International Journal of Advanced Mathematical Sciences, 5 (1) (2017) 36-42

International Journal of Advanced Mathematical Sciences

Website: www.sciencepubco.com/index.php/IJAMS
doi: 10.14419/ijams.v5i1.7571
Research paper

The Kumaraswamy compound Rayleigh distribution : properties and estimation

Hesham M. Reyad [1] *, Soha A. Othman [2], Adil M Younis [1], and Ahmed M. Hashish [1]

[1] *College of Business and Economics, Qassim University, Kingdom of Saudi Arabia*
[2] *Institute of Statistical Studies and Research, Cairo University, Egypt*
**Corresponding author E-mail: hesham_reyad@yahoo.com*

Abstract

We introduce a new four parameter continuous model, called the Kumaraswamy compound Rayleigh (KwCR) distribution that extends the compound Rayleigh distribution. We study some mathematical properties of this distribution such as; mean, variance, coefficient of variation, quantile function, median, ordinary and incomplete moments, skewness, kurtosis, moment and probability generating functions, reliability analysis, Lorenz, Bonferroni and Zenga curves, Rényi of entropy, order statistics and record statistics. We consider the methods of moments and maximum likelihood for estimating the model parameters.

Keywords: *Kumaraswamy Distribution; Compound Rayleigh Distribution; Order Statistics; Record Statistics; Moments Estimation; Maximum Likelihood Estimation.*

1. Introduction

The compound Rayleigh distribution plays a vital role for modelling and analysis in different areas of statistics including reliability study and life time data espically in biological and medical science. In the last couple of decades, statisticans have much attention to study this distribution. Abushal [1] applied the maximum likelihood and Bayes approches to estimate parameters, reliability and hazard functions of compound Rayleigh distribution based on progressive first-failure censord data. Shajaee et al. [2] obtained the empirical Bayes estimates for parameter and reliability function assciated to compound Rayleigh distribution under record data. Barot and Patal [3] compared the maximum likelihood and Bayes estimates of the reliability parameters corresponding to compound Rayleigh distribution under progressive type-ii censored data. Abd-Elmougod and Mahmoud [4] studied the compound Rayleigh distribution with constant partially accelerated life tests under an adaptive type-ii propgressive hybrid censored data.
The random variable X with compound Rayleigh distribution (CR) distribution has the cumulative distribution function (cdf) given by

$$G(x;\alpha,\beta)=1-\beta^{\alpha}(\beta+x^{2})^{-\alpha},\quad x>0,\ \alpha,\beta>0 \tag{1}$$

where β and α are the scale and shape parameters respectively. The probability density function (pdf) corresponding to Eq. (1) takes the form

$$g(x;\alpha,\beta)=2\alpha\beta^{\alpha}x(\beta+x^{2})^{-(\alpha+1)},\quad x>0,\alpha,\beta>0 \tag{2}$$

This study aims to propose a new distribution so called, Kumaraswamy compound Rayleigh (KwCR) distribution and investigate some of its statistical properties. The model parameters of the new model are estimated by using the moments and maximum likelihood approaches.
The rest of this paper is as follows. In Section 2, we define the Kumaraswamy compound Rayleigh (KwCR) distribution and obtain some corresponding reliability functions . The limit of the KwCR distribution is studied in Section 3. The expansion of the KwCR distribution is discussed in Section 4. In Section 5, some mathematical properties of the new distribution are discussed. The moments and maximum likelihood estimates are derived in Section 6. Some concluding remarks have been given in the last Section.

2. The KwCR distribution

In this section, we present the Kumaraswamy compound Rayleigh distribution and its sub-models. Some reliability functions corresponding to this distribution are also discussed.
Let $G(x)$ be the cdf of any random variable X . Cordeiro and Castro [5] introduced a new procedure for building a new distribution from $G(x)$ known as the Kw generalized (Kw-G) distribution given by

$$F(x)=1-\left\{1-[G(x)]^{a}\right\}^{b} \tag{3}$$

Where $a>0$ andare two additional shape parameters for $b>0$ the F distribution. The corresponding pdf for Eq.(3) is given by

$$f(x)=abg(x)[G(x)]^{a-1}\left\{1-[G(x)]^{a}\right\}^{b-1} \tag{4}$$

Where $g(x)=\partial G(x)/\partial x$ is the baseline density function. Replacing Eq. (1) in Eq. (3), we obtain a new distribution, called the Kumaraswamy compound Rayleigh (KwCR) distribution with cdf given from

$$F(x;\alpha,\beta,a,b)=1-\left\{1-\left[1-\beta^{\alpha}\left(\beta+x^{2}\right)^{-\alpha}\right]^{a}\right\}^{b} \qquad (5)$$

The pdf corresponding to Eq. (5) is given by

$$f(x;\alpha,\beta,a,b)=2ab\alpha\,\beta^{\alpha}x\left(\beta+x^{2}\right)^{-(\alpha+1)}\left[1-\beta^{\alpha}\left(\beta+x^{2}\right)^{-\alpha}\right]^{a-1}$$
$$\times\left\{1-\left[1-\beta^{\alpha}\left(\beta+x^{2}\right)^{-\alpha}\right]^{a}\right\}^{b-1} \qquad (6)$$

For $x>0,\ \alpha>0,\ \beta>0,\ a>0$ and $b>0$.

