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Research paper

Type II half logistic Ibrahim distribution with applications

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Abstract

In this paper, we introduce a new distribution called type II half logistic Ibrahim (TIIHLI) distribution. Some properties of this distribution will be discussed. The estimation of unknown parameters for TIIHLI distribution will be handled using real data.

Keywords: *Ibrahim Distribution; Type II Half Logistic Distributions; Marshall-Olkin Distribution; Moments, Parameter Estimation.*

1. Introduction

A new family of distributions called the type II half logistic (TIIHL) distributions is introduced and studied by Hassan et al. [3]. The cumulative distributions function (cdf) of TIIHL distributions is

$$F(x)=\frac{2\left[G(x;\xi)\right]^{\lambda}}{1+\left[G(x;\xi)\right]^{\lambda}};\quad x>0 \text{ and } \lambda>0, \tag{1}$$

Where, λ is a scale parameter and $G(x;\xi)$ is a baseline cdf, which depends on a parameter vector ζ .
The probability density function (pdf) is given by

$$f(x)=\frac{2\lambda g(x;\xi)\left[G(x;\xi)\right]^{\lambda-1}}{\left\{1+\left[G(x;\xi)\right]^{\lambda}\right\}^{2}};\quad x>0 \text{ and } \lambda>0, \tag{2}$$

Where $g(x;\xi)$ is the pdf of base distribution.
The survival function $\bar{F}(x)$, hazard rate function $h(x)$ and reverse hazard rate function $h^{*}(x)$ are, respectively, given by

$$\bar{F}(x)=\frac{1-\left[G(x;\xi)\right]^{\lambda}}{1+\left[G(x;\xi)\right]^{\lambda}};\quad x>0 \text{ and } \lambda>0, \tag{3}$$

$$h(x)=\frac{2\lambda g(x;\xi)\left[G(x;\xi)\right]^{\lambda-1}}{1-\left[G(x;\xi)\right]^{2\lambda}};\quad x>0 \text{ and } \lambda>0, \tag{4}$$

and

$$h^{*}(x)=\frac{\lambda g(x;\xi)}{G(x;\xi)\left\{1+\left[G(x;\xi)\right]^{\lambda}\right\}}=\frac{\lambda h_{1}^{*}(x)}{1+\left[G(x;\xi)\right]^{\lambda}};\quad x>0 \text{ and } \lambda>0, \tag{5}$$

Where $h_1^{*}(x)$ is the reverse hazard rate function of base distribution. More details on the family of TIIHL distributions see Hassan et al. [3].

Lemma: The TIIHL distribution at $\lambda = 1$ tends to Marshall-Olkin (M-O) distribution with $\alpha = 0.5$
Proof: Put $\lambda = 1$ in (1), we get

$$F(x) = \frac{2G(x;\xi)}{1+G(x;\xi)}. \tag{6}$$

The cdf of M-O distribution (Marshall and Olkin [4]) is

$$F(x) = \frac{G(x;\xi)}{1-\bar{\alpha}\bar{G}(x;\xi)}; \quad \bar{\alpha} = 1-\alpha. \tag{7}$$

Set $\alpha = 0.5$ in (7) and after simplified, we get (6).□
A random variable X has Ibrahim distribution with two parameters β and θ if its probability density function (pdf) is (Abdul-Moniem [1])

$$g(x) = \frac{\beta}{\theta+1}\left[e^{-\beta x} + 2\theta\beta x e^{-(\beta x)^2}\right]; \quad x \geq 0, \ (\beta \text{ and } \theta > 0). \tag{8}$$

The cdf corresponding (8) is

$$G(x) = 1 - \frac{1}{\theta+1}\left(e^{-\beta x} + \theta e^{-(\beta x)^2}\right); \quad x \geq 0, \ (\beta \text{ and } \theta > 0). \tag{9}$$

For properties of this distribution see Abdul-Moniem [1].
The rest of the paper is organized as follows. In Section 2 we introduce TIIHL distribution. The traditional moments for TIIHL distribution are given in Section 3. In Section 4 we discussed the parameters estimators using maximum likelihood method. We use a real data set to show that the TIIHL distribution can be a better model than one based on the Ibrahim distribution in Section 5.

