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On a probability distribution with fractional moments arising from generalized Pearson system of differential equation and its characterization

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Abstract

This paper derives a probability distribution with fractional moments arising from generalized Pearson system of differential equation. The expressions for the probability density function, cumulative distribution function and moments have been obtained. The plots for the probability density function, cumulative distribution function, and survival and hazard functions are also provided. Some distributional relationships of the proposed distribution have been established. A characterization of the new distribution is given. It is hoped that the findings of the paper will be useful for researchers in different fields like economics, engineering, environmental science, finance, medical sciences and physical sciences among others, where fractional moments become required to be computed if the integer moments $k \geq 1$ do not exist.

Keywords: *Characterization; Distribution; Differential Equations; Moments; Hazard Function; Probability Density Function.*

2010 Mathematics Subject Classifications: *33C15; 60E05; 62E15.*

1 Introduction

Various systems of distributions have been constructed to provide approximations to a wide variety of distributions, see, e.g., Johnson et al. [12]. One of these systems is the Pearson system. A continuous distribution belongs to this system if its probability density function (*pdf*) *f(x)* satisfies a differential equation of the form

$$\frac{1}{f_X(x)}\frac{d\,f(x)}{d\,x} = -\frac{x+a}{bx^2+cx+d}\ , \tag{1}$$

where a, b, c, and d are real parameters such that *f(x)* is a *pdf* . The shapes of the *pdf* . depend on the values of these parameters based on which Pearson [18-19] classified these distributions into a number of types known as Pearson Types I – VI. Later in another paper, Pearson [20] defined more special cases and subtypes, known as Pearson Types VII - XII. Many well-known distributions are special cases of Pearson Type distributions which include Normal and Student t distributions (Pearson Type VII), Beta distribution (Pearson Type I), Gamma distribution (Pearson Type III), among others.

1.1 Generalized Pearson Distribution

In recent years, some researchers have considered a generalization of (1), known as generalized Pearson differential equation (GPE), given by

$$\frac{1}{f_X(x)}\frac{d\,f(x)}{d\,x} = \frac{\sum_{j=0}^{m} a_j\,x^j}{\sum_{j=0}^{n} b_j\,x^j}\ , \tag{2}$$

where m , $n \in \mathrm{N}/\{0\}$ and the coefficients a_j and b_j are real parameters. The system of continuous univariate *pdf's* generated by GPE is called a generalized Pearson system, which includes a vast majority of continuous *pdf* 's , by proper choices of these parameters. For example:

1) Roy [22] studied GPE, when $m = 2, n = 3, b_0 = 0$, to derive five frequency curves whose parameters depends on the first seven population moments.
2) Dunning and Hanson [8] used GPE in his paper on generalized Pearson distributions and nonlinear programming.
3) Cobb et al. [5] extended Pearson's class of distributions to generate multimodal distributions by taking the polynomial in the numerator of GPE of degree higher than one, and the denominator, say, $v(x)$, having one of the following forms:

(I) $v(x) = 1, \quad -\infty < x < \infty$,

(II) $v(x) = x, \quad 0 < x < \infty$,

(III) $v(x) = x^2, \quad 0 < x < \infty$, (IV) $v(x) = x(1-x), \quad 0 < x < 1$.

4) Chaudhry and Ahmad [3] studied another class of generalized Pearson's distributions when

$$m = 4, n = 3, b_0 = b_1 = b_2 = 0, \frac{a_4}{2b_3} = -2\alpha, \frac{a_0}{2b_3} = 2\beta, b_3 \neq 0 .$$

5) Rossani and Scarfone [21] have studied the GPE in the following form

$$\frac{1}{f_X(x)} \frac{d f(x)}{d x} = -\frac{a_0 + a_1 x + a_2 x^2}{b_0 + b_1 x + b_2 x^2},$$

and used it to generate generalized Pearson distributions in order to study charged particles interacting with an electric and/or a magnetic field.
6) Shakil and Kibria [23], and Shakil et al. [24-25] have defined some new classes of generalized Pearson distributions for different choices of the parameters m, $n \in \mathrm{N} / \{0\}$ and the coefficients a_j and b_j in GPE. Recently, these distributions have been characterized by Hamedani [10-11]. Also, see Ahsanullah et al. [2], Shakil et al. [26], Kibria and Shakil [13], Lahcene [14], and Lee et al. [16], among others.

