International Journal of Advanced Mathematical Sciences, 4 (2) (2016) 44-46

International Journal of Advanced Mathematical Sciences

Website: www.sciencepubco.com/index.php/IJAMS

Research paper

Vertex and edge Co-PI indices of bridge graphs

Sharmila devi.G[1*] and Kaladevi.V[2]

[1]*Research Scholar, Research and Development Centre, Bharathiar University, Coimbatore. Department of Mathematics, Kongu Arts and Science college(Autonomous), Erode, India.*
[2]*Department of Mathematics, Bishop Heber College, Trichy, India*
[*]*Corresponding author E-mail:sharmilashamritha@gmail.com*

Abstract

The Co-PI index of a graph G, denoted by $Co-PI(G)$, is defined as $Co-PI(G) = \sum_{uv=e\in E(G)} \left|n_u^G(e) - n_v^G(e)\right|$, where $n_u^G(e)$ is number of vertices of G whose distance to the vertex u is less than the distance to the vertex v in G. Similarly, the edge Co-PI index of G is defined as $Co-PI_e(G) = \sum_{uv=e\in E(G)} \left|m_u^G(e) - m_v^G(e)\right|$, where $m_u^G(e)$ is number of edges of G whose distance to the vertex u is less than the distance to the vertex v in G. In this paper, the upperbound for the Co-PI and edge Co-PI indices of bridge graph are obtained.

Keywords: *Bridge Graph; Co-PI Index; Edge Co-PI Index.*

1. Introduction

All the graphs considered in this paper are connected and simple.A vertex $x \in V(G)$ is said to be *equidistant* from the edge $e = uv$ of G if $d_G(u,x) = d_G(v,x)$, where $d_G(u,x)$ denotes the distance between u and x in G. The degree of the vertex u in G is denoted by $d_G(u)$. The edges $e = uv$ and $f = xy$ of G are said to be *equidistant edges* if $min\{d_G(u,x), d_G(u,y)\} = min\{d_G(v,x), d_G(v,y)\}$.

For an edge $uv = e \in E(G)$, the number of vertices of G whose distance to the vertex u is smaller than the distance to the vertex v in G is denoted by $n_u^G(e)$; analogously, $n_v^G(e)$ is the number of vertices of G whose distance to the vertex v in G is smaller than the distance to the vertex u; the vertices equidistant from both the ends of the edge $e = uv$ are not counted. Similarly, $m_u(e)$ denotes the number of edges of G whose distance to the vertex u is less than the distance to the vertex v.

The *vertex PI index* of G, denoted by $PI(G)$, is defined as $PI(G) = \sum_{e=uv\in E(G)} \left(n_u^G(e) + n_v^G(e)\right)$ and the *edge PI index* of G, denoted by $PI_e(G)$, is defined as $PI_e(G) = \sum_{e=uv\in E(G)} (m_u^G(e) + m_v^G(e))$.

Similarly, the *Co-PI index* of G, denoted by *Co-PI(G)*, is defined as $Co-PI(G) = \sum_{e=uv\in E(G)} \left|n_u^G(e) - n_v^G(e)\right|$ and the *edge Co-PI index* of G, denoted by $Co-PI_e(G)$, is defined as $Co-PI_e(G) = \sum_{e=uv\in E(G)} \left|m_u^G(e) - m_v^G(e)\right|$.