For the reliability analysis, the reliability function, hazard $R(x)$ function $h(x)$, inverse hazard function $h_r(x)$ and cumulative hazard function $H(x)$ for the KwCR distribution are given respectively from

$$R(x)=1-F(x)=\left\{1-\left[1-\beta^{\alpha}\left(\beta+x^{2}\right)^{-\alpha}\right]^{a}\right\}^{b}, \qquad (7)$$

$$h(x)=\frac{f(x)}{R(x)}=2ab\alpha\beta^{\alpha}x\left(\beta+x^{2}\right)^{-(\alpha+1)}\left[1-\beta^{\alpha}\left(\beta+x^{2}\right)^{-\alpha}\right]^{a-1}$$
$$\times\left\{1-\left[1-\beta^{\alpha}\left(\beta+x^{2}\right)^{-\alpha}\right]^{a}\right\}^{-(2b-1)} \qquad (8)$$

$$h_r(x)=\frac{f(x)}{F(x)}=2ab\alpha\beta^{\alpha}x\left(\beta+x^{2}\right)^{-(\alpha+1)}\times\left[1-\beta^{\alpha}\left(\beta+x^{2}\right)^{-\alpha}\right]^{a-1}$$
$$\times\left\{1-\left[1-\beta^{\alpha}\left(\beta+x^{2}\right)^{-\alpha}\right]^{a}\right\}^{-(b-1)}$$
$$\times\left(1-\left\{1-\left[1-\beta^{\alpha}\left(\beta+x^{2}\right)^{-\alpha}\right]^{a}\right\}^{b}\right)^{-1} \qquad (9)$$

And

$$H(x)=-\ln R(x)$$
$$=-\ln\left\{1-\left[1-\beta^{\alpha}\left(\beta+x^{2}\right)^{-\alpha}\right]^{a}\right\}^{b} \qquad (10)$$

2.1. Sub-models

The following distributions can be obtained as special cases of the KwCR distribution:

1) If $a=b=1$, Eq. (6) reduces to the compound Rayleigh distribution, CR(α,β).
2) When $a=b=\alpha=1$, Eq. (6) represents the compound Rayleigh distribution, CR($1,\beta$).
3) Setting $a=b=\beta=1$, the KwCR distribution is reduced to the Burr-XII distribution, BXII($\alpha,2$).
4) Suppose $a=b=\alpha=\beta=1$, then we obtain the Burr-XII distribution, BXII($1,2$).

3. The limit of the KwCR distribution

The limit of the Kumaraswamy compound Rayleigh distribution when $x\to 0$ is 0 and when $x\to\infty$ is 0. We can show this by taking the limit of Eq. (6) as follows:

$$\lim_{x\to 0} f(x)=\left(\lim_{x\to 0}2ab\alpha\beta^{\alpha}\right)\left(\lim_{x\to 0}x\right)\left(\lim_{x\to 0}(\beta+x^{2})^{-(\alpha+1)}\right)$$
$$\times\left(\lim_{x\to 0}\left[1-\beta^{\alpha}(\beta+x^{2})^{-\alpha}\right]^{a-1}\right)\left(\lim_{x\to 0}\left\{1-\left[1-\beta^{\alpha}(\beta+x^{2})^{-\alpha}\right]^{a}\right\}^{b-1}\right)=0$$

Because $\lim_{x\to 0}x=0$ and $\lim_{x\to 0}\left[1-\beta^{\alpha}(\beta+x^{2})^{-\alpha}\right]^{a-1}=0$.

Similarly, replacing $x\to\infty$ in above, we obtain

$$\lim_{x\to\infty} f(x)=0$$

Because $\lim_{x\to\infty}(\beta+x^{2})^{-(\alpha+1)}=0$ and $\left(\lim_{x\to\infty}\left\{1-\left[1-\beta^{\alpha}(\beta+x^{2})^{-\alpha}\right]^{a}\right\}^{b-1}\right)=0$

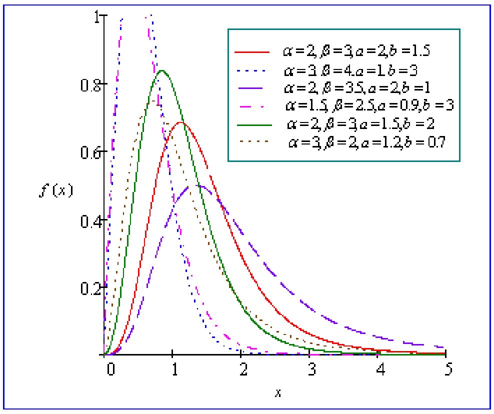

Fig. 1: The pdf of The KwCR distribution for different values of the parameters.

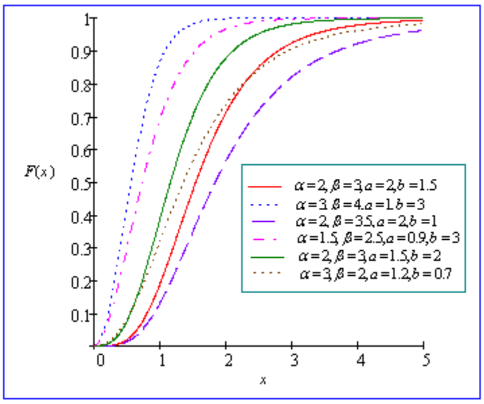

Fig. 1: The cdf of The KwCR distribution for different values of the parameters.