The organization of the paper is as follows. We give a new class of distribution as a solution of the GPE in section 2. Distributional properties of this is documented in section 3. Characterization of the distribution is provided in section 4. This paper ends up with some conclusions in section 5.

2 Derivation of new probability model

We give a new class of distributions as solutions of the GPE. We consider the following differential equation

$$\frac{df_X(x)}{dx} = \left(\frac{a_1 + a_2 x + a_3 x^2}{b_3 x^2 + b_4 x^3}\right) f_X(x), \tag{3}$$

which is a special case of the generalized Pearson Eq. (2) when $m = 2, n = 3$, Putting $b_3 = 1$, $b_4 = \gamma$, $a_1 = \beta\gamma$, $a_2 = \beta - \gamma + \gamma\nu$, $a_3 = \nu + \mu - 2$, $x > 0$; in (3), we have

$$\frac{1}{f_X(x)} \frac{df_X(x)}{dx} = \frac{\beta\gamma + (\beta - \gamma + \gamma\nu)x + (\mu + \nu - 2)x^2}{x^3 + \gamma x^2} = \frac{\nu - 1}{x} + \frac{\mu - 1}{x + \gamma} + \frac{\beta}{x^2},$$

where we assume that $\beta > 0, \gamma > 0, 0 < \nu < 1, 0 < \mu < 1, 1 - \mu > \nu > 0$.
Integrating the above equation, we have

$$f_X(x) = C x^{\nu - 1} (x + \gamma)^{(\mu - 1)} \exp\left(-\beta x^{-1}\right), \quad 0 < x < \infty, \tag{4}$$

Using the equation (3.471.7), Page 340 of Gradshteyn and Ryzhik [9], we easily obtain the following normalizing constant as

$$\frac{1}{C} = \beta^{(\nu-1)/2} \gamma^{\frac{\nu-1}{2} + \mu} \Gamma(1 - \mu - \nu) \exp\left(\frac{\beta}{2\gamma}\right) W_{\frac{\nu-1}{2} + \mu, \frac{-\nu}{2}}\left(\frac{\beta}{\gamma}\right), \tag{5}$$

where W(.) denotes the Whittaker function which is defined as the solution of the following differential equation

$$\frac{d^2 W}{dx^2} + \left(-\frac{1}{4} + \frac{\mu}{x} + \frac{1/4 - \mu^2}{x^2}\right) W = 0,$$

(see, for details, page 505, chapter 13, Abramowitz and Stegun [1]). Also, see Lebedev [15], and Chaudhry and Zubair [4], among others. Using the relation between Whittaker function and confluent hypergeometric function, that is,

$$W_{a,b}(z)=e^{-z/2}z^{b+1/2}\,U(b+1/2-a,1+2b,z),$$

where $U(b+1/2-a,1+2b,z)$, sometimes denoted by $\psi(b+1/2-a,1+2b,z)$, is known as the confluent hypergeometric function of Tricomi, which is another solution of Kummer's differential equation (see page 504, Abramowitz and Stegun [1]), the normalizing function C can be written as

$$\frac{1}{C}=\beta^{(\nu-1)/2}\,\gamma^{(\nu-1)/2+\mu}\,\Gamma(1-\mu-\nu)\exp\left(\frac{\beta}{2\gamma}\right)W_{(\nu-1)/2+\mu,\frac{-\nu}{2}}\left(\frac{\beta}{\gamma}\right)$$

$$=\gamma^{\mu+\nu-1}\,\Gamma(1-\mu-\nu)\,U\left(1-\mu-\nu;1-\nu;\frac{\beta}{\gamma}\right)$$

$$=\gamma^{\mu+\nu-1}\,\Gamma(1-\mu-\nu)\,\psi\left(1-\mu-\nu;1-\nu;\frac{\beta}{\gamma}\right) \tag{6}$$

