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Research paper

Logarithmic ratio and product-type estimators of population mean in simple random sampling

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Abstract

Based on the natural logarithm of known population mean of an auxiliary variable, x, the study introduces logarithmic ratio and product-type estimators of the population mean of the study variable, y, in simple random sampling without replacement (SRSWOR) scheme. Part of the efficiency conditions for the proposed logarithmic estimators to be more efficient than the existing exponential ratio and product-type estimators, as well as the customary ratio and product-type estimators, is that the natural logarithm of the known population mean of the auxiliary variable, x, must be greater than 2. Generally, there is a high tendency for the proposed logarithmic estimators to be more efficient than existing customary and exponential ratio and product-type estimators when the natural logarithm of the auxiliary variable population mean is greater than 2. The theoretical results are illustrated and confirmed using some numerical datasets.

Keywords: *Auxiliary information; exponential; logarithmic; product; ratio estimators.*

1. Introduction

The use of auxiliary information to improve estimates of population parameters of the study variable is well-known in sample surveys. The ratio-type estimators are preferred when there is a highly positive linear relationship between the study and auxiliary variables described by a straight line that passes through the origin. Regression-type estimators are preferred if the straight line does not pass through the origin. When there is a highly negative linear relationship between the study and auxiliary variables with the regression line passing through the origin, the product-type estimators are most preferable. Cochran (1940) introduced the customary ratio-type estimator, while Murthy (1964) proposed the customary product-type estimator. Srivastava (1967) discussed a power transformation estimator for which the customary ratio and product-type estimators are special cases. The use of auxiliary information in form of exponential ratio and product-type estimators was initiated by Bahl and Tuteja (1991). Many other scholarly works on the use of auxiliary variables exist in literature [Kadilar and Cingi (2004), Khoshnevisan et al. (2007), Onyeka (2012), Chaun and Singh (2014), Yadav et al. (2014), Onyeka et al. (2015), Etebong, P.C. (2016), Subramani and Ajith (2016), Madhulika et al. (2017)]. The present study utilizes auxiliary information to improve estimates of the population mean of the study variable by introducing logarithmic ratio and product-type estimators.

Consider a random sample of n units drawn from a population of N units using simple random sampling without replacement (SRSWOR) method. Let y and x respectively denote the study and auxiliary variables, where the population mean ($\bar{X}$) of the auxiliary variable is assumed known. Cochran (1940) proposed the customary ratio-type estimator as:

$$\bar{y}_R = \frac{\bar{y}}{\bar{x}}\bar{X} \tag{1}$$

where $\bar{y}$ and $\bar{x}$ are sample means of the study and auxiliary variables respectively. The estimator, $\bar{y}_R$ is biased for the population mean ($\bar{Y}$) of the study variable, with its bias and mean square error obtained up to first order of approximations as:

$$Bias(\bar{y}_R) = \left(\frac{1-f}{n}\right)\left(\frac{1}{\bar{X}}\right)(RS_x^2 - S_{yx}) \tag{2}$$

$$MSE(\bar{y}_R) = \left(\frac{1-f}{n}\right)(S_y^2 + R^2S_x^2 - 2RS_{yx}) \tag{3}$$

where $S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(Y_i-\bar{Y})^2$ is the population variance of y
$S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}(X_i-\bar{X})^2$ is the population variance of x
$S_{yx} = \frac{1}{N-1}\sum_{i=1}^{N}(Y_i-\bar{Y})(X_i-\bar{X})$ is the population covariance of y and x
$f = n/N$ and $R = \bar{Y}/\bar{X}$ are sampling fraction and population ratio respectively. Murthy (1964) proposed the customary product-type estimator as:

$$\bar{y}_P = \frac{\bar{y}\bar{x}}{\bar{X}} \tag{4}$$

The estimator, $\bar{y}_P$ is biased for the population mean ($\bar{Y}$) of the study variable, with its exact bias and mean square error obtained up to first order of approximations as:

$$Bias(\bar{y}_P) = \left(\frac{1-f}{n}\right)\left(\frac{1}{\bar{X}}\right)S_{yx} \tag{5}$$

$$MSE(\bar{y}_P) = \left(\frac{1-f}{n}\right)(S_y^2 + R^2S_x^2 + 2RS_{yx}) \tag{6}$$