2. Type II half logistic Ibrahim distribution

Using (1) and (9), we can define the cdf of type II half logistic Ibrahim (TIIHLI) distribution as follows

$$F(x) = \frac{2[G(x)]^{\lambda}}{1+[G(x)]^{\lambda}}; \qquad x \geq 0, \ (\theta, \beta \text{ and } \lambda > 0). \tag{10}$$

The pdf of TIIHLI distribution is

$$f(x) = \frac{2\lambda\beta\left[e^{-\beta x} + 2\theta\beta x e^{-(\beta x)^2}\right][G(x)]^{\lambda-1}}{(\theta+1)\left\{1+[G(x)]^{\lambda}\right\}^2}; \qquad x \geq 0, \ (\theta, \beta \text{ and } \lambda > 0), \tag{11}$$

The survival function $\bar{F}(x)$, hazard rate function $h(x)$ and reverse hazard rate function $h^*(x)$ corresponding (11) are, respectively, given by

$$\bar{F}(x) = \frac{1-[G(x)]^{\lambda}}{1+[G(x)]^{\lambda}}; \qquad x \geq 0, \ (\theta, \beta \text{ and } \lambda > 0). \tag{12}$$

$$h(x) = \frac{2\lambda\frac{\beta}{\theta+1}\left[e^{-\beta x} + 2\theta\beta x e^{-(\beta x)^2}\right][G(x)]^{\lambda-1}}{1-[G(x)]^{2\lambda}}; \qquad x \geq 0, \ (\theta, \beta \text{ and } \lambda > 0). \tag{13}$$

and

$$h^*(x) = \frac{\lambda\beta\left[e^{-\beta x} + 2\theta\beta x e^{-(\beta x)^2}\right][G(x)]^{-1}}{(\theta+1)\left\{1+[G(x)]^{\lambda}\right\}}; \qquad x \geq 0, \ (\theta, \beta \text{ and } \lambda > 0). \tag{14}$$

Where $G(x)$ is given by (9).
Figures 1, 2 and 3 depict the behavior of the distribution for some parameter values.

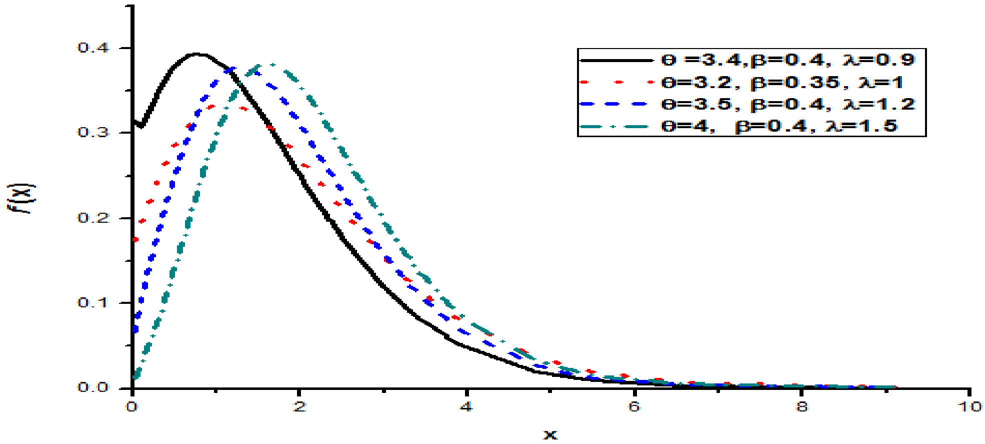

Fig. 1: Plots of PDF.

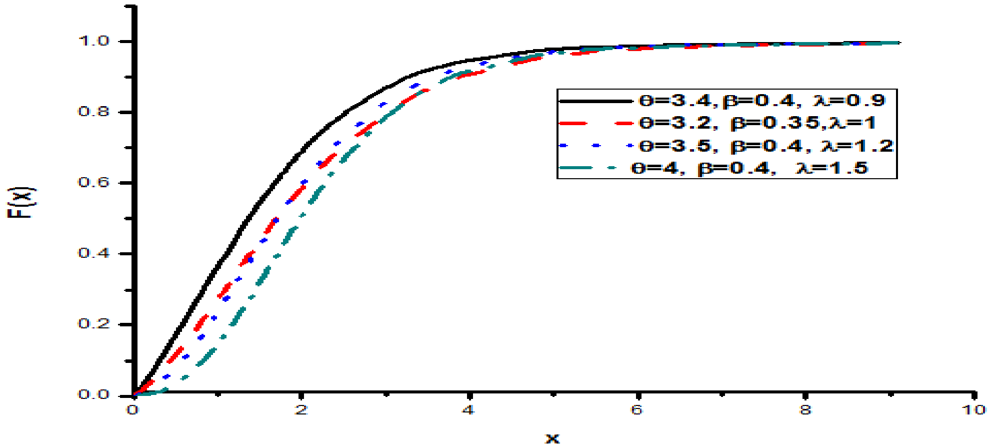

Fig. 2: Plots of CDF.