4. Expansion for the KwCR distribution

We can expand the cdf and pdf corresponding to KwCR distribution in terms of an infinite (or finite) weighted sums of cdf 's and pdf 's of random variables having CR distributions respectively. For c is real non-integer, then we have the series representation

$$(1-v)^{c}=\sum_{j=0}^{\infty}(-1)^{j}\binom{c}{j}v^{j},|v|<1 \qquad (11)$$

Therefore, the cdf of KwCR distribution can be expressed as follows:

$$F(x;\alpha,\beta,a,b)=1-\sum_{j,i=0}^{\infty}z_{j,i}\,\bar{G}(x;\alpha i,\beta) \qquad (12)$$

Where

$$z_{j,i}=\frac{(-1)^{j+i}\Gamma(b+1)\Gamma(aj+1)}{j!\,i!\,\Gamma(b-j+1)\Gamma(aj-i+1)}$$

And $\bar{G}(x;\alpha i,\beta)=1-G(x;\alpha i,\beta)$, $G(x;\alpha i,\beta)$ deotes the cdf of CR distribution with parameters αi and β. If a and b are integers,

then the summations in Eq. (12) are stoped at b and aj respectively.
Likewise, we can rewrite; the pdf of the KwCR distribution in Eq. (6) as follows:

$$f(x;\alpha,\beta,a,b) = \sum_{j,i,h=0}^{\infty} w_{j,i,h}\, H\left(x;\alpha(j+h+1),\beta\right) \tag{13}$$

Where

$$w_{j,i,h} = \frac{(-1)^{j+i+h}\Gamma(a)\Gamma(b)\Gamma(ai+1)ab}{(j+h+1)j!i!h!\Gamma(a-j)\Gamma(b-i)\Gamma(ai-h+1)}$$

And $H\left(x;\alpha(j+h+1),\beta\right)$ deotes the pdf of CR distribution with parameters $\alpha(j+h+1)$ and β. Also, if a and b are integers, then the summations in Eq. (13) are stoped at a,b and ai respectively.

5. Mathematical properties

In this section, we discuss some mathematical properties of the proposed distribution such as; quantile function, ordinary and incomplete moments, coefficient of variation, skewness, kurtosis, median, moment and probability generating functions, Lorenz, Bonferroni and Zenga curves, Rényi of entropy, order statistics and record statistics.

5.1. Quantile function and simulation

The quantile function of the KwCR distribution, say $Q(\mu)=F^{-1}(\mu)$ random variable can be obtained by inverting Eq. (5) as

$$Q(\mu)=\left\{\beta\left[\left\{1-\left[1-(1-u)^{1/b}\right]^{1/a}\right\}^{-1/\alpha}\right]-1\right\}^{1/2} \tag{14}$$

Simulating the KwCR random variable is straightforward. If U is a uniform variate on the interval (0,1), then the random variable $X=Q(U)$ follows Eq. (6). Based on Eq. (14), we can deduct that the median m of X is $m=Q(1/2)$.

5.2. Ordinary moments

Suppose X is a random variable distributed according to KwCR distribution, then the ordinary moments, , say μ_r' is given by

$$\mu_r' = E(X^r) = 2ab\alpha\beta^{\alpha}\int_0^{\infty} x^{r+1}\left(\beta+x^2\right)^{-(\alpha+1)}$$

$$\times\left[1-\beta^{\alpha}\left(\beta+x^2\right)^{-\alpha}\right]^{a-1}\left\{1-\left[1-\beta^{\alpha}\left(\beta+x^2\right)^{-\alpha}\right]^{a}\right\}^{b-1}dx$$

$$=2\alpha\sum_{j,i,h=0}^{\infty} w_{j,i,h}(j+h+1)\beta^{\alpha(j+h+1)}\int_0^{\infty} x^{r+1}\left(\beta+x^2\right)^{-[\alpha(j+h+1)+1]}dx$$

$$=\frac{2\alpha}{\beta}\sum_{j,i,h=0}^{\infty} w_{j,i,h}(j+h+1)\int_0^{\infty} x^{r+1}\left(1+\frac{x^2}{\beta}\right)^{-[\alpha(j+h+1)+1]}dx$$

Using $z=\left(1+x^2/\beta\right)^{-1}$, then we obtain

$$\mu_r' = \alpha\beta^{r/2}\sum_{j,i,h=0}^{\infty} w_{j,i,h}(j+h+1)\int_0^1 z^{\alpha(j+h+1)-(r/2)-1}\left(1-z\right)^{r/2}dz$$

$$=\alpha\beta^{r/2}\sum_{j,i,h=0}^{\infty} w_{j,i,h}(j+h+1)\beta\left[\alpha(j+h+1)-r/2,\, r/2+1\right] \tag{15}$$

Where $\beta(.,.)$ is the beta function. Substituting $r=1,2$ in Eq. (15), then we get the mean and variance respectively as follows:

$$\mu_1' = \alpha\beta^{1/2}\sum_{j,i,h=0}^{\infty} w_{j,i,h}(j+h+1)\beta\left[\alpha(j+h+1)-1/2,3/2\right] \tag{16}$$

And

$$v(x) = \mu_2' - \mu_1'^2$$

$$=\alpha\beta\sum_{j,i,h=0}^{\infty} w_{j,i,h}(j+h+1)\beta\left[\alpha(j+h+1)-1,2\right]$$

$$-\alpha^2\beta\left[\sum_{j,i,h=0}^{\infty} w_{j,i,h}(j+h+1)\beta\left[\alpha(j+h+1)-1/2,3/2\right]\right]^2$$

(17)

