interval

$$M_\lambda = M_\lambda(x; F) = \int_a^b \frac{\delta^\lambda(y)}{|x - y|^{1+\lambda}} dy$$

is finite a.e in F. Moreover, $M_\lambda \in L(F)$ and

$$\int_F M_\lambda dx \leq 2\lambda^{-1}|G|$$

where $G = (a, b) - F$.
To prove this we note the following:
(i) $\delta(x) = \delta(x; F) = inf\{|x - y| : y \in F\}$
(ii) $\delta(x) = o$ iff $x \in F$
sincce $\delta=0$ in F, integration in the integral defining M_λ can be restricted to the set $G = (a, b) - F$ without changing M_λ since the complement of F is an open interval which is a union of $\cup_k(a_k, b_k)$ of disjoint open intervals, at most two of these interval can be infinite. Thus,

$$
\begin{aligned}
\int_F M_\lambda(x) dx &= \int_F \int_a^b \frac{\delta^\lambda(y)}{|x - y|^{1+\lambda}} dy dx \\
&= \int_G \delta^\lambda(y) \int_F \frac{dx}{|x - y|^{1+\lambda}} dy \quad by \;\; Fubini's \;\; theorem
\end{aligned}
$$

$$\tag{16}$$

Suppose, we fix $y \in G$ and for any $y \in F$ we have $|x - y| \geq \delta(y) > 0$, Thus,

$$
\begin{aligned}
&\leq \int_G \delta^\lambda(y) \left(\int_{|x-y| \geq \delta(y)} \frac{dx}{|x - y|^{1+\lambda}} \right) dy \\
&= \int_G \delta^\lambda(y) \left(2 \int_{\delta(y)}^\infty \frac{dt}{t^{1+\lambda}} \right) dy \\
&= 2 \int_G \delta^\lambda(y) \left(\left[\frac{-\lambda^{-1}}{t^\lambda} \right]_{\delta(y)}^\infty \right) dy
\end{aligned}
$$

$$= 2 \int_G \delta^\lambda(y) \left(\frac{\lambda^{-1}}{\delta^\lambda(y)} \right) dy$$

$$= 2 \int_G \lambda^{-1} dy$$

$$= 2\lambda^{-1}|G| \leq \infty$$

Hence $M_\lambda(x)$ is finite and $M_\lambda \in L(F)$.

Remark: The finiteness of $M_\lambda(x)$ simply means F is very Dense near x.

Example 4.2 We prove the Gaussian integral which is the basis for much of probability theory

We prove that

$$\int_{R^n} e^{-|x|^2} dx = \pi^{\frac{n}{2}}$$

Proof: Suppose n=1

$$\int_R e^{-|x|^2} dx = \pi^{\frac{1}{2}}$$

then we have

$$\left(\int_R e^{-|x|^2} dx \right)^2 = \pi$$

we show that the LHS equals the RHS

$$\left(\int_R e^{-|x|^2} dx \right)^2 = \left(\int_{-\infty}^{\infty} e^{-|x|^2} dx \right)^2$$

$$= \left(2 \int_0^{\infty} e^{-|x|^2} dx \right)^2$$

$$= 4 \int_0^{\infty} \int_0^{\infty} e^{-|x|^2} e^{-|y|^2} dxdy$$

$$= 4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dxdy$$

let $r^2 = x^2 + y^2$ $dxdy = rdrd\theta$, $x = r\cos\theta, y = r\sin\theta$ $x = 0$ to ∞ implies $r = 0$ to ∞ and $y = 0$ to ∞ implies $\theta = 0$ to $\frac{\pi}{2}$

$$4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dxdy = 4 \int_0^{\infty} \int_0^{\frac{\pi}{2}} e^{-r^2} rdrd\theta$$