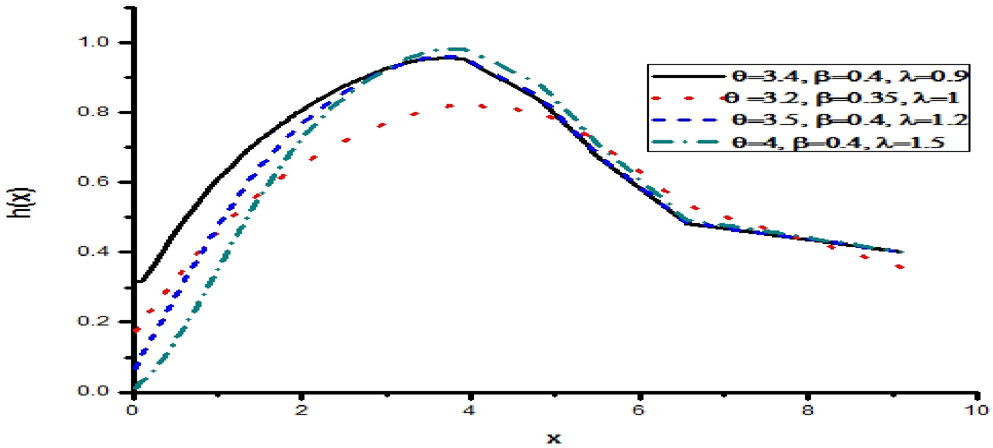

Fig. 3: Plot of HRF.

3. Statistical properties

In this section some statistical properties of TIIHLI distribution are discussed.

3.1. Moments

The r^{th} traditional moments for TIIHLI distribution is

$$\mu_r' = \frac{2\lambda\beta}{(\theta+1)}\int_0^\infty \frac{x^r\left[e^{-\beta x}+2\theta\beta x e^{-(\beta x)^2}\right]\left[1-\frac{1}{\theta+1}\left(e^{-\beta x}+\theta e^{-(\beta x)^2}\right)\right]^{\lambda-1}}{\left\{1+\left[1-\frac{1}{\theta+1}\left(e^{-\beta x}+\theta e^{-(\beta x)^2}\right)\right]^{\lambda}\right\}^2}dx$$

$$= \frac{2\lambda\beta}{(\theta+1)}\sum_{i=0}^{\infty}(-1)^i(i+1)\int_0^\infty x^r\left[e^{-\beta x}+2\theta\beta x e^{-(\beta x)^2}\right]\left[1-\frac{1}{\theta+1}\left(e^{-\beta x}+\theta e^{-(\beta x)^2}\right)\right]^{\lambda(i+1)-1}dx$$

$$= 2\lambda\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}(i+1)}{(\theta+1)^{j+1}}\binom{\lambda(i+1)-1}{j}\int_0^\infty x^r\left[e^{-\beta x}+2\theta\beta x e^{-(\beta x)^2}\right]\left(e^{-\beta x}+\theta e^{-(\beta x)^2}\right)^j dx$$

$$= 2\lambda\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^{i+j}(i+1)\theta^j}{(\theta+1)^{k+1}}\binom{j}{k}\binom{\lambda(i+1)-1}{j}$$

$$\int_0^\infty x^r\left[e^{-\beta x}+2\theta\beta x e^{-(\beta x)^2}\right]e^{-\beta(j-k)x-k(\beta x)^2}dx$$

$$= 2\lambda\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^{i+j}(i+1)\theta^j}{(\theta+1)^{j+1}}\binom{j}{k}\binom{\lambda(i+1)-1}{j}\left\{I_1+2\theta\beta I_2\right\} \tag{15}$$