5.3. Coefficients of variation, skewness and kurtosis

The coefficients of variation, skewness and kurtosis of the KwCR distribution are given respectively as follows:

$$C.V = \frac{\sigma}{\mu}$$

$$=\frac{\sqrt{\left\{\begin{array}{l}\sum_{j,i,h=0}^{\infty} w_{j,i,h}(j+h+1)\beta\left[\alpha(j+h+1)-1,2\right]\\ -\alpha\left[\sum_{j,i,h=0}^{\infty} w_{j,i,h}(j+h+1)\beta\left[\alpha(j+h+1)-1/2,3/2\right]\right]^2\end{array}\right\}}}{\alpha^{1/2}\sum_{j,i,h=0}^{\infty} w_{j,i,h}(j+h+1)\beta\left[\alpha(j+h+1)-1/2,3/2\right]}, \tag{18}$$

$$\varpi_1 = \frac{\mu_3'}{(\mu_2')^{3/2}}$$

$$=\frac{\beta^{1/2}\sum_{j,i,h=0}^{\infty} w_{j,i,h}(j+h+1)\beta\left[\alpha(j+h+1)-3/2,5/2\right]}{\alpha^{1/2}\left\{\sum_{j,i,h=0}^{\infty} w_{j,i,h}(j+h+1)\beta\left[\alpha(j+h+1)-1/2,3/2\right]\right\}^{3/2}} \tag{19}$$

And

$$\varpi_2 = \frac{\mu_4'}{(\mu_2')^2} = \frac{\beta\sum_{j,i,h=0}^{\infty} w_{j,i,h}(j+h+1)\beta\left[\alpha(j+h+1)-2,3\right]}{\alpha\left\{\sum_{j,i,h=0}^{\infty} w_{j,i,h}(j+h+1)\beta\left[\alpha(j+h+1)-1/2,3/2\right]\right\}^2} \tag{20}$$

5.4. Incomplete moments

Suppose X is a random variable has the KwCR distribution, then the r^{th} incomplete moments denoted as $m_r(z)$ can be obtained as follows:

$$m_r(z) = \int_0^z x^r f(x)dx$$

$$=\frac{2\alpha}{\beta}\sum_{j,i,h=0}^{\infty} w_{j,i,h}(j+h+1)\int_0^z x^{r+1}\left(1+\frac{x^2}{\beta}\right)^{-[\alpha(j+h+1)+1]}dx$$

Insert $w=\left(1+x^2/\beta\right)^{-1}$, then we get

$$m_r(z) = \alpha\beta^{r/2}\sum_{j,i,h=0}^{\infty} w_{j,i,h}(j+h+1)\int_{\left(1+z^2/\beta\right)^{-1}}^{1} w^{\alpha(j+h+1)-(r/2)-1}\left(1-w\right)^{r/2}dw$$

Based on the binomial expansion to the last factor, we get

$$m_r(z) = \alpha\beta^{r/2}\sum_{j,i,h=0}^{\infty}\sum_{\iota=0}^{r/2}\binom{r/2}{\iota}(-1)^{\iota}w_{j,i,h}(j+h+1)\int_{\left(1+z^2/\beta\right)^{-1}}^{1} w^{\alpha(j+h+1)+\iota-(r/2)-1}dw$$

$$= \alpha\beta^{r/2} \sum_{j,i,h=0}^{\infty} \sum_{\text{l}=0}^{r/2} \binom{r/2}{\text{l}} \frac{(-1)^{\text{l}} w_{j,i,h}(j+h+1)}{\alpha(j+h+1)+\text{l}-(r/2)} \times \left\{ 1-\left[1+\frac{z^2}{\beta}\right]^{r/2-[\alpha(j+h+1)+\text{l}]} \right\} \quad (21)$$

5.5. Generating functions

The moment generating function, say $M_x(t)$ of the KwCR distribution can be obtained as follows:

$$M_x(t) = E(e^{tx}) = 2ab\alpha\beta^{\alpha} \int_0^{\infty} e^{tx} x \left(\beta+x^2\right)^{-(\alpha+1)}$$

$$\times \left[1-\beta^{\alpha}\left(\beta+x^2\right)^{-\alpha}\right]^{a-1} \left\{1-\left[1-\beta^{\alpha}\left(\beta+x^2\right)^{-\alpha}\right]^{a}\right\}^{b-1} dx$$

$$= \frac{2\alpha}{\beta} \sum_{j,i,h=0}^{\infty} w_{j,i,h}(j+h+1) \int_0^{\infty} e^{tx} x \left(1+\frac{x^2}{\beta}\right)^{-[\alpha(j+h+1)+1]} dx$$

Using $e^{tx} = \sum_{\text{l}=0}^{\infty} \frac{t^{\text{l}} x^{\text{l}}}{\text{l}!}$, then we get

$$M_x(t) = \alpha \sum_{j,i,h,\text{l}=0}^{\infty} \frac{w_{j,i,h}(j+h+1) t^{\text{l}} \beta^{\text{l}/2}}{\text{l}!} \int_0^1 z^{\alpha(j+h+1)-(\text{l}/2)-1} (1-z)^{\text{l}/2} dz$$

$$= \alpha \sum_{j,i,h,\text{l}=0}^{\infty} \frac{w_{j,i,h}(j+h+1) t^{\text{l}} \beta^{\text{l}/2}}{\text{l}!} \beta\left[\alpha(j+h+1)-\text{l}/2, \text{l}/2+1\right] \quad (22)$$

Similarly, the probability generating function, denoted as $M_{[x]}(t)$ of the KwCR distribution can be derived as below

$$M_{[x]}(t) = E(t^x) = 2ab\alpha\beta^{\alpha} \int_0^{\infty} t^x x \left(\beta+x^2\right)^{-(\alpha+1)}$$

$$\times \left[1-\beta^{\alpha}\left(\beta+x^2\right)^{-\alpha}\right]^{a-1} \left\{1-\left[1-\beta^{\alpha}\left(\beta+x^2\right)^{-\alpha}\right]^{a}\right\}^{b-1} dx$$

$$= \frac{2\alpha}{\beta} \sum_{j,i,h=0}^{\infty} w_{j,i,h}(j+h+1) \int_0^{\infty} t^x x \left(1+\frac{x^2}{\beta}\right)^{-[\alpha(j+h+1)+1]} dx$$