Using the Kummer transformation $U(a;b;z)=z^{1-b}\,U(1+a-b;2-b;z)$, Equation 13.1.29, page 505, Abramowitz and Stegun [1], the normalizing function C in Equation (6) can easily be expressed as

$$\frac{1}{C}=\beta^{\nu}\gamma^{\mu-1}\Gamma(1-\mu-\nu)\,\psi\left(1-\mu;1+\nu;\frac{\beta}{\gamma}\right) \tag{7}$$

$$F_X(x)=C\sum_{k=0}^{\infty}\left\{\frac{(-1)^k\,\beta^{\nu+k}\,\gamma^{\mu-k-1}(1-\mu)_k\,\Gamma\left(-(\nu+k),\frac{\beta}{x}\right)}{k\,!}\right\}, \tag{8}$$

where C is the normalizing constant given by (6), $(1-\mu)_k=\frac{\Gamma(1-\mu+k)}{\Gamma(1-\mu)}$ denotes the Pochhammer symbol, and $\Gamma\left(-(\nu+k),\frac{\beta}{x}\right)$ denotes the incomplete gamma function. By direct differentiation of the c.d.f. in (8), and noting that $\frac{\partial\Gamma(a,t)}{\partial t}=-t^{a-1}e^{-t}$, it can be easily verified that $\frac{dF_X(x)}{dx}=f_X(x)$, where $f_X(x)$ denotes the p.d.f. of the random variable X as given in (4). The possible shapes of the *pdf f(x)* as given in (4) and the *cdf F(x)* as given in (8) are provided for some selected values of the parameters in the following Figures 1(a, b) and 2(a, b), respectively. The proposed distribution is right skewed and the effects of the parameters can easily be seen from these graphs.

3 Distributional properties

In this section, we derive moments for the proposed distribution. Some distributional relationships are established.

3.1 Fractional moments:

Using Eq. (4), the non-integer (or fractional) moment of order k of the proposed distribution is easily obtained as follows.

$$E\left(X^k\right)=\int_0^{\infty}x^k f(x)dx=C\int_0^{\infty}x^{\nu+k-1}(x+\gamma)^{(\mu-1)}\exp\left(-\beta x^{-1}\right),\quad 0<x<\infty.$$

Since, $\int_0^{\infty}x^{\nu-1}(x+\gamma)^{(\mu-1)}\exp\left(-\beta x^{-1}\right)=\beta^{\nu}\gamma^{\mu-2}\,\Gamma(1-\mu-\nu)\psi\left(1-\mu-\nu;1-\nu;\frac{\beta}{\gamma}\right),$

we will have

$$E\left(X^k\right)=\frac{\beta^k\,\Gamma(1-\mu-\nu-k)\,\psi\left(1-\mu;1+\nu+k;\frac{\beta}{\gamma}\right)}{\Gamma(1-\mu-\nu)\,\psi\left(1-\mu;1+\nu;\frac{\beta}{\gamma}\right)},$$

or, equivalently,

$$E\left(X^{k}\right)=\frac{\gamma^{k}\,\Gamma(1-\mu-\nu-k)\,\psi\left(1-\mu-\nu-k\,;1-\nu-k\,;\frac{\beta}{\gamma}\right)}{\Gamma(1-\mu-\nu)\,\psi\left(1-\mu-\nu;1-\nu;\frac{\beta}{\gamma}\right)},$$

where $\beta>0$, $\gamma>0$, $0<\nu<1$, $0<\mu<1$, $1-\mu>\nu>0$, $\left(\nu\neq\frac{1}{2},\mu\neq\frac{1}{2}\right)$ (at the same time). Evidently, only non-integer moments (NIM), that is, fractional moments, in $0<k<1$ exist because $k<1-\mu-\nu$. One of the earliest examples in which non-integer moments (NIM) were calculated related to the spans of random walks, but more recently the properties of non-integer moments have found application in the study of random resistor networks, chaos, and diffusion-limited aggregation, see Weiss et al. [27], and references therein. Also, see Cottone and Paola [6], and Cottone et al. [7] for recent developments on fractional moments. Fractional Moments for different parameters are given in Table 3.1 and presented in Figure 3. From Table 3.1 and Figure 3, it is observed that the fractional moments $E(X^k)$ is an increasing function in k and ν for the given values of parameters.