Khadikar [12] first introduced edge PI index of graphs and they investigated the chemical applications of the PI index. The PI index of the graph G is a topological index related to equidistant vertices. Another topological index of G related to distance of G is the wiener index of G, first introduced by wiener, see[18]. Khadikar, Karmarkar and Agarwal [12] first introduced edge Padmakar-Ivan index of graphs and they investigated the chemical applications of the Padmakar-Ivan index. The mathematical properties of the PI_v and its applications in chemistry and nanoscience are well studied by Ashrafi and Loghman [1,3]. Ashrafi and Rezari [2], Deng, Chen and Zhang [6], Khadikar [10], Khalifeh, Yousefi-Azari and Ashrafi [11], Klavžar [13] and Yousefi-Azari, Manoochehrian and Ashrafi [17]. The vertex PI indices of the tensor and strong products of graphs are studied in [14, 16]. In [20, 21, 22] the PI indices of bridge graphs and chain graphs are discussed. In this paper, the upper bounds for the Co-PI and edge Co-PI indices of bridge graphs are obtained. Let $\{G_i\}_{i=1}^s$ be a set of finite pairwise vertex disjoint connected graphs with $v_i \in V(G_i)$. The bridge graph $B(G_1, G_2, \ldots, G_s) = B(G_1, G_2, \ldots, G_s; v_1, v_2, \ldots, v_s)$ of $\{G_i\}_{i=1}^s$ with respect to the vertices $\{v_i\}_{i=1}^s$ is the graph obtained from the graphs $G_1, G_2, \ldots, G_s$ by connecting the vertices v_i and v_{i+1} by an edge for all $i = 1, 2, \ldots, s-1$.

2. Co-PI Index of Bridge Graph

Let G be a graph and let $v \in V(G)$. The set of all edges xy such that $d_G(x,v) = d_G(y,v)$ is denoted by $N_v(G)$. Define $K(G_i) = \{e = xy \in E(G_i) \setminus N_v | d(x,v) < d(y,v)\}$ and
$L(G_i) = \{e = xy \in E(G_i) \setminus N_v | d(x,v) > d(y,v)\}$.

Theorem 2.1
Let $G = B(G_1, G_2, \ldots, G_s)$ of $\{G_i\}_{i=1}^s$ with respect to the vertices $\{v_i\}_{i=1}^s$ and $|V(G)| = a$. Then $Co-PI(G) \le \sum_{i=1}^{s} (Co-PI(G_i)) + \sum_{i=1}^{s} (|V(G)| - |V(G_i)|)(k_i + \ell_i) + \sum_{i=1}^{s-1} \left(2a_i - a\right)$, where $a_i = \sum_{j=1}^{i} |V(G_j)|$, $k_i = |E(K(G_i))|$ and $l_i = |E(L(G_i))|$.

Proof. From the definition of $Co-PI(G)$,

$$\begin{aligned} Co-PI(G) &= \sum_{e=uv\in E(G)} \left|n_u^G(e)-n_v^G(e)\right| \\ &= \sum_{i=1}^{s}\sum_{e=uv\in E(G_i)} \left|n_u^G(e)-n_v^G(e)\right| \\ &\quad +\sum_{i=1}^{s-1}\left|n_{v_i}^G(v_iv_{i+1})-n_{v_{i+1}}^G(v_iv_{i+1})\right| \\ &= \sum_{i=1}^{s}\sum_{e=uv\in N_{v_i}(G_i)} \left|n_u^G(e)-n_v^G(e)\right| \\ &\quad +\sum_{i=1}^{s}\sum_{e=uv\in E(G_i)\setminus N_{v_i}(G_i)} \left|n_u^G(e)-n_v^G(e)\right| \\ &\quad +\sum_{i=1}^{s-1}\left|n_{v_i}^G(v_iv_{i+1})-n_{v_{i+1}}^G(v_iv_{i+1})\right|. \qquad (1) \end{aligned}$$

The summations in equation (1) are computed separately.
Case (A):
If $e=uv\in N_{v_i}(G_i)$, then all the vertices in $V(G)\setminus V(G_i)$ are equidistant from the ends of the edge $e=uv$. This implies $n_u^G(e)=n_u^{G_i}(e)$ and $n_v^G(e)=n_v^{G_i}(e)$. Then $\sum_{e=uv\in N_{v_i}(G_i)}\left|n_u^G(e)-n_v^G(e)\right|=\sum_{e=uv\in N_{v_i}(G_i)}\left|n_u^{G_i}(e)-n_v^{G_i}(e)\right|$.
Thus $\sum_{i=1}^{s}\sum_{e=uv\in N_{v_i}(G_i)}\left|n_u^G(e)-n_v^G(e)\right| = \sum_{i=1}^{s}\sum_{e=uv\in N_{v_i}(G_i)}\left|n_u^{G_i}(e)-n_v^{G_i}(e)\right|$.
Case (B):
If $e=uv\in E(G_i)\setminus N_{v_i}(G_i)$ then, the following cases are arise:
(i) If $e=uv\in K(G_i)$, then