Where

$$I_1=\int_0^\infty x^r e^{-\beta(j+1-k)x-k(\beta x)^2}dx$$

and

$$I_2=\int_0^\infty x^{r+1}e^{-\beta(j-k)x-(k+1)(\beta x)^2}dx$$

$$I_1=\int_0^\infty x^r e^{-\beta(j+1-k)x-k(\beta x)^2}dx$$

Let $y=\beta(j+1-k)x+k(\beta x)^2 \Rightarrow x=\dfrac{\sqrt{4ky+(j+1-k)^2}-(j+1-k)}{2\beta k}$

and $dx=\dfrac{dy}{\beta\sqrt{4ky+(j+1-k)^2}}$

$$I_1=\frac{1}{(2\beta k)^r}\int_0^\infty\left[\sqrt{4ky+(j+1-k)^2}-(j+1-k)\right]^r e^{-y}\frac{dy}{\beta\sqrt{4ky+(j+1-k)^2}}$$

$$=\frac{2k}{(2\beta k)^{r+1}}\sum_{s=0}^{r}(-1)^s\binom{r}{s}(j+1-k)^s\int_0^\infty\left[\sqrt{4ky+(j+1-k)^2}\right]^{r-s-1}e^{-y}dy$$

$$=\frac{2k}{(2\beta k)^{r+1}}\sum_{s=0}^{r}\sum_{m=0}^{\infty}(-1)^s\binom{r}{s}\binom{\frac{r-s-1}{2}}{m}(j+1-k)^{s+m}(4k)^{\frac{r-s-1}{2}-m}\int_0^\infty y^{\frac{r-s-1}{2}-m}e^{-y}dy$$

$$=\frac{2k}{(2\beta k)^{r+1}}\sum_{s=0}^{r}\sum_{m=0}^{\infty}(-1)^s\binom{r}{s}\binom{\frac{r-s-1}{2}}{m}(j+1-k)^{s+m}(4k)^{\frac{r-s-1}{2}-m}\,\Gamma\left(\frac{r-s+1}{2}-m\right) \tag{16}$$

$$I_2=\int_0^\infty x^{r+1}e^{-\beta(j-k)x-(k+1)(\beta x)^2}dx$$

Let $y=\beta(j-k)x+(k+1)(\beta x)^2 \Rightarrow x=\dfrac{\sqrt{4(k+1)y+(j-k)^2}-(j-k)}{2\beta(k+1)}$

and $dx=\dfrac{dy}{\beta\sqrt{4(k+1)y+(j-k)^2}}$

$$I_2=\frac{1}{\left[2\beta(k+1)\right]^{r+1}}\int_0^\infty\left[\sqrt{4(k+1)y+(j-k)^2}-(j-k)\right]^{r+1}e^{-y}\,\frac{dy}{\beta\sqrt{4(k+1)y+(j-k)^2}}$$

$$=\frac{1}{\beta\left[2\beta(k+1)\right]^{r+1}}\sum_{l=0}^{r+1}(-1)^l\binom{r+1}{l}(j-k)^l\int_0^\infty\left[\sqrt{4(k+1)y+(j-k)^2}\right]^{r-l}e^{-y}dy$$

$$=\frac{1}{\beta\left[2\beta(k+1)\right]^{r+1}}\sum_{l=0}^{r+1}\sum_{n=0}^{\infty}(-1)^l\binom{r+1}{l}\binom{\frac{r-l}{2}}{n}(j-k)^{l+2n}\left[4(k+1)\right]^{\frac{r-l}{2}-n}\int_0^\infty y^{\frac{r-l}{2}-n}e^{-y}dy$$

$$I_2=\frac{1}{\beta\left[2\beta(k+1)\right]^{r+1}}\sum_{l=0}^{r+1}\sum_{n=0}^{\infty}(-1)^l\binom{r+1}{l}\binom{\frac{r-l}{2}}{n}(j-k)^{l+2n}\left[4(k+1)\right]^{\frac{r-l}{2}-n}\Gamma\left(\frac{r-l}{2}-n+1\right) \tag{17}$$

Substituting from (16) and (17) in (15), we get

$$\mu_r'=2\lambda\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^{i+j}(i+1)\theta^j}{(\theta+1)^{j+1}}\binom{j}{k}\binom{\lambda(i+1)-1}{j}\left\{\frac{1}{\beta^{r+1}}\sum_{s=0}^{r}\sum_{m=0}^{\infty}(-1)^s\binom{r}{s}\binom{\frac{r-s-1}{2}}{m}\frac{(j+1-k)^{s+m}}{2^{s+1}k^{\frac{r+s+1}{2}+m}}\Gamma\left(\frac{r-s+1}{2}-m\right)\right.$$
$$\left.+\frac{\theta}{\beta^{r+1}}\sum_{l=0}^{r+1}\sum_{n=0}^{\infty}(-1)^l\binom{r+1}{l}\binom{\frac{r-l}{2}}{n}\frac{(j-k)^{l+2n}}{2^{2n+l+1}(k+1)^{\frac{r+2+l}{2}+n}}\Gamma\left(\frac{r-l}{2}-n+1\right)\right\} \tag{18}$$