Table 3.1: table for fractional moments

k	ν	$\mu=0.2,\ \beta=2,\ \gamma=1$	$\nu=0.2,\ \beta=2,\ \gamma=1$
0.1			
	0.1	1.261431458	1.265137937
	0.2	1.309431046	1.309431046
	0.3	1.377644178	1.373385208
	0.4	1.485576009	1.476137246
	0.5	1.691455140	1.674866820
	0.6	2.285613328	2.254809943
0.2			
	0.1	1.651757512	1.661559089
	0.2	1.803930056	1.803930056
	0.3	2.046595138	2.033849652
	0.4	2.512785173	2.480667190
	0.5	3.866012410	3.789845477
0.3			
	0.1	2.275534118	2.295986971
	0.2	2.679875211	2.679875211
	0.3	3.461723864	3.429153427
	0.4	5.743255284	5.632500198
0.4			
	0.1	3.380478894	3.421388272
	0.2	4.532888700	4.532888700
	0.3	7.912162208	7.812192904
0.5			
	0.1	5.717928397	5.805272281
	0.2	10.36043082	10.36043082
0.6			
	0.1	13.06897336	13.31088520

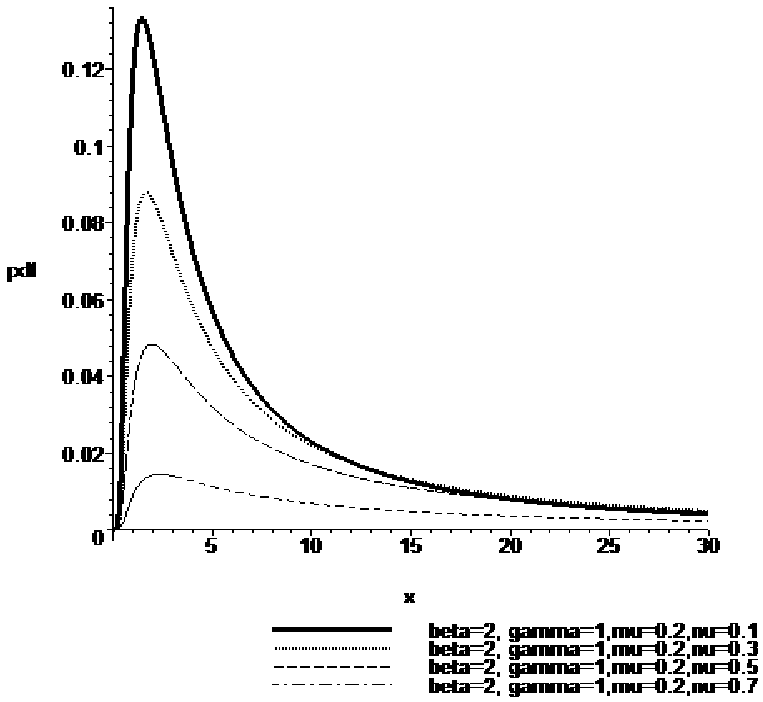

Figure 1(a): PDF for $\nu = 0.1, 0.3, 0.5, 0.7$ when $\beta = 2, \gamma = 1, \mu = 0.2$