Bahl and Tuteja (1991) introduced exponential ratio and product-type estimators of the popoulation mean, $\bar{Y}$, respectively given by:

$$t_R = \bar{y}exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{7}$$

$$t_P = \bar{y}exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right) \tag{8}$$

The exponential ratio-type estimator, t_R is biased for $\bar{Y}$ with bias and mean square error obtained up to first order of approximations as:

$$Bias(t_R) = \left(\frac{1-f}{n}\right)\left(\frac{1}{\bar{X}}\right)\left(\frac{3}{8}RS_x^2 - \frac{1}{2}S_{yx}\right) \tag{9}$$

$$MSE(t_R) = \left(\frac{1-f}{n}\right)\left(S_y^2 + \frac{1}{4}R^2S_x^2 - RS_{yx}\right) \tag{10}$$

Similarly, the exponential product-type estimator, t_P is biased for $\bar{Y}$ with bias and mean square error obtained up to first order of approximations as:

$$Bias(t_P) = \left(\frac{1-f}{n}\right)\left(\frac{1}{\bar{X}}\right)\left(\frac{1}{2}RS_{yx} - \frac{1}{8}S_x^2\right) \tag{11}$$

$$MSE(t_P) = \left(\frac{1-f}{n}\right)\left(S_y^2 + \frac{1}{4}R^2S_x^2 + RS_{yx}\right) \tag{12}$$

The present study looks beyond the customary and exponential ratio and product-type estimators proposed by Cochran (1940), Murthy (1964) and Bahl and Tuteja (1991), all of which uses the known value of the auxiliary variable population mean. The study introduces logarithmic ratio and product-type estimators based on the natural logarithm of the known population mean of the auxiliary variable. The exponential-type estimators are known to perform better than the corresponding customary ratio and product-type estimators, in terms of having smaller mean square errors, under certain efficiency conditions. The question therefore arises as to what happens when the conditions that favor exponential-type estimators over the customary estimators are not readily satisfied. The answer, of course, lies in the use of other efficient estimators that would perform better than both the existing exponential and customary estimators. The search for such efficient estimators leads us to consider logarithmic-type estimators, knowing that logarithm is the inverse operation to exponentiation. Hence, the relevance of the present study.

2. The proposed logarithmic estimators

Based on the natural logarithm of the known population mean ($Ln(\bar{X})$) of the auxiliary variable (x), we propose logarithmic ratio and product-type estimators of the population mean ($\bar{Y}$) of the study variable (y) in simple random sampling without replacement (SRSWOR) scheme respectively as:

$$L_R = \frac{\bar{y}Ln(\bar{X})}{Ln(\bar{x})}\ ; \qquad Ln(\bar{X}) \neq 0\ ; \qquad Ln(\bar{x}) \neq 0 \tag{13}$$

$$L_P = \frac{\bar{y}Ln(\bar{x})}{Ln(\bar{X})}\ ; \qquad Ln(\bar{X}) \neq 0\ ; \qquad Ln(\bar{x}) \neq 0 \tag{14}$$

The proposed logarithmic ratio-type estimator, L_R, is constructed by taking the ratio of the sample mean, $\bar{y}$ of the study variable and the natural logarithm of the sample mean, $Ln(\bar{x})$ of the auxiliary variable and multiplying by the natural logarithm of the known population mean, $Ln(\bar{X})$ of the auxiliary variable. On the other hand, the proposed logarithmic product-type estimator, L_P, is formulated by taking the product of the sample mean, $\bar{y}$ of the study variable and the natural logarithm of the sample mean, $Ln(\bar{x})$ of the auxiliary variable and dividing by the natural logarithm of the known population mean, $Ln(\bar{X})$ of the auxiliary variable. Both estimators are defined and meaningful only if $Ln(\bar{X}) \neq 0$ *and* $Ln(\bar{x}) \neq 0$, otherwise, they are both undefined.