$$= 4 \int_0^{\infty} \int_0^{\frac{\pi}{2}} \frac{e^{-t}}{2} dtd\theta$$

$$= 2 \int_0^{\infty} \int_0^{\frac{\pi}{2}} e^{-t} dtd\theta$$

$$= 2 \int_0^{\frac{\pi}{2}} \left[e^{-t} \right]_0^{\infty} d\theta$$

$$= 2 \int_0^{\frac{\pi}{2}} d\theta = \pi$$

hence $\left(\int_R e^{-|x|^2} dx \right)^2 = \pi$ suppose $n \geq 1$, $\left(\int_{R^n} e^{-|x|^2} dx \right)^2$ let $e^{-|x|^2} = e^{-x_1^2} e^{-x_2^2} e^{-x_3^2} \cdots e^{-x_n^2}$ then we have

$$\int_{R^n} e^{-|x|^2} dx = \int_{R^1} \int_{R^2} \cdots \int_{R^n} e^{-x_1^2} e^{-x_2^2} \cdots e^{-x_n^2} dx_1 dx_2 \cdots dx_n$$

$$\left(\int_{R^n} e^{-|x|^2} dx \right)^2 = \left(\int_{R_1} \int_{R_2} \cdots \int_{R_n} e^{-x_1^2} e^{-x_2^2} \cdots e^{-x_n^2} dx_1 dx_2 \cdots dx_n \right)^2$$

$$
\begin{aligned}
&= \int_{R_1}\int_{R_1}\int_{R_2}\int_{R_2}\cdots\int_{R_n}\int_{R_n} e^{-x_1^2}e^{-y_1^2}e^{-x_2^2}e^{-y_2^2}\cdots e^{-x_n^2}e^{-y_n^2}dx_1dy_1dx_2dy_2\cdots dx_ndy_n \\
&= \int_{R_1}\int_{R_1}\int_{R_2}\int_{R_2}\cdots\int_{R_n}\int_{R_n} e^{-(x_1^2+y_1^2)}e^{-(x_2^2+y_2^2)}\cdots e^{-(x_n^2+y_n^2)}dx_1dy_1dx_2dy_2\cdots dx_ndy_n \\
&= \int_{R_1}\int_{R_1}\cdots\int_{R_{n-1}}\int_{R_{n-1}}(\int_{R_n}\int_{R_n} e^{-(x_1^2+y_1^2)}e^{-(x_2^2+y_2^2)}\cdots e^{-(x_n^2+y_n^2)}dx_ndy_n)dx_{n-1}dy_{n-1}\cdots dx_1dy_1 \\
&= \int_{R_1}\int_{R_1}\cdots\int_{R_{n-1}}\int_{R_{n-1}}(\pi e^{-(x_1^2+y_1^2)}e^{-(x_2^2+y_2^2)}\cdots e^{-(x_{n-1}^2+y_{n-1}^2)})dx_{n-1}dy_{n-1}\cdots dx_1dy_1 \\
&= \int_{R_1}\int_{R_1}\cdots\int_{R_{n-2}}\int_{R_{n-2}}(\pi^2 e^{-(x_1^2+y_1^2)}e^{-(x_2^2+y_2^2)}\cdots e^{-(x_{n-2}^2+y_{n-2}^2)})dx_{n-2}dy_{n-2}\cdots dx_2dy_2dx_1dy_1 \\
&= \ldots\ldots \\
&= \int_{R_1}\int_{R_1}\int_{R_2}\int_{R_2} \pi^{n-2}e^{-(x_1^2+y_1^2)}e^{-(x_2^2+y_2^2)}dx_2dy_2dx_1dy_1 \\
&= \int_{R_1}\int_{R_1} \pi^{n-1}e^{-(x_1^2+y_1^2)}dx_1dy_1 \\
&= \pi^{n-1}(\pi)
\end{aligned}
$$

therefore

$$
\left(\int_{R^n} e^{-|x|^2}dx\right)^2 = \pi^n
$$
$$
\int_{R^n} e^{-|x|^2}dx = \pi^{\frac{n}{2}}
$$

