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Fixed point theorem of Zamfirescus type in generalized cone metric spaces

O. K. Adewale*, B.V. Akinremi

University of Lagos, Lagos, Nigeria
**Corresponding author E-mail: adewalekayode2@yahoo.com*

Abstract

In this paper we establish and prove the existence of Zamfirescus fixed point theorem in G-Cone Metric Spaces.The uniqueness is also shown.

Keywords: Cauchy sequence, complete G-cone metric spaces, complete symmetric G-cone metric spaces, *G-cone metric spaces, unique fixed point.*

1 Introduction

The importance of fixed point theorems cannot be overemphasized. The study of metric fixed point theory has been researched extensively in the past decades [4]. Different researchers have attempted to generalise the usual notion of metric space (X, d) to extend the known metric space theorems in a more general setting. In 2004, Mustafa and Sims [7] introduced the generalised metric space as generalisation of the usual metric space (X, d). Furthermore, Beg, Abbas and Nazir [3], introduced the concept of G- cone metric space by replacing the set of real numbers by ordered Banach space. The authors in [3] also introduced new fixed point theorems in this new structure.

2 Preliminary notes

We briefly give some basic definitions of concepts which serves as background to this work.

Definition 1.1 [1,4]. Let X be a non-empty set. Suppose that $d : X \times X \to E$ satisfies:

(i) $0 \leq d(x, y)\ \forall x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)\ \forall x, y \in X$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)\ \forall x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.2 [6]. Let X be a non-empty set and $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following properties:

(i) $G(x, y, z) = 0$ if and only if $x = y = z$

(ii) $G(x, x, y) > 0$, $\forall x, y \in X$, with $x \neq y$

(iii) $G(x, x, y) < G(x, y, z)$, $\forall x, y, z \in X$, with $z \neq y$

(iv) $G(x, y, z) = G(p(x, y, z))$ (symmetry). Where p denotes the permutation function.

(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z) \quad \forall a, x, y, z \in X$ (rectangle inequality)

Then the function G is called a G-metric.

Definition 1.3 [3]. Let X be a non-empty set. Suppose that $d : X \times X \times X \to E$ satisfies:

1. (G_1) $G(x, y, z) = 0$ if $x = y = z$,
2. (G_2) $0 < G(x, x, y)$; whenever $x \neq y$ $\forall x, y \in X$,
3. (G_3) $G(x, x, y) \leq G(x, y, z)$; whenever $y \neq z$.
4. (G_4) $G(x, y, z) = G(x, z, y) = G(y, x, z)$=...(Symmetric in all the three variables),
5. (G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ $\forall x, y, z, a \in X$.

Then G is called a generalized cone metric on X and X is called a generalized cone metric space or G-cone metric space.

Definition 1.4 [3]. A G-cone metric space X is symmetric if $G(x, y, y) = G(y, x, x)$ $\forall x, y \in X$.

Proposition 1.5 [3]. Let X be a G-cone metric space, define $d_G : X \times X \to E$ by $d_G(x, y) = G(x, y, y) + G(y, x, x)$. Then (X, d_G) is a cone metric space. It can be noted that $G(x, y, y) \leq \frac{2}{3} d_G(x, y)$. If X is a symmetric G-cone metric space, then $d_G(x, y) = 2G(x, y, y)$ $\forall x, y \in X$.

Definition 1.6 [3]. Let X be a G-cone metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is:

(i) Cauchy sequence if for every $c \in E$, there is N such that $\forall m, n > N$, $G(x_n, x_m, x_l) \ll c$.

(ii) Convergent sequence if for every $c \in E$ with $0 \ll c$, there is N such that $\forall m, n > N$, $G(x_n, x_m, x) \ll c$ for some fixed $x \in X$. Here x is called the limit of the sequence $\{x_n\}$ and is denoted by $\lim x_n = x$ or $x_n \to x$ as $n \to \infty$.

A G-cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Proposition 1.7 [3]. Let X be a G-cone metric space then the following are equivalent.

(i) x_n is convergent to x.

(ii) $G(x_n, x_n, x) \to 0$, as $n \to \infty$.

(iii) $G(x_n, x, x) \to 0$, as $n \to \infty$.

(iv) $G(x_n, x_m, x) \to 0$, as $n, m \to \infty$.

Lemma 1.8 [3]. Let X be a G cone metric space. x_m, y_n and z_l be sequences in X such that $x_m \to x$, $y_n \to y$ and $z_l \to z$, then $G(x_m, y_n, z_l) \to G(x, y, z)$ as $m, n, l \to \infty$.

Lemma 1.9 [3]. Let $\{x_n\}$ be a sequence in G-cone metric space X and $x \in X$. If $\{x_n\}$ converges to x and x_n converges to y, then $x = y$.

Lemma 1.10 [3]. Let $\{x_n\}$ be a sequence in G-cone metric space X and if $\{x_n\}$ converges to x $\forall$ $x \in X$, then $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

Lemma 1.11 [3]. Let $\{x_n\}$ be a sequence in G-cone metric space X and $x \in X$. If $\{x_n\}$ converges to $x \in X$,then x_n is a Cauchy sequence.