2.1. Properties of logarithmic ratio-type estimator

The proposed logarithmic ratio-type estimator, L_R, is biased for the study variable population mean, $\bar{Y}$. Theorem 1 gives expressions of its bias and mean square error up to first order of approximations.

Theorem 1: Let $k = 1/Ln(\bar{X})$, then the proposed logarithmic ratio-type estimator, L_R is biased for $\bar{Y}$ with its bias and mean square error obtained up to first order of approximations as:

$$Bias(L_R) = \left(\frac{1-f}{n}\right)\left(\frac{1}{\bar{X}}\right)\left(\frac{1}{2}k(1+2k)RS_x^2 - kS_{yx}\right) \tag{15}$$

$$MSE(L_R) = \left(\frac{1-f}{n}\right)(S_y^2 + k^2R^2S_x^2 - 2kRS_{yx}) \tag{16}$$

Proof: Let

$$e_0 = \frac{\bar{y}-\bar{Y}}{\bar{Y}} \quad and \quad e_1 = \frac{\bar{x}-\bar{X}}{\bar{X}} \tag{17}$$

Then

$$E(e_0) = E(e_1) = 0 \tag{18}$$

$$E(e_0^2) = V(\bar{y})/\bar{Y}^2 = \left(\frac{1-f}{n}\right)\frac{S_y^2}{\bar{Y}^2} \tag{19}$$

$$E(e_1^2) = V(\bar{x})/\bar{X}^2 = \left(\frac{1-f}{n}\right)\frac{S_x^2}{\bar{X}^2} \tag{20}$$

$$E(e_0e_1) = Cov(\bar{y},\bar{x})/\bar{Y}\bar{X} = \left(\frac{1-f}{n}\right)\frac{S_{yx}}{\bar{Y}\bar{X}} \tag{21}$$

Now, we can write the natural logarithm of the sample mean, $\bar{x}$ as:

$$Ln(\bar{x}) = Ln(\bar{X} + (\bar{x}-\bar{X})) = Ln(\bar{X}) + Ln(1+e_1) \tag{22}$$

Assuming that $|e_1| < 1$, which is trivial and easily satisfied in most practical surveys, we can expand $Ln(\bar{x})$ up to first order of approximations in expected value as:

$$Ln(\bar{x}) = Ln(\bar{X}) + (e_1 - e_1^2/2 + e_1^3/3 - e_1^4/4 + \ldots) \approx Ln(\bar{X})(1 + ke_1 - ke_1^2/2) \tag{23}$$

Dividing both sides of (23) by $Ln(\bar{X})$ gives

$$\frac{Ln(\bar{x})}{Ln(\bar{X})} \approx (1 + ke_1 - ke_1^2/2) \tag{24}$$

Taking the reciprocal of (24) and expanding up to first order of approximations in expected values gives:

$$\frac{Ln(\bar{X})}{Ln(\bar{x})} \approx (1 + ke_1 - ke_1^2/2)^{-1} \approx 1 - ke_1 + k(1+2k)e_1^2/2 \tag{25}$$

Using (13), (17) and (25), we rewrite the proposed logarithmic ratio-type estimator, L_R up to first order of approximations in expected value as:

$$L_R \approx \bar{Y}(1+e_0)(1 - ke_1 + k(1+2k)e_1^2/2) = \bar{Y}(1 + e_0 - ke_1 - ke_0e_1 + k(1+2k)e_1^2/2) \tag{26}$$

Or

$$(L_R - \bar{Y}) \approx \bar{Y}(e_0 - ke_1 - ke_0e_1 + k(1+2k)e_1^2/2) \tag{27}$$

Taking the expectation of both sides of (27) and using (18) - (21) to make the necessary substitutions gives the approximate bias of L_R as:

$$Bias(L_R) = E(L_R - \bar{Y}) \approx \bar{Y}\{E(e_0) - kE(e_1) - kE(e_0e_1) + k(1+2k)E(e_1^2)/2)\}$$