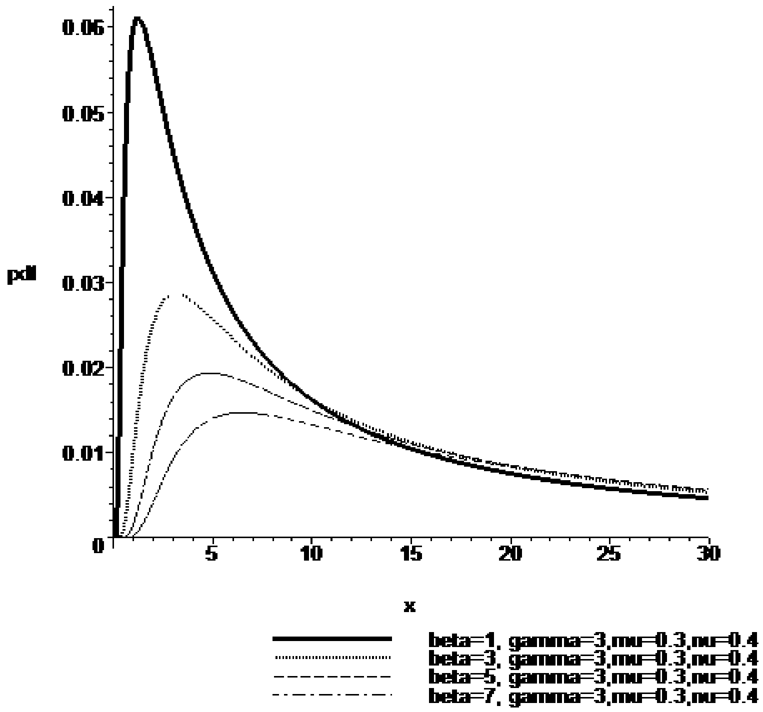

Figure 1(b): PDF for $\beta = 1, 3, 5, 7$ when $\gamma = 3, \mu = 0.3, \nu = 0.4$

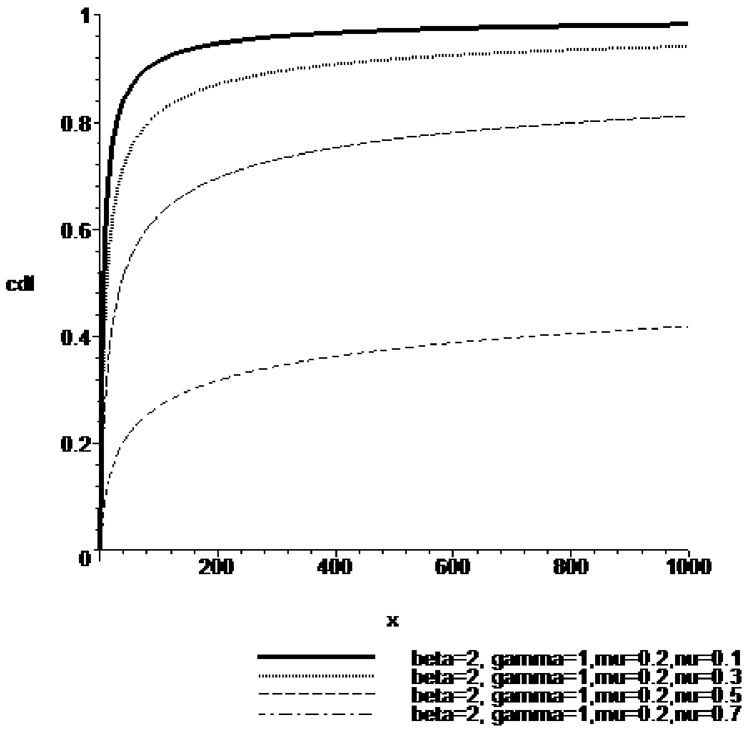

Figure 2(a): CDF for $\nu = 0.1, 0.3, 0.5, 0.7$ when $\beta = 2, \gamma = 1, \mu = 0.2$