Lemma 1.12 [3]. Let $\{x_n\}$ be a sequence in a G-cone metric space X and if x_n is a Cauchy sequence in X, then $G(x_m, x_n, x_l) \to 0$, as $m, n, l \to \infty$.

Theorem 1.13 [11]. Each self map T of a complete metric space (X, d) such that

$$d(Tx, Ty) \leq kd(x, y) \quad (x \neq y, 0 \leq k < 1)$$

has a unique fixed point.

Theorem 1.14 [3]. Let X be a complete symmetric G-cone metric space and $T : X \to X$ be a mapping satisfying one of the following conditions:

$$\begin{aligned} G(Tx, Ty, Tz) &\leq aG(x, y, z) + bG(x, Tx, Tx) + cG(y, Ty, Ty) + dG(z, Tz, Tz) \\ &\textit{or} \\ G(Tx, Ty, Tz) &\leq aG(x, y, z) + bG(x, Tx, x) + cG(y, y, Ty) + dG(z, z, Tz) \end{aligned}$$

$\forall\ x, y, z \in X$, where $0 \leq a + b + c + d < 1$. Then T has a unique fixed point.

Theorem 1.15 [3]. Let X be a complete G-cone metric space and $T : X \to X$ be a mapping satisfying one of the following conditions:

$$\begin{aligned} G(Tx, Ty, Tz) &\leq a[G(x, Ty, Ty) + G(y, Tx, Tx)] \\ &\textit{or} \\ G(Tx, Ty, Tz) &\leq a[G(x, x, Ty) + G(y, y, Tx)] \end{aligned}$$

$\forall\ x, y, z \in X$, where $a \in [0, \frac{1}{2})$. Then T has a unique fixed point.

Example 1.16 [3].. Let $E = R^3$; $P = \{(x, y, z) \in R^3 : x, y, z \geq 0\}$, and
$X = \{(x, 0, 0) \in R^3 : 0 \leq x \leq 1\} \bigcup \{(0, x, 0) \in R^3 : 0 \leq x \leq 1\} \bigcup \{(0, 0, x) \in R^3 : 0 \leq x \leq 1\}$.
Define mapping $G : X \times X \times X \to E$ by

$$\begin{aligned} &G((x, 0, 0), (y, 0, 0), (z, 0, 0)) \\ &= (\tfrac{4}{3}(/x - y/ + /y - z/), /x - y/ + /y - z/, (/x - y/ + /y - z/)), G((0, x, 0), (0, y, 0), (0, z, 0)) \\ &= ((/x - y/ + /y - z/), \tfrac{2}{3}(/x - y/ + /y - z/), (/x - y/ + /y - z/)), G((0, 0, x), (0, 0, y), (0, 0, z)) \\ &= ((/x - y/ + /y - z/), /x - y/ + /y - z/, \tfrac{1}{3}(/x - y/ + /y - z/)) \end{aligned}$$

and

$$\begin{aligned} G((x, 0, 0), (0, y, 0), (0, 0, z)) &= G((0, 0, z), (0, y, 0), (x, 0, 0)) = ... \\ &= (\tfrac{4}{3}x + y + z, x + \tfrac{2}{3}y + z, x + y + \tfrac{1}{3}z) \end{aligned}.$$

Then X is a Complete G-cone metric space. Let $T : X \to X$
$T(x, 0, 0) = (0, x, 0), T(0, x, 0) = (0, 0, \frac{1}{3}x)$ and $T(0, 0, x) = (\frac{2}{3}x, 0, 0)$.
Then T satisfies the contractive condition given in Theorem 3.5 of [3] with constant $a = \frac{3}{4} \in [0, 1)$.
Note that T has a unique fixed point $(0, 0, 0) \in X$.

3 The Main result

Theorem 2.1. Let X be a complete symmetric $G-$ cone metric space and $T : X \to X$ a map for which there exist the real numbers a, b and c satisfying $0 \leq a < 1$, $0 \leq b$, $c < \frac{1}{2}$, such that for each pair $x, y, z \in X$ at least one of the following is true.

$$\begin{aligned} (GZ_1) \quad G(Tx, Ty, Tz) &\leq aG(x, y, z) \\ (GZ_2) \quad G(Tx, Ty, Tz) &\leq b[G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)] \\ (GZ_3) \quad G(Tx, Ty, Tz) &\leq c[G(x, Ty, Ty) + G(x, Tz, Tz) + G(y, Tz, Tz) \\ &\quad +G(y, Tx, Tx) + G(z, Tx, Tx) + G(z, Ty, Ty)]. \end{aligned}$$

Then T has a unique fixed point.