Or $Bias(L_R) = \left(\frac{1-f}{n}\right)\left(\frac{1}{\bar{X}}\right)\left(\frac{1}{2}k(1+2k)RS_x^2 - kS_{yx}\right)$ as stated in the theorem.
To obtain the mean square error (MSE) of L_R, we first square both sides of (27) and expand up to first order of approximations in expected value to obtain:

$$(L_R - \bar{Y})^2 \approx \bar{Y}^2\{e_0 - ke_1 - ke_0e_1 + k(1+2k)e_1^2/2)\}^2 = \bar{Y}^2(e_0^2 + k^2e_1^2 - 2ke_0e_1) \tag{28}$$

Taking the expectation of both sides of (28) and using (18) - (21) to make the necessary substitutions gives the approximate mean square error of L_R as:

$$MSE(L_R) = E(L_R - \bar{Y})^2 \approx \bar{Y}^2\{E(e_0^2) + k^2E(e_1^2) - 2kE(e_0e_1)\}$$

Or $MSE(L_R) = \left(\frac{1-f}{n}\right)(S_y^2 + k^2R^2S_x^2 - 2kRS_{yx})$ as stated in the theorem. This completes the proof.

Remark 1: The proposed logarithmic ratio-type estimator, L_R, would be as efficient as the customary ratio-type estimator, $\bar{y}_R$, in terms of having the same mean square error if:

$$k = 1 \; or \; 1/Ln(\bar{X}) = 1 \; or \; Ln(\bar{X}) = 1 \; or \; \bar{X} = exp(1) \approx 2.718 \tag{29}$$

Remark 2: The proposed logarithmic ratio-type estimator, L_R, would be as efficient as the exponential ratio-type estimator, t_R, in terms of having the same mean square error if:

$$k = 1/2 \; or \; 1/Ln(\bar{X}) = 1/2 \; or \; Ln(\bar{X}) = 2 \; or \; \bar{X} = exp(2) \approx 7.389 \tag{30}$$

2.2. Properties of logarithmic product-type estimator

The proposed logarithmic product-type estimator, L_P, is biased for the study variable population mean, $\bar{Y}$. Theorem 2 gives expressions of its bias and mean square error up to first order of approximations.

Theorem 2: Let $k = 1/Ln(\bar{X})$, then the proposed logarithmic product-type estimator, L_P is biased for $\bar{Y}$ with its bias and mean square error obtained up to first order of approximations as:

$$Bias(L_P) = \left(\frac{1-f}{n}\right)\left(\frac{k}{\bar{X}}\right)\left(S_{yx} - \frac{1}{2}RS_x^2\right) \tag{31}$$

$$MSE(L_P) = \left(\frac{1-f}{n}\right)(S_y^2 + k^2R^2S_x^2 + 2kRS_{yx}) \tag{32}$$

Proof: Using (14), (17) and (24), we rewrite the proposed logarithmic product-type estimator, L_P, up to first order of approximations in expected value as:

$$L_P \approx \bar{Y}(1+e_0)(1+ke_1 - ke_1^2/2) = \bar{Y}(1 + e_0 + ke_1 + ke_0e_1 - ke_1^2/2) \tag{33}$$

Or

$$(L_P - \bar{Y}) \approx \bar{Y}(e_0 + ke_1 + ke_0e_1 - ke_1^2/2) \tag{34}$$

Taking the expectation of both sides of (34) and using (18) - (21) to make the necessary substitutions gives the approximate bias of L_P as:

$$Bias(L_P) = E(L_P - \bar{Y}) \approx \bar{Y}\{E(e_0) + kE(e_1) + kE(e_0e_1) - kE(e_1^2)/2)\}$$