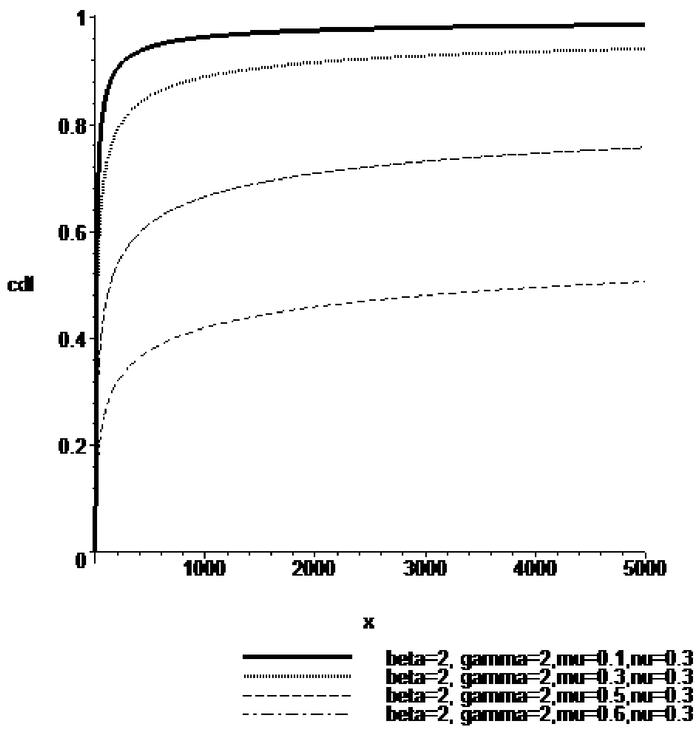

Figure 2(b): CDF for $\nu = 0.1, 0.3, 0.5, 0.6$ when $\beta = 2, \gamma = 2, \mu = 0.3$

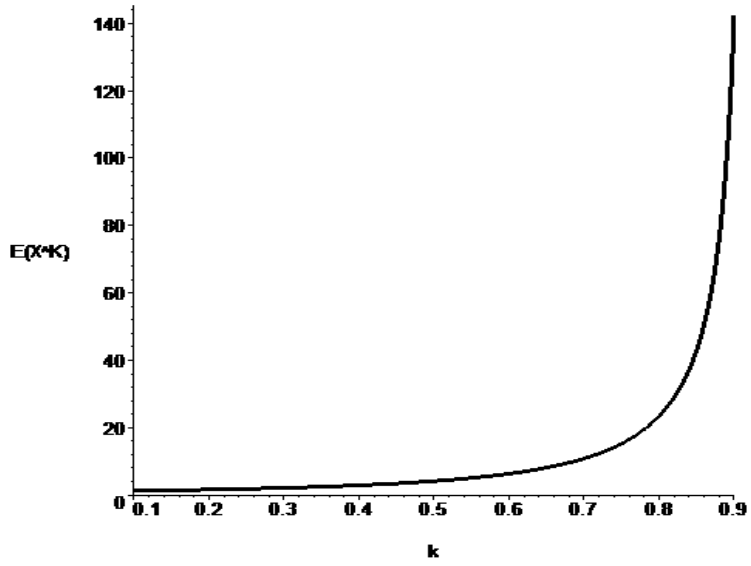

Figure 3: Plot of $E(X^k)$, when $\mu = 0.045, \nu = 0.05, \beta = 3, \gamma = 2$.

3.2 Survival and Hazard functions

The survival and hazard functions for our newly proposed distribution are respectively given by $S(x) = 1 - F_X(x)$, and $h(x) = \dfrac{f_X(x)}{1 - F_X(x)}$, where f(x) and F(x) are given by the equations (4) and (8) respectively. The possible shapes of the survival $S(x)$ and hazard rate $h(x)$ functions corresponding to the *pdf* (4) are provided for some selected values of the parameters in the following Figures 4(a) and 4(b), respectively. The effects of the parameters can easily be seen from these graphs.

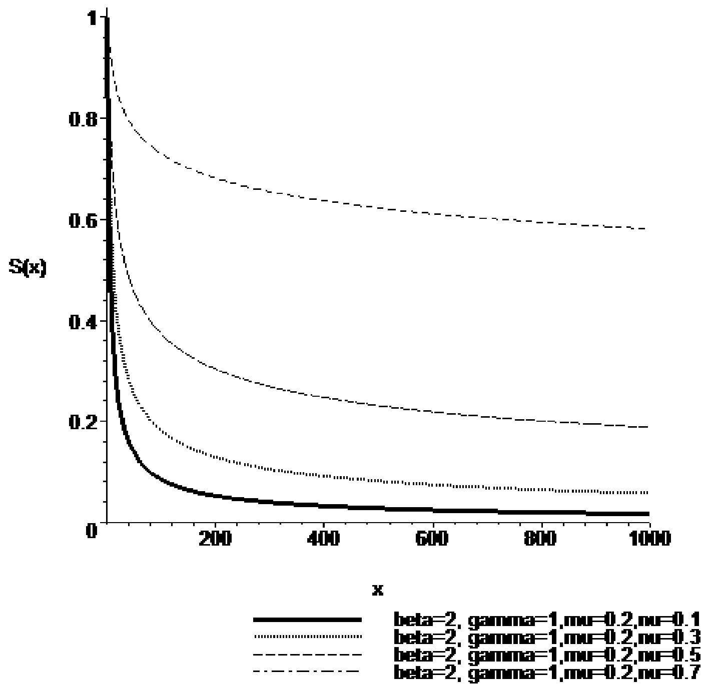

Figure 4(a) $S(x)$ for $\nu = 0.1, 0.3, 0.5, 0.7$ when $\beta = 2, \gamma = 1, \mu = 0.2$