3.2. The mode

We can get the mode for TIIHLI distribution by solving the following equation

$$\left\{(\theta+1)^{\lambda}+\left[\theta+1-\left(e^{-\beta x}+\theta e^{-(\beta x)^2}\right)\right]^{\lambda}\right\}\left\{(\theta+1)\left(-\beta e^{-\beta x}+2\theta\left[\beta-2(\beta x)^2\right]e^{-(\beta x)^2}\right)\right\}+(\theta+1)^{\lambda}$$

$$\left\{\beta\lambda e^{-2\beta x}+\theta\beta\left[4\beta x^2+(2\beta\lambda+\lambda-2\beta-1)x-1\right]e^{-\beta x-(\beta x)^2}+2\theta^2\left[(\lambda+1)(\beta x)^2-\beta\right]e^{-2(\beta x)^2}\right\}$$

$$+(\beta\lambda-2\lambda)e^{-2\beta x}+2\theta^2\left[(\lambda+1)(\beta x)^2-2\lambda\beta(x)^2-\beta\right]e^{-2(\beta x)^2}\Big\}+\left[\theta+1-\left(e^{-\beta x}+\theta e^{-(\beta x)^2}\right)\right]^{\lambda}$$

$$\left\{\left[4\theta(\beta x)^2+\theta\beta(2\beta\lambda-2\beta-1)x-\lambda\theta x(3\beta+2)-\theta\beta\right]e^{-\beta x-(\beta x)^2}=0\right. \tag{19}$$

3.3. The median

The median of TIIHLI distribution is the solving of the following equation

$$e^{-\beta x_{0.5}}+\theta e^{-(\beta x_{0.5})^2}=(\theta+1)\left[1-\left(\frac{1}{3}\right)^{\frac{1}{\lambda}}\right] \tag{20}$$

4. Parameters estimators

In this section, we consider maximum likelihood estimators (MLE) of TIIHLI. Let $x_1,x_2,\ldots,x_n$ be a random sample of size n from TIIHLI, then the log-likelihood function LL can be written as

$$LL\propto n\left[\ln(\lambda)+\ln(\beta)-\ln(\alpha+1)\right]+\sum_{i=1}^{n}\ln\left(e^{-\beta x_i}+2\alpha\beta x_i e^{-(\beta x_i)^2}\right)$$
$$+(\lambda-1)\sum_{i=1}^{n}\ln\left[1-\frac{1}{\alpha+1}\left(e^{-\beta x_i}+\alpha e^{-(\beta x_i)^2}\right)\right]-2\sum_{i=1}^{n}\ln\left\{1+\left[1-\frac{1}{\alpha+1}\left(e^{-\beta x_i}+\alpha e^{-(\beta x_i)^2}\right)\right]^{\lambda}\right\}$$

The normal equations become

$$\frac{\partial LL}{\partial \alpha}=\frac{-n}{\alpha+1}+\sum_{i=1}^{n}\frac{2\beta x_i e^{-(\beta x_i)^2}}{e^{-\beta x_i}+2\alpha\beta x_i e^{-(\beta x_i)^2}}+(\lambda-1)\sum_{i=1}^{n}\frac{e^{-\beta x_i}-e^{-(\beta x_i)^2}}{(\alpha+1)^2-(\alpha+1)\left(e^{-\beta x_i}+\alpha e^{-(\beta x_i)^2}\right)}-2\sum_{i=1}^{n}\frac{\lambda\left(e^{-\beta x_i}-e^{-(\beta x_i)^2}\right)\left[1-\frac{1}{\alpha+1}\left(e^{-\beta x_i}+\alpha e^{-(\beta x_i)^2}\right)\right]^{\lambda-1}}{(\alpha+1)^2+(\alpha+1)^2\left[1-\frac{1}{\alpha+1}\left(e^{-\beta x_i}+\alpha e^{-(\beta x_i)^2}\right)\right]^{\lambda}} \tag{21}$$