Theorem 4.3 Let $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\lambda)$ be two probability spaces and $w: X\times Y \to [0,1)$ be a weight function on $X\times Y$. If I is an interval of the real line $f(x)\in L'(\mu)$, $f(x)\in I$ $\forall x\in X$ and φ is a real convex function on I then

$$
\int_Y \varphi \int_X f(x)w(x,y)d\mu d\lambda
$$

and

$$
\varphi\int_X f(x)d\mu \le \int_Y \varphi\int_X f(x)w(x,y)d\mu d\lambda \le \int_X (\varphi\cdot f)(x)d\mu \tag{17}
$$

Let $X=\{0,1\}$ with $\mathcal{A}=2^X$ and $\mu\{0\}=\mu\{1\}=\frac{1}{2}$, $Y=[0,1]$ with Lebesgue measure λ. If φ is a convex function on a closed interval I such that $I=[a,b]$ then

$$
\varphi\left(\frac{a+b}{2}\right) \le \frac{1}{(b-a)}\int_a^b \varphi(t)dt \le \frac{\varphi(a)+\varphi(b)}{2}
$$

Proof:
$X=\{0,1\}$ $\mu\{0\}=\mu\{1\}=\frac{1}{2}$ and $I=[a,b]$ and by (10)

$$
\varphi\int_X f(x)d\mu \le \int_Y \varphi\int_X f(x)w(x,y)d\mu d\lambda \le \int_X (\varphi\cdot f)(x)d\mu
$$

taking $w: X\times Y \to [0,\infty)$ as a weight function defined $w(x,y)=|2(x-1)+2y|$ then we have $w(0,y)=2(1-y)$ and $w(1,y)=2y$ for $y\in[0,1]$, by theorem 4.2 $f(x)\in I$, let $f(0)=a$ and $f(1)=b$ since $I=[a,b]$ then

$$
\begin{aligned}
\varphi\int_X f d\mu &= \varphi\left(\int_{x=0} f(x)dx + \int_{x=1} f(x)dx\right) \\
&= \varphi\left(\frac{f(0)}{2} + \frac{f(1)}{2}\right) \\
&= \varphi\left(\frac{a+b}{2}\right) \tag{18}
\end{aligned}
$$

$$
\begin{aligned}
\int_Y \varphi \int_X f(x)w(x,y)d\mu d\lambda &= \int_Y \varphi \left(\int_{x=0} f(x)w(x,y)dx + \int_{x=1} f(x)w(x,y)dx \right) d\lambda \\
&= \int_Y \varphi \left(\int f(x)w(0,y)dx + \int f(x)w(1,y)dx \right) dy \\
&= \int_Y \varphi \left(a\frac{2(1-y)}{2} + b\frac{2y}{2} \right) dy \\
&= \int_Y \varphi \left(a(1-y) + by \right) dy
\end{aligned}
$$

let $t = a(1-y) + by$ then $\frac{dt}{dy} = b - a$ and $dy = \frac{dt}{b-a}$

$$
\begin{aligned}
&= \int_a^b \frac{\varphi(t)}{b-a}dt \\
&= \frac{1}{b-a} \int_a^b \varphi(t)dt
\end{aligned}
\tag{19}
$$

and

$$
\begin{aligned}
\int_X (\varphi \cdot f)d\mu &= \int_{x=0} (\varphi f(x)) + \int_{x=1} (\varphi \cdot f(x))dx \\
&= \frac{\varphi(a)}{2} + \frac{\varphi(b)}{2} \\
&= \frac{\varphi(a) + \varphi(b)}{2}
\end{aligned}
\tag{20}
$$

hence the result follows from (18), (19) and (20).

Remark: If φ is a real convex function on a closed interval $[a,b]$, then we have the Hermite-Hadamard inequality.

5 Conclusion

In this work we discussed the concepts of measure space and integration of functions of two variables, where definitions, some theorems and proofs were treated. We considered integrating functions of two variables in measure space using Fubini's theorem with some applications. Futhermore, we were able to use the generalized Jensen's inequality to prove the Hermite-Hadamard inequality. In this study, we conclude that the Fubini's theorem is of great importance in the integration of function of two variables in measure space.

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