Using $t^x = \sum_{\text{l}=0}^{\infty} \frac{(\ln t)^{\text{l}} x^{\text{l}}}{\text{l}!}$, then we get

$$M_{[x]}(t) = \alpha \sum_{j,i,h,\text{l}=0}^{\infty} \frac{w_{j,i,h}(j+h+1)(\ln t)^{\text{l}} \beta^{\text{l}/2}}{\text{l}!} \int_0^1 z^{\alpha(j+h+1)-(\text{l}/2)-1} (1-z)^{\text{l}/2} dz$$

$$= \alpha \sum_{j,i,h,\text{l}=0}^{\infty} \frac{w_{j,i,h}(j+h+1)(\ln t)^{\text{l}} \beta^{\text{l}/2}}{\text{l}!} \beta\left[\alpha(j+h+1)-\text{l}/2, \text{l}/2+1\right]$$

(23)

5.6. Lorenz, Bonferroni and Zenga curves

The Lorenz, Bonferroni and Zenga curves have been used in different fields such as demography, insurance, reliability, medicine and economics. Oluyede and Rajasooriya [6] defined the Lorenz $L_F(x)$, Bonferroni $B\left(F(x)\right)$ and Zenga $A(x)$ curves respectively as follows:

$$L_F(x) = \frac{1}{E(x)} \int_0^x t f(t) dt,$$

$$B\left(F(x)\right) = \frac{1}{F(x)E(x)} \int_0^x t f(t) dt = \frac{L_F(x)}{F(x)}$$

And

$$A(x) = 1-\left[\frac{M^-(x)}{M^+(x)}\right]$$

Where $M^-(x) = \frac{1}{F(x)} \int_0^x t f(t) dt$

And $M^+(x) = \frac{1}{1-F(x)} \int_x^{\infty} t f(t) dt$

Therefore, these quantities for the KwCR distribution are obtained below

$$L_F(x) = \frac{\Upsilon_1}{\Upsilon_2}, \quad (24)$$

$$B\left(F(x)\right) = \frac{\Upsilon_1}{\Upsilon_2 \Upsilon_3} \quad (25)$$

And

$$A(x) = 1-\frac{\Upsilon_1 \Upsilon_4}{\Upsilon_3 \Upsilon_5} \quad (26)$$

Where

$$\Upsilon_1 = \sum_{j,i,h=0}^{\infty} \sum_{k=0}^{1/2} \binom{1/2}{k} \frac{(-1)^k w_{j,i,h}(j+h+1)}{\alpha(j+h+1)+k-(1/2)} \left\{1-\left[1+\frac{x^2}{\beta}\right]^{1/2-[\alpha(j+h+1)+k]}\right\},$$

$$\Upsilon_2 = \sum_{j,i,h=0}^{\infty} w_{j,i,h}(j+h+1)\beta\left[\alpha(j+h+1)-1/2, 3/2\right],$$

$$\Upsilon_3 = 1-\left\{1-\left[1-\beta^{\alpha}\left(\beta+x^2\right)^{-\alpha}\right]^{a}\right\}^{b},$$

$$\Upsilon_4 = \left\{1-\left[1-\beta^{\alpha}\left(\beta+x^2\right)^{-\alpha}\right]^{a}\right\}^{b},$$

$$\Upsilon_5 = \sum_{j,i,h=0}^{\infty} w_{j,i,h}(j+h+1)\beta_{(1+x^2/\beta)^{-1}}\left[\alpha(j+\text{l}+1)-1/2, 3/2\right]$$

And $\beta_y(a,b) = \int_0^y u^{a-1}(1-u)^{b-1} du$ is the incomplete beta function.

5.7. Rényi entropy

The entropy of a random variable X is a measure of randomness contained in a probability distribution. The Rényi entropy is defined as

$$I_R(\delta) = \frac{1}{1-\delta}\left[\log I(\delta)\right],$$

Where $I(\delta) = \int f^{\delta}(x) dx$, $\delta > 0$ and $\delta \neq 0$.

Using Eq. (6) yields

$$I(\delta) = 2^{\delta} a^{\delta} b^{\delta} \alpha^{\delta} \beta^{\delta\alpha} \int_0^{\infty} x^{\delta} \left(\beta+x^2\right)^{-\delta(\alpha+1)} \left[1-\beta^{\alpha}\left(\beta+x^2\right)^{-\alpha}\right]^{\delta(a-1)}$$

$$\times \left\{1-\left[1-\beta^{\alpha}\left(\beta+x^2\right)^{-\alpha}\right]^{a}\right\}^{\delta(b-1)} dx$$

$$= 2^{\delta-1} a^{\delta} b^{\delta} \alpha^{\delta} \beta^{(\delta-1)/2} \sum_{j,i,\text{l}=0}^{\infty} \binom{\delta(a-1)}{j} \binom{\delta(b-1)}{i} \binom{ai}{\text{l}} (-1)^{j+i+\text{l}}$$

$$\times \beta\left[\alpha(\delta+j+\text{l})+(\delta-1)/2, (\delta+1/2)\right] \quad (27)$$

Consequently, the Rényi entropy is given by using Eq. (27) in the last definition.