$$\begin{aligned} n_u^G(e)-n_v^G(e) &= n_u^{G_i}(e)+|V(G)|-|V(G_i)|-n_v^{G_i}(e) \\ &= n_u^{G_i}(e)-n_v^{G_i}(e)+|V(G)|-|V(G_i)|. \qquad (2) \end{aligned}$$

(ii) If $e=uv\in L(G_i)$, then

$$\begin{aligned} n_u^G(e)-n_v^G(e) &= n_u^{G_i}(e)-(n_v^{G_i}(e)+|V(G)|-|V(G_i)|) \\ &= n_u^{G_i}(e)-n_v^{G_i}(e)-(|V(G)|-|V(G_i)|). \quad (3) \end{aligned}$$

Thus, $\sum_{e=uv\in E(G_i)\setminus N_{v_i}(G_i)}\left|n_u^G(e)-n_v^G(e)\right|$

$$\begin{aligned} &= \sum_{e=uv\in K(G_i)}\left|n_u^G(e)-n_v^G(e)\right|+\sum_{e=uv\in L(G_i)}\left|n_u^G(e)-n_v^G(e)\right| \\ &\le \sum_{e=uv\in E(G_i)\setminus N_{v_i}(G_i)}\left|n_u^{G_i}(e)-n_v^{G_i}(e)\right| \\ &\quad + \sum_{e=uv\in K(G_i)}|(|V(G)|-|V(G_i)|)| \\ &\quad + \sum_{e=uv\in L(G_i)}|(|V(G)|-|V(G_i)|)|. \end{aligned}$$

Case (C):
If e is an edge $e=v_iv_{i+1}$, then there exists no vertex t which is equidistant from the ends of the edge e. Since $n_{v_i}^G(e)-n_{v_{i+1}}^G(e)=\sum_{j=1}^{i}\left|V(G_j)\right|-\sum_{j=i+1}^{s}\left|V(G_j)\right|=a_i-(|V(G)|-a_i)=2a_i-|V(G)|$, the last summation in (1) becomes

$$\sum_{i=1}^{s-1}\left|n_{v_i}^G(v_iv_{i+1})-n_{v_{i+1}}^G(v_iv_{i+1})\right|=\sum_{i=1}^{s-1}\left|\sum_{j=1}^{i}\left|V(G_j)\right|-\sum_{j=i+1}^{s}\left|V(G_j)\right|\right|$$

$$\le \sum_{i=1}^{s-1}\Big(2a_i-|V(G)|\Big).$$

Hence,

$$\begin{aligned} Co-PI(G) &= \sum_{i=1}^{s}\sum_{e=uv\in N_{v_i}(G_i)}\left|n_u^{G_i}(e)-n_v^{G_i}(e)\right| \\ &\quad + \sum_{e=uv\in E(G_i)\setminus N_{v_i}(G_i)}\left|n_u^{G_i}(e)-n_v^{G_i}(e)\right| \\ &\quad + \sum_{e=uv\in K(G_i)}|(|V(G)|-|V(G_i)|)| \\ &\quad + \sum_{e=uv\in L(G_i)}|(|V(G)|-|V(G_i)|)| \\ &\quad +\sum_{i=1}^{s-1}\Big(2a_i-|V(G)|\Big) \\ &\le \sum_{i=1}^{s}(Co-PI(G_i))+\sum_{i=1}^{s}(|V(G)|-|V(G_i)|)(k_i+\ell_i) \\ &\quad +\sum_{i=1}^{s-1}\Big(2a_i-|V(G)|\Big). \end{aligned}$$