Proof:
Considering (GZ_1),

$$G(Tx, Ty, Ty) \leq aG(x, y, y). \tag{1}$$

Considering (GZ_2),

$$G(Tx, Ty, Ty) \leq b[G(x, Tx, Tx) + 2G(y, Ty, Ty)]. \tag{2}$$

Considering (GZ_3),

$$G(Tx, Ty, Ty) \leq c[2G(x, Ty, Ty) + 2G(y, Tx, Tx) + 2G(y, Ty, Ty)]. \tag{3}$$

By adding (1), (2) and (3),we have

$$G(Tx, Ty, Ty) \leq qG(x, y, y) + rG(x, Tx, Tx) + sG(y, Ty, Ty) + t[G(x, Ty, Ty) + G(y, Tx, Tx)], \tag{4}$$

where $q = \dfrac{a}{3}$, $r = \dfrac{b}{3}$, $s = \dfrac{2b+2c}{3}$ and $t = \frac{2c}{3}$.
Suppose T satisfies condition (4) and $x_0 \in X$ be an arbitrary point and define the sequence x_n by $x_n = T^n x_0$, then we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\leq qG(x_{n-1}, x_n, x_n) + rG(x_{n-1}, x_n, x_n) \\ &\quad +sG(x_n, x_{n+1}, x_{n+1}) + tG(x_{n-1}, x_{n+1}, x_{n+1}) \\ [\text{1-s-t}]G(x_n, x_{n+1}, x_{n+1}) &\leq [\text{q+r+t}][G(x_{n-1}, x_n, x_n) \\ G(x_n, x_{n+1}, x_{n+1}) &\leq \tfrac{q+r+t}{1-s-t} G(x_{n-1}, x_n, x_n). \end{aligned} \tag{5}$$

Let $j = \dfrac{q+r+t}{1-s-t} < 1$ i.e $q \in [0, 1)$,we deduce that

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\leq jG(x_{n-1}, x_n, x_n) \\ &\leq j^2 G(x_{n-2}, x_{n-1}, x_{n-1}) \\ &\leq j^3 G(x_{n-3}, x_{n-2}, x_{n-2}) \\ &\leq j^n G(x_0, x_1, x_1). \end{aligned} \tag{6}$$

By repeated use of rectangle inequality, we have

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad +G(x_{n+2}, x_{n+3}, x_{n+3}) + ... + G(x_{m-1}, x_m, x_m). \end{aligned} \tag{7}$$

From (6) and (7), we have

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad +G(x_{n+2}, x_{n+3}, x_{n+3}) + ... + G(x_{m-1}, x_m, x_m) \\ &\leq [j^n + j^{n+1} + j^{n+2} + j^{n+3} + ... + j^{m-1}]G(x_0, x_1, x_1) \\ &\leq j^n[1 + j + j^2 + j^3 + j^4 + ... + j^{m-n-1}]G(x_0, x_1, x_1) \\ &\leq j^n[1 - j]^{-1}G(x_0, x_1, x_1) \\ &\leq [\tfrac{j^n}{1-j}]G(x_0, x_1, x_1). \end{aligned} \tag{8}$$

Let $0 \ll c$ be given. Choose $\delta > 0$ such that $c + N_\delta(0) \subseteq P$,where $N_\delta(0) = [y \in E : ||y|| < \delta]$. Also, choose a natural number N_1 such that $\dfrac{j^n}{1-j}G(x_0, x_1, x_1) \in N_\delta(0)$, $\forall m \geq N_1$. Then $[\dfrac{j^n}{1-j}]G(x_0, x_1, x_1) \ll c \ \forall m \geq N_1$. i.e. $G(x_n, x_m, x_l) \ll c \ \forall m \geq N_1$. Therefore $\{x_n\}$ is a G-Cauchy Sequence.
Next we will show that $Tu = u$. Suppose $Tu \neq u$ and $\{x_n\} \to u$.

$$\begin{aligned} G(x_n, Tu, Tu) &\leq qG(x_{n-1}, u, u) + rG(x_{n-1}, x_n, x_n) + sG(u, Tu, Tu) \\ &\quad +t[G(x_{n-1}, Tu, Tu) + G(u, x_n, x_n)]. \\ G(u, Tu, Tu) &\leq [\text{s+t}]G(u, Tu, Tu). \end{aligned} \tag{9}$$

This is a contradiction. So, $Tu = u$.
To show the uniqueness, suppose $v \neq u$ is such that $Tv = v$, then

$$\begin{aligned} G(Tu, Tv, Tv) &\leq qG(u, v, v) + rG(u, Tu, Tu) + sG(v, Tv, Tv) \\ &\quad +t[G(u, Tv, Tv) + G(v, Tu, Tu)]. \end{aligned} \tag{10}$$

Since $Tu = u$ and $Tv = v$, we have

$$\begin{aligned} G(u, v, v) &\leq qG(u, v, v) + rG(u, u, u) + sG(v, v, v) \\ &\quad +t[G(u, v, v) + G(v, u, u)]. \\ G(u, v, v) &\leq [q + 2t]G(u, v, v), \end{aligned} \tag{11}$$

A contracdiction, which implies that $v = u$. Hence the proof.

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