5.8. Order statistics

Order statistics play an important role in probability and statistics. Let $x_{1,n} \le x_{2:n}, \ldots \le x_{n:n}$ be the ordered sample from a continuous population with pdf $f(x)$ and cdf $F(x)$. The pdf of $X_{k:n}$, the k^{th} order statistics is from

$$f_{X_{k:n}}(x) = \frac{n!}{(k-1)!(n-k)!} f(x)\left[F(x)\right]^{k-1} \times \left[1-F(x)\right]^{n-k}$$

Then, the pdf the k^{th} order KwCR random variable $X_{k:n}$ can be obtained by using Eqs. (5) and (6) in above equation to be

$$f_{X_{k:n}}(x) = \frac{2n!ab\alpha\beta^{\alpha}}{(k-1)!(n-k)!}\Omega_1 x\left(\beta+x^2\right)^{-(\alpha+1)} \times \left[1-\beta^{\alpha}\left(\beta+x^2\right)^{-\alpha}\right]^{a-1} \quad (28)$$

Where

$$\Omega_1 = \sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j\left\{1-\left[1-\beta^{\alpha}\left(\beta+x^2\right)^{-\alpha}\right]^a\right\}^{b(n-k+j+1)-1}$$

Therefore, the pdf the 1^{th} order KwCR random variable $X_{1:n}$ is given by

$$f_{X_{1:n}}(x) = 2n\,ab\alpha\beta^{\alpha}\Omega_2 x\left(\beta+x^2\right)^{-(\alpha+1)} \times \left[1-\beta^{\alpha}\left(\beta+x^2\right)^{-\alpha}\right]^{a-1} \quad (29)$$

Where

$$\Omega_2 = \left\{1-\left[1-\beta^{\alpha}\left(\beta+x^2\right)^{-\alpha}\right]^a\right\}^{nb-1}$$

Also, the pdf the n^{th} order KwCR random variable $X_{n:n}$ is given by

$$f_{X_{n:n}}(x) = 2nab\alpha\beta^{\alpha}\Omega_3 x\left(\beta+x^2\right)^{-(\alpha+1)}\left[1-\beta^{\alpha}\left(\beta+x^2\right)^{-\alpha}\right]^{a-1} \quad (30)$$

Where

$$\Omega_3 = \left\{1-\left[1-\beta^{\alpha}\left(\beta+x^2\right)^{-\alpha}\right]^a\right\}^{b-1}\left(1-\left\{1-\left[1-\beta^{\alpha}\left(\beta+x^2\right)^{-\alpha}\right]^a\right\}^{b-1}\right)^{n-1}$$

Moreover, the joint distribution of two order statistics $X_{k:n} \le X_{s:n}$ is given as

$$f_{X_{k:s:n}}(x_1,x_2) = \frac{n!}{(k-1)!(s-k-1)!(n-s)!} f(x_1)$$

$$\times f(x_2)\left[F(x_1)\right]^{k-1}\left[F(x_2)-F(x_1)\right]^{s-k-1}\left[1-F(x_2)\right]^{n-s}$$

Using Eqs. (5) and (6) in the last equation, we obtain

$$f_{X_{k:s:n}}(x_1,x_2) = \frac{4n!a^2b^2\alpha^2\beta^{2\alpha}}{(k-1)!(s-k-1)!(n-s)!}\Omega_4$$

$$\times x_1x_2\left(\beta+x_1^2\right)^{-(\alpha+1)}\left(\beta+x_2^2\right)^{-(\alpha+1)}$$

$$\times\left[1-\beta^{\alpha}\left(\beta+x_1^2\right)^{-\alpha}\right]^{a-1}\left[1-\beta^{\alpha}\left(\beta+x_2^2\right)^{-\alpha}\right]^{a-1} \quad (31)$$

Where

$$\Omega_4 = \sum_{\ell=0}^{s-k-1}\sum_{j=0}^{k-1}\binom{s-k-1}{\ell}\binom{k-1}{j}(-1)^{\ell+j}\left\{1-\left[1-\beta^{\alpha}\left(\beta+x_1^2\right)^{-\alpha}\right]^a\right\}^{b(j+1)+s-(k+\ell+2)}$$

$$\times\left\{1-\left[1-\beta^{\alpha}\left(\beta+x_2^2\right)^{-\alpha}\right]^a\right\}^{b(n+\ell-s+1)-1}$$

5.9. Record statistics

Record values and the associated statistics are of great interest in many real life applications such as industrial stress testing, meteorological analysis, hydrology and athletic events. Suppose $X_1, X_2, \ldots, X_n$ is a sequence of independent and identical distributed random variables having cdf $F(x)$ and pdf $f(x)$. Let $Y_n = \max(\min)\{X_1, X_2, \ldots, X_n\}$ for $n \ge 1$. We say X_j is an upper (lower) record value of this sequence if $Y_j > (<) Y_{j-1}, j > 1$. Thus X_j will be called an upper (lower) record value if its value exceeds (is lower than) that of all previous observations.
The pdf of $X_{U(r)}$, the r^{th} upper record is given as (see Ahsanullah [7] and Arnold et al. [8])

$$f_{X_{U(r)}}(x) = \frac{1}{(r-1)!}\left[R(x)\right]^{r-1} f(x)$$

Where

$$R(x) = -\ln\left[1-F(x)\right]$$

Then, the pdf the r^{th} upper record KwCR random variable $X_{U(r)}$ can be obtained to be

$$f_{X_{U(r)}}(x) = \frac{2ab^r\alpha\beta^{\alpha}}{(r-1)!}\Omega_5 x\left(\beta+x^2\right)^{-(\alpha+1)} \times\left[1-\beta^{\alpha}\left(\beta+x^2\right)^{-\alpha}\right]^{a-1} \quad (32)$$

Where

$$\Omega_5 = \left\{1-\left[1-\beta^{\alpha}\left(\beta+x^2\right)^{-\alpha}\right]^a\right\}^{b-1}\left[\ln\left\{1-\left[1-\beta^{\alpha}\left(\beta+x^2\right)^{-\alpha}\right]^a\right\}\right]^{-(r-1)}$$