3. Edge Co-PI Index of Bridge Graph

For a graph G with $v\in V(G)$, let $T_v(G)$ be the set of edges uv of G such that $d_G(x,v)=d_G(x,u)$. For a bridge graph $B(G_1,G_2,\ldots,G_s)$, $i=1,2,\ldots,s-1$, let $K(G_i)$ be the set of edges $e=xy\in E(G_i)\setminus T_{v_i}(G_i)$ such that $d_G(x,v_i)<d_G(y,v_i)$ and $L(G_i)$ the set of edges $e=uv\in E(G_i)\setminus T_{v_i}(G_i)$ such that $d_G(x,v_i)>d_G(y,v_i)$.
Theorem 3.1
Let $G=B(G_1,G_2,\ldots,G_s)$ of $\{G_i\}_{i=1}^s$ with respect to the vertices $\{v_i\}_{i=1}^s$. Then $Co-PI_e(G)\le\sum_{i=1}^{s}(Co-PI_e(G_i))+\sum_{i=1}^{s}(|E(G)|-|E(G_i)|)(k_i+\ell_i)+\sum_{i=1}^{s-1}\Big(2a_i-\left|E(G_j)\right|+1\Big)$, where $a_i=\sum_{j=1}^{i}\left|E(G_j)\right|+i-1, k_i=|E(K(G_i))|$ and $l_i=|E(L(G_i))|$.
Proof. Let $G=B(G_1,G_2,\ldots,G_s)$. Observe that $E(G_i)=T_{v_i}(G_i)\cup K(G_i)\cup L(G_i)$ for $i=1,2,\ldots,s$. By the definition of the edge Co-PI index,

$$\begin{aligned} Co-PI_e(G) &= \sum_{e=uv\in E(G)}\left|m_u^G(e)-m_v^G(e)\right| \\ &= \sum_{i=1}^{s}\sum_{e=uv\in T_{v_i}(G_i)}\left|m_u^G(e)-m_v^G(e)\right| \\ &\quad +\sum_{i=1}^{s}\sum_{e=uv\in K(G_i)\cup L(G_i)}\left|m_u^G(e)-m_v^G(e)\right| \\ &\quad +\sum_{i=1}^{s-1}\left|m_{v_i}^G(v_iv_{i+1})-m_{v_{i+1}}^G(v_iv_{i+1})\right|. \qquad (4) \end{aligned}$$

- For $i=1,2,\ldots,s$, if $e=uv\in T_{v_i}(G_i)$, then $d_G(v_i,u)=d_G(v_i,v)$, and for any edge $e_1\in E(G)\setminus E(G_i)$, then $d_G(u,e_1)=d_G(v,e_1)$. This implies $m_u^G(e)=m_u^{G_i}(e)$ and $m_v^G(e)=m_v^{G_i}(e)$. Then $\sum_{e=uv\in T_{v_i}(G_i)}\left|m_u^G(e)-m_v^G(e)\right|=\sum_{e=uv\in T_{v_i}(G_i)}\left|m_u^{G_i}(e)-m_v^{G_i}(e)\right|$.
- For $i=1,2,\ldots,s$, if $e=uv\in K(G_i)$, then $d_G(v_i,u)<d_G(v_i,v)$, thus

$$\begin{aligned} m_u^G(e)-m_v^G(e) &= m_u^{G_i}(e)+s-1+\sum_{1\le j\le s,\ j\ne i}\left|E(G_j)\right|-m_v^{G_i}(e) \\ &= m_u^{G_i}(e)-m_v^{G_i}(e)+(|E(G)|-|E(G_i)|). \quad (5) \end{aligned}$$

Similarly, if $e=uv\in L(G_i)$, then $d_G(v_i,u)>d_G(v_i,v)$, thus

$$\begin{aligned} m_u^G(e)-m_v^G(e) &= m_u^{G_i}(e)-(m_v^{G_i}(e)+s-1+\sum_{1\le j\le s,\ j\ne i}\left|E(G_j)\right|) \\ &= m_u^{G_i}(e)-m_v^{G_i}(e)-(|E(G)|-|E(G_i)|). \qquad (6) \end{aligned}$$