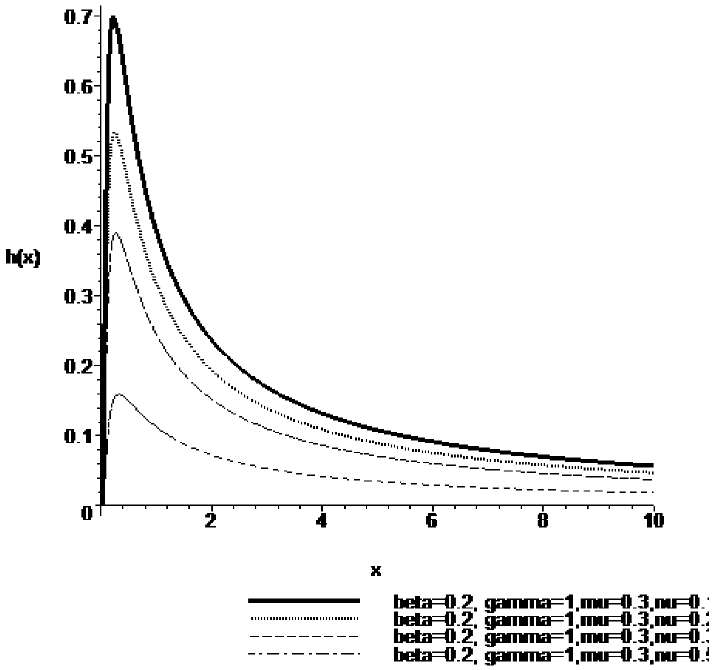

Figure 4(b) $h(x)$ for $\nu = 0.1, 0.2, 0.3, 0.5$ when $\beta = 0.2, \gamma = 1, \mu = 0.3$

3.3 Distributional relationships

The following probability distributions are a special case of the proposed distribution with the *pdf* given by (4).

Inverted Kummer-Gamma Distribution: The Kummer-gamma distribution is defined by the probability density function (see NG and Kotz [17])

$$f_X(x) = x^{\nu-1}(1+x)^{-\mu}\exp(-\beta x), \; x > 0, \beta > 0, \; \nu > 0, -\infty < \mu < \infty \quad (9)$$

Let $Y = \frac{1}{X}$, then it is easy to see from (9) that the inverted Kummer-gamma distribution is defined by the following probability density function

$$g_Y(y) = C\, y^{\mu-\nu-1}(1+y)^{-\mu}\exp\left(-\frac{\beta}{y}\right), y > 0, \beta > 0, \; \nu > 0, -\infty < \mu < \infty \quad (10)$$

where $C = \beta^{\frac{\mu-\nu-1}{2}}\Gamma(1+\nu)\exp\left(\frac{\beta}{2}\right)W_{\frac{1-\mu-\nu}{2},-\frac{\mu-\nu}{2}}(\beta)$ is the normalizing constant, and $W_{\frac{1-\mu-\nu}{2},-\frac{\mu-\nu}{2}}(\beta)$ denotes the Whittaker's Function, Gradshteyn & Ryzhik [9], Equation 3.471.7, Page 340. Comparing Equations (4) and (10), it follows that the inverted Kummer-Gamma distribution (10) is a special case of the proposed distribution (4) when $\gamma = 1$, $\beta > 0$, $0 < \nu < 1$, and $0 < \mu < 1$, in Equation (10).