Furthermore, the joint distribution of the first n upper record values $x \equiv \left(x_{U(1)}, x_{U(2)}, \ldots, x_{U(n)}\right)$ is given by (see Ahsanullah [7])

$$f_{1,2,\ldots,n}(x_{U(1)}, x_{U(2)}, \ldots, x_{U(n)}) = f(x_{U(n)})\prod_{i=1}^{n-1}\frac{f(x_{U(i)})}{1-F(x_{U(i)})}$$

Consequently, for the KwBC distribution we get

$$f_{1,2,\ldots,n}(x_{U(1)}, x_{U(2)}, \ldots, x_{U(n)}) = 2^n a^n b^n \alpha^n \beta^{n\alpha}\left\{1-\left[1-\beta^{\alpha}\left(\beta+x^2\right)^{-\alpha}\right]^a\right\}^b$$

$$\times\exp\left(\Omega_6+\Omega_7-\Omega_8\right) \quad (33)$$

Where

$$\Omega_6 = \sum_{i=1}^{n}\ln x_i - (\alpha+1)\sum_{i=1}^{n}\ln(\beta+x_i^2),$$

$$\Omega_7 = (a-1)\sum_{i=1}^{n}\ln\left[1-\beta^{\alpha}(\beta+x_i^2)^{-\alpha}\right]$$

And

$$\Omega_8 = \sum_{i=1}^{n}\ln\left\{1-\left[1-\beta^{\alpha}\left(\beta+x_i^2\right)^{-\alpha}\right]^a\right\}$$

Furthermore, the pdf of $X_{L(r)}$, the r^{th} lower record is given as (see Ahsanullah [7] and Arnold et al. [8])

$$f_{X_{U(r)}}(x)=\frac{1}{(r-1)!}\left[H(x)\right]^{r-1}f(x)$$

Where

$$H(x)=-\ln\left[F(x)\right]$$

So, for the KwCR distribution we have

$$f_{X_{L(r)}}(x)=\frac{2ab\alpha\beta^{\alpha}}{(r-1)!}\Omega_9^{r-1}x(\beta+x^2)^{-(\alpha+1)}$$

$$\times\left[1-\beta^{\alpha}(\beta+x^2)^{-\alpha}\right]^{a-1}\left\{1-\left[1-\beta^{\alpha}(\beta+x^2)^{-\alpha}\right]^{a}\right\}^{b-1} \quad (34)$$

Where

$$\Omega_9=-\ln\left[1-\left\{1-\left[1-\beta^{\alpha}\left(\beta+x^2\right)^{-\alpha}\right]^{a}\right\}^{b}\right]$$

Moreover, the joint distribution of the first n lower record values $x\equiv\left(x_{L(1)},x_{L(2)},...,x_{L(n)}\right)$ is given by (see Ahsanullah [7])

$$f_{1,2,...,n}(x_{L(1)},x_{L(2)},...,x_{L(n)})=f(x_{L(n)})\prod_{i=1}^{n-1}\frac{f(x_{L(i)})}{F(x_{L(i)})}$$

Then, the joint distribution of the first n lower record values of the KwCR distribution is given from

$$f_{1,2,...,n}(x_{L(1)},x_{L(2)},...,x_{L(n)})=2^n a^n b^n \alpha^n \beta^{n\alpha}\exp\left(\Omega_6+\Omega_7+(b-1)\Omega_8-\Omega_{10}\right) \quad (35)$$

Where

$$\Omega_{10}=\sum_{i=1}^{n-1}\ln\left[1-\left\{1-\left[1-\beta^{\alpha}(\beta+x_i^2)^{-\alpha}\right]^{a}\right\}^{b}\right]$$

6. Methods of estimation

In this section we obtain the moments and the maximum likelihood estimates for the model parameters corresponding to the KwCR distribution.

6.1. Moments estimation

The moment estimators for the vector of parameters $\Theta=(\alpha,\beta,a,b)^T$ can be obtained by equating population moments with the sample moments as follows:

$$\mu_r'=\frac{1}{n}\sum_{i=1}^{n}x_i^r$$

Then, we have:

$$\alpha\sqrt{\beta}\sum_{j,i,h=0}^{\infty}w_{j,i,h}(j+h+1)\ \times\beta\left[\alpha(j+h+1)-1/2,3/2\right]=\frac{1}{n}\sum_{i=1}^{n}x_i, \quad (36)$$

$$\alpha\beta\sum_{j,i,h=0}^{\infty}w_{j,i,h}(j+h+1)\beta\left[\alpha(j+h+1)-1,2\right]=\frac{1}{n}\sum_{i=1}^{n}x_i^2, \quad (37)$$

$$\alpha\sqrt{\beta^3}\sum_{j,i,h=0}^{\infty}w_{j,i,h}(j+h+1)\ \times\beta\left[\alpha(j+h+1)-3/2,5/2\right]=\frac{1}{n}\sum_{i=1}^{n}x_i^3 \quad (38)$$

And

$$\alpha\beta^2\sum_{j,i,h=0}^{\infty}w_{j,i,h}(j+h+1)\ \times\beta\left[\alpha(j+h+1)-2,3\right]=\frac{1}{n}\sum_{i=1}^{n}x_i^4 \quad (39)$$

It is impossible to solve Eqs. (36-39) analytically to obtain the moments estimates for α,β,a and b. We can use statistical software to obtain the moments estimates numerically like Newton Raphson iteration.