From (5) and (6)

$$\sum_{e=uv\in K(G_i)\cup L(G_i)} \left|m_u^G(e)-m_v^G(e)\right|$$

$$\begin{aligned}
&= \sum_{e=uv\in K(G_i)} \left|m_u^G(e)-m_v^G(e)\right| + \sum_{e=uv\in L(G_i)} \left|m_u^G(e)-m_v^G(e)\right| \\
&= \sum_{e=uv\in K(G_i)} \left|m_u^{G_i}(e)-m_v^{G_i}(e)+(|E(G)|-|E(G_i)|)\right| \\
&\quad + \sum_{e=uv\in L(G_i)} \left|m_u^{G_i}(e)-m_v^{G_i}(e)-(|E(G)|-|E(G_i)|)\right| \\
&\leq \sum_{e=uv\in K(G_i)} \left|m_u^{G_i}(e)-m_v^{G_i}(e)\right| + \sum_{e=uv\in K(G_i)} (|E(G)|-|E(G_i)|) \\
&\quad + \sum_{e=uv\in L(G_i)} \left|m_u^{G_i}(e)-m_v^{G_i}(e)\right| + \sum_{e=uv\in L(G_i)} (|E(G)|-|E(G_i)|).
\end{aligned}$$

- For an edge $e=v_iv_{i+1}$, $i=1,2,\ldots,s-1$, one can easily observe that $m_{v_i}^G(e)-m_{v_{i+1}}^G(e)=\left(\sum_{j=1}^{i}\left|E(G_j)\right|+i-1\right)-\left(\sum_{j=i+1}^{s}\left|E(G_j)\right|+s-(i+1)\right)$.

$$\sum_{i=1}^{s-1}\left|m_{v_i}^G(v_iv_{i+1})-m_{v_{i+1}}^G(v_iv_{i+1})\right|$$

$$\begin{aligned}
&= \sum_{i=1}^{s-1}\left|\left(\sum_{j=1}^{i}\left|E(G_j)\right|+i-1\right)-\left(\sum_{j=i+1}^{s}\left|E(G_j)\right|+s-(i+1)\right)\right| \\
&= \sum_{i=1}^{s-1}\left|a_i-(\left|E(G_j)\right|-1-a_i)\right| \\
&\leq \sum_{i=1}^{s-1}\left(2a_i-\left|E(G_j)\right|+1\right).
\end{aligned}$$

Hence the edge Co-PI index of the bridge graph is given by,

$$\begin{aligned}
&Co-PI_e(G)=\sum_{i=1}^{s}\sum_{e=uv\in T_{v_i}(G_i)}\left|m_u^{G_i}(e)-m_v^{G_i}(e)\right| \\
&+\sum_{i=1}^{s}\sum_{e=uv\in K(G_i)}\left|m_u^{G_i}(e)-m_v^{G_i}(e)\right| \\
&+\sum_{i=1}^{s}\sum_{e=uv\in K(G_i)}(|E(G)|-|E(G_i)|) \\
&+\sum_{i=1}^{s}\sum_{e=uv\in L(G_i)}\left|m_u^{G_i}(e)-m_v^{G_i}(e)\right| \\
&+\sum_{i=1}^{s}\sum_{e=uv\in L(G_i)}(|E(G)|-|E(G_i)|)+\sum_{i=1}^{s-1}\left(2a_i-\left|E(G_j)\right|+1\right) \\
&= \sum_{i=1}^{s}Co-PI_e(G_i)+\sum_{i=1}^{s}(|E(G)|-|E(G_i)|)(k_i+\ell_i) \\
&+\sum_{i=1}^{s-1}\left(2a_i-\left|E(G_j)\right|+1\right).
\end{aligned}$$

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