4 A characterization

To prove the characterization theorem (Theorem (4.1)) we need the following Lemma**.**

Lemma 4.1 *Suppose that a non-negative random variable X has an absolutely continuous (with respect to Lebesgue measure) cumulative distribution function (CDF) F(x) and probability density function (PDF) f(x). We assume that f'(x) exits for all x and 0 < E(X) <* ∞. *Further, if*

$E(X \mid X \le t) = g(t)\tau(t)$, where $\tau(t) = \frac{f(t)}{F(t)}$, for all t > 0 and $g'(t)$ exists for all t > 0, then

$$f(x) = c e^{\int \frac{x - g'(x))}{g(x)} dx}$$, for all x > 0, where c is a constant and $c = \int_0^\infty f(x)dx$.

Proof: It is well known that

$$\text{E (X|X} \leq \text{t)} = \frac{\int_0^t xf(x)dx}{F(t)}.$$

Thus

$$\int_0^t xf(x) = g(t)f(t).$$

Differentiating both sides of the equation, we obtain

$$t f(t) = g^{/}(t) f(t) + g(t) f^{/}(t).$$

On simplification we obtain

$$\frac{f^{/}(t)}{f(t)} = \frac{t - g^{/}(t)}{g(t)}$$

Integrating the above expression, we obtain

$$f(x) = ce^{\int \frac{x - g^{/}(x)}{g(x)} dx}.$$

Remark 1: In the Lemma, the left truncated conditional expectation of X considers a product of reverse hazard rate and another function of the truncated point.

Theorem 4.1: *Suppose that a non-negative random variable X has an absolutely continuous (with respect to Lebesgue measure) cumulative distribution function F(x) and probability density function f(x). If f(x) is as given in (4) if and only if*

$$g(t) = t^{1-\nu} \gamma^{1-\mu} \exp(\beta x^{-1}) \sum_{k=0}^{\infty} \frac{(-1)^k \beta^{\nu+k+1} (1-\eta)_k \, \Gamma(-(\nu+k+1), \frac{\beta}{x})}{k!}. \tag{11}$$

Proof:

We have $g(t) = \frac{\int_0^t xf(x)dx}{f(x)}$, then substituting f(x) from (4), it is easy to show that g(x) is as given in (11).

Suppose

$$g(t) = t^{1-\nu} \gamma^{1-\mu} \exp(\beta x^{-1}) \sum_{k=0}^{\infty} \frac{(-1)^k \beta^{\nu+k+1} (1-\eta)_k \, \Gamma(-(\nu+k+1), \frac{\beta}{x})}{k!},$$

Then

$$g^{/}(t) = t - g(t)[\frac{\nu-1}{t} + \frac{\mu-1}{x+\gamma} + \frac{\beta}{x^2}].$$

Thus

$$\frac{f^{/}(t)}{f(t)} = \frac{t - g^{/}(t)}{g(t)} = [\frac{\nu-1}{t} + \frac{\mu-1}{x+\gamma} + \frac{\beta}{x^2}]$$

On integrating both sides of the above expression from 0 to x, we obtain

$$f_X(x) = c \, e^{\int \{\frac{\nu-1}{x} + \frac{\mu-1}{x+\gamma} + \frac{\beta}{x}) dx}$$

$$= cx^{\nu-1}(x+\gamma)^{\mu-1} \exp(-\frac{\beta}{x}),$$

where c is a normalizing constant such that

$$\frac{1}{c} = \int_0^{\infty} x^{\nu-1}(x+\gamma)^{\mu-1} \exp\left(-\frac{\beta}{x}\right) dx.$$

This completes the proof of Theorem 4.1.

5 Conclusion

This paper derives a new class of generalized Pearson distribution. The expressions for the PDF, CDF, and moments have been obtained. The plots for the PDF, CDF, and hazard rate and survival functions are given. Some distributional relationships and the characterization of the proposed distribution have been established. For the proposed distribution, evidently, only non-integer moments (NIM), that is, fractional moments of order k, where $0 < k < 1$, exist. We hope that the findings of the paper will be useful for researchers in different fields like economics, engineering, environmental science, finance, life testing, medical sciences, reliability theory, physical sciences, traffic data, among others, where fractional moments become required to be computed if the integer moments $k \geq 1$ do not exist. Also, it is hoped that the proposed attempt will be helpful in designing a new approach of unifying different families of distributions based on generalized Pearson differential equation.

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