6.2. Maximum likelihood estimation

Letbe an independent random sample from the this $x_1,x_2,...,x_n$ distribution, then the corresponding log-likelihood function is given by

$$\ell=n\left[\ln(2)+\ln(a)+\ln(b)+\ln(\alpha)+\alpha\ln(\beta)\right]+\sum_{i=1}^{n}\ln(x_i)-(\alpha+1)\sum_{i=1}^{n}\ln(\eta_i)$$

$$+(a-1)\sum_{i=1}^{n}\ln(1-\beta^{\alpha}\eta_i^{-\alpha})+(b-1)\sum_{i=1}^{n}\ln\left[1-(1-\beta^{\alpha}\eta_i^{-\alpha})^{a}\right] \quad (40)$$

Where $\eta_i=\beta+x_i^2$.

The components of the score vector $\nabla\ell=\left(\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\beta},\frac{\partial\ell}{\partial a},\frac{\partial\ell}{\partial b}\right)$ are given below:

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}+n\ln(\beta)-\sum_{i=1}^{n}\ln(\eta_i)\ -(a-1)\sum_{i=1}^{n}\left[\beta^{\alpha}\eta_i^{-\alpha}\left(1-\beta^{\alpha}\eta_i^{-\alpha}\right)^{-1}\ln\left(\beta\eta_i^{-1}\right)\right]$$

$$+a(b-1)\sum_{i=1}^{n}\left\{\begin{matrix}\beta^{\alpha}\eta_i^{-\alpha}\left(1-\beta^{\alpha}\eta_i^{-\alpha}\right)^{a-1}\ln\left(\beta\eta_i^{-1}\right)\\ \times\left[1-\left(1-\beta^{\alpha}\eta_i^{-\alpha}\right)^{a}\right]^{-1}\end{matrix}\right\} \quad (41)$$

$$\frac{\partial\ell}{\partial\beta}=\frac{n\alpha}{\beta}-(\alpha+1)\sum_{i=1}^{n}\eta_i^{-1}-\alpha(a-1)\beta^{\alpha-1}\sum_{i=1}^{n}\left[\frac{x_i^2\eta_i^{-(\alpha+1)}}{1-\beta^{\alpha}\eta_i^{-\alpha}}\right]+a\alpha(b-1)\beta^{\alpha-1}$$

$$\times\sum_{i=1}^{n}\left[\frac{x_i^2\eta_i^{-(\alpha+1)}\left[\left(1-\beta^{\alpha}\eta_i^{-\alpha}\right)^{-\alpha}\right]^{a-1}}{1-\left[1-\beta^{\alpha}\eta_i^{-\alpha}\right]^{a}}\right], \quad (42)$$

$$\frac{\partial\ell}{\partial a}=\frac{n}{a}+\sum_{i=1}^{n}\ln\left[1-\beta^{\alpha}\eta_i^{-\alpha}\right]\ -(b-1)\sum_{i=1}^{n}\left[\frac{\left[\left(1-\beta^{\alpha}\eta_i^{-\alpha}\right)^{-\alpha}\right]^{a}\ln\left(1-\beta^{\alpha}\eta_i^{-\alpha}\right)}{1-\left(1-\beta^{\alpha}\eta_i^{-\alpha}\right)^{a}}\right] \quad (43)$$

And

$$\frac{\partial\ell}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}\ln\left[1-\left(1-\beta^{\alpha}\eta_i^{-\alpha}\right)^{a}\right] \quad (44)$$

The maximum likelihood estimates, say $\hat{\Theta}=(\hat{\alpha},\hat{\beta},\hat{a},\hat{b})^T$ of $\Theta=(\alpha,\beta,a,b)^T$ can be obtained by solving the system of nonlinear equations (41) through (44). These equations cannot be solved algebraically and it needed numeric iteration techniques.

7. Conclusion

This paper presents a new continuous distribution namely the Kumaraswamy compound Rayleigh (KwCR) distribution that generalizes the compound Rayleigh (CR) distribution. Basic properties of the new distribution are investigated including the mean, variance, coefficient of variation, ordinary and incomplete moments, skewness, kurtosis, moment and probability generating functions, reliability, hazard, reverse hazard and cumulative hazard functions, Lorenz, Bonferroni and Zenga curves, Rényi of entropy, order statistics and record statistics. We used the moments and maximum likelihood methods to estimate the model parameters.

References

[1] T.A. Abushal, Estimation of the unknown parameters for the compound Rayleigh distribution based on progressive first-failure-censored sampling, Open Journal of Statistics, 1 (2011) 161-171. https://doi.org/10.4236/ojs.2011.13020.

[2] O. Shajaee, R. Azimi and M. Babanezhad, Empirical Bayes estimators of parameter and reliability function for compound Rayleigh distribution under record data, American Journal of Theoretical and Applied Statistics, 1 (2012) 12-15. https://doi.org/10.11648/j.ajtas.20120101.12.

[3] D.R. Barot, and M.N. Patal, Performance of estimates of reliability parameters for compound Rayleigh progressive type-ii censored data, Control Theory and Informatics, 5 (2015) 33-41.

[4] G.A. Abd-Elmougod, and E.E. Mahmoud, Parameters estimation of compound Ratleigh distribution under an adaptive type-ii progressive hybrid censored data for constant partially accelerated life tests. Global Journal of Pure and Applied Mathematics, 12(2016) 3253-3273.

[5] G.M. Cordeiro and M. de Castro, A new family of generalized distributions, Journal of Statistical Computation and Simulation, (2009) 1-17.

[6] B.O. Oluyede, and S. Rajasooriya, The Mc-Dagum distribution and its statistical properties with applications, Asian Journal of Mathematics and Applications, 44 (2013) 1-16.

[7] M. Ahsanullah, Record Statistics, New York: Nova Science Publishers, Inc. Commack.1995.

[8] B.C. Arnold, N. Balakrishan, and H.N. Nagaraja, Records. New York: Wiley, USA.1998.