$$\frac{\partial LL}{\partial \beta}=\frac{n}{\beta}-\sum_{i=1}^{n}\frac{x_i e^{-\beta x_i}-2\alpha x_i\left[1-2(\beta x_i)^2\right]e^{-(\beta x_i)^2}}{e^{-\beta x_i}+2\alpha\beta x_i e^{-(\beta x_i)^2}}+(\lambda-1)\sum_{i=1}^{n}\frac{\left(x_i e^{-\beta x_i}+2\alpha\beta(x_i)^2 e^{-(\beta x_i)^2}\right)}{\alpha+1-\left(e^{-\beta x_i}+\alpha e^{-(\beta x_i)^2}\right)}$$

$$-2\sum_{i=1}^{n}\frac{\lambda\left[1-\frac{1}{\alpha+1}\left(e^{-\beta x_i}+\alpha e^{-(\beta x_i)^2}\right)\right]^{\lambda-1}\left(x_i e^{-\beta x_i}+2\alpha\beta(x_i)^2 e^{-(\beta x_i)^2}\right)}{\alpha+1+(\alpha+1)\left[1-\frac{1}{\alpha+1}\left(e^{-\beta x_i}+\alpha e^{-(\beta x_i)^2}\right)\right]^{\lambda}}, \tag{22}$$

$$\frac{\partial LL}{\partial \lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}\ln\left[1-\frac{1}{\alpha+1}\left(e^{-\beta x_i}+\alpha e^{-(\beta x_i)^2}\right)\right]-2\sum_{i=1}^{n}\frac{\left[1-\frac{1}{\alpha+1}\left(e^{-\beta x_i}+\alpha e^{-(\beta x_i)^2}\right)\right]^{\lambda}\ln\left[1-\frac{1}{\alpha+1}\left(e^{-\beta x_i}+\alpha e^{-(\beta x_i)^2}\right)\right]}{1+\left[1-\frac{1}{\alpha+1}\left(e^{-\beta x_i}+\alpha e^{-(\beta x_i)^2}\right)\right]^{\lambda}}. \tag{23}$$

The MLE of α, θ and λ can be obtain by solving the equations (21), (22), and (23) using $\frac{\partial \ln L}{\partial \alpha}=0$, $\frac{\partial \ln L}{\partial \theta}=0$ and $\frac{\partial \ln L}{\partial \lambda}=0$.

5. Application of TIIHLI distribution

In this section, we fit TIIHLI to real data set and compare the fitness with the Marshall-Olkin and Ibrahim distributions. The set of data represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal [2]. In order to compare distributions, we consider the K-S (Kolmogorov-Smirnov) statistic, – 2logL, AIC (Akaike Information Criterion), AICC (Akaike Information Criterion Corrected), BIC (Bayesian Information Criterion). The best distribution corresponds to lower K-S, – 2logL, AIC, BIC, AICC statistics value.

Table 1: Maximum-Likelihood Estimates, K-S Statistics and -2LL, AIC, AICC and BIC Values for the A Random Sample of 72 Guinea Pigs Infected with Virulent Tubercle Bacilli

Model	MLEs			Measures				
	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	K-S	-2LL	AIC	AICC	BIC
TIIHLID	0.203	0.046	0.809	0.1	503.397	509. 397	509.75	516.227
M-OID	0.399	0.054	-	0.141	505.64	509.64	509.814	514.193
ID	0.19	0.078	-	0.161	508.509	512.509	512.683	517.063

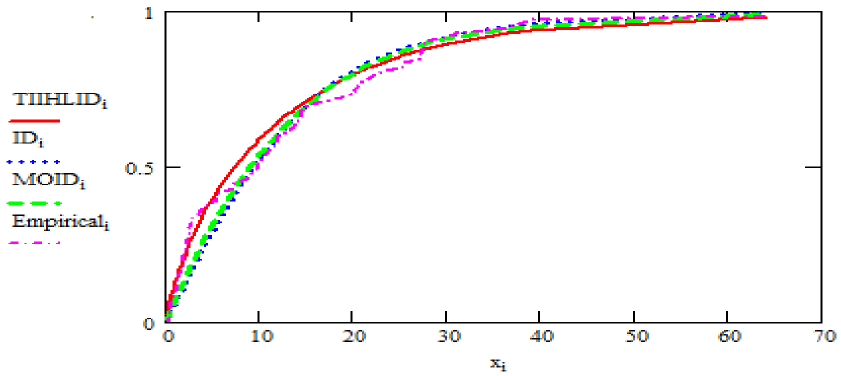

Fig. 4: Empirical, Fitted TIIHLID, ID and M-OID CDF of the Above Data.

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