Or $Bias(L_P) = \left(\frac{1-f}{n}\right)\left(\frac{k}{\bar{X}}\right)\left(S_{yx} - \frac{1}{2}RS_x^2\right)$ as stated in the theorem.
To obtain the mean square error (MSE) of L_P, we first square both sides of (34) and expand up to first order of approximations in expected value to obtain:

$$(L_P - \bar{Y})^2 \approx \bar{Y}^2\{e_0 + ke_1 + ke_0e_1 - ke_1^2/2)\}^2 = \bar{Y}^2(e_0^2 + k^2e_1^2 + 2ke_0e_1) \tag{35}$$

Taking the expectation of both sides of (35) and using (18) - (21) to make the necessary substitutions gives the approximate mean square error of L_P as:

$$MSE(L_P) = E(L_P - \bar{Y})^2 \approx \bar{Y}^2\{E(e_0^2) + k^2E(e_1^2) + 2kE(e_0e_1)\}$$

Or $MSE(L_P) = \left(\frac{1-f}{n}\right)(S_y^2 + k^2R^2S_x^2 + 2kRS_{yx})$ as stated in the theorem. This completes the proof.

Remark 3: The proposed logarithmic product-type estimator, L_P, would be as efficient as the customary product-type estimator, $\bar{y}_P$, in terms of having the same mean square error if:

$$k = 1 \; or \; 1/Ln(\bar{X}) = 1 \; or \; Ln(\bar{X}) = 1 \; or \; \bar{X} = exp(1) \approx 2.718 \tag{36}$$

Remark 4: The proposed logarithmic product-type estimator, L_P, would be as efficient as the exponential product-type estimator, t_P, in terms of having the same mean square error if:

$$k = 1/2 \; or \; 1/Ln(\bar{X}) = 1/2 \; or \; Ln(\bar{X}) = 2 \; or \; \bar{X} = exp(2) \approx 7.389 \tag{37}$$

3. Efficiency comparison

Remarks 1 and 2 stated the conditions under which the proposed logarithmic ratio-type estimator would be as efficient as the existing customary and exponential ratio-type estimators. Similarly, Remarks 3 and 4 stated the conditions under which the proposed logarithmic product-type estimator would be as efficient as the existing customary and exponential product-type estimators. In this section, we establish conditions under which the proposed logarithmic estimators would be more efficient than the existing customary and exponential ratio and product-type estimators, in terms of having smaller mean square errors. It is interesting to observe that the efficiencies of the proposed logarithmic estimators are affected by the natural logarithm (and the value) of the known population mean of the auxiliary variable.

3.1. Logarithmic versus customary ratio-type estimators

The efficiency conditions of the logarithmic ratio-type estimator over the customary ratio-type estimator are as follows:
Theorem 3: The proposed logarithmic ratio-type estimator, L_R is more efficient than the customary ratio-type estimator, $\bar{y}_R$, in terms of having a smaller mean square error, if:

$$Ln(\bar{X}) > 1 \; [or\ \bar{X} > exp(1) \approx 2.718] \quad and \quad \frac{\beta_{yx}}{R} < \frac{1}{2}\left(\frac{1+Ln(\bar{X})}{Ln(\bar{X})}\right) \tag{38}$$

Proof: Using (3) and (16), the proposed logarithmic ratio-type estimator, L_R would be more efficient than the customary ratio-type estimator, $\bar{y}_R$, if:

$\Delta_1 = MSE(\bar{y}_R) - MSE(L_R) > 0$

Or $(1-k^2)R^2S_x^2 - 2(1-k)RS_{yx} > 0$
Or $(1-k)[(1+k) - 2\beta_{yx}/R] > 0$
Or $(1-k) > 0 \quad and \quad [(1+k) - 2\beta_{yx}/R] > 0$
Or $k < 1 \quad and \quad \frac{\beta_{yx}}{R} < \frac{1}{2}(1+k)$
Or $Ln(\bar{X}) > 1 \; [or\ \bar{X} > exp(1) \approx 2.718] \quad and \quad \frac{\beta_{yx}}{R} < \frac{1}{2}\left(\frac{1+Ln(\bar{X})}{Ln(\bar{X})}\right)$, since $k = 1/Ln(\bar{X})$. This completes the proof.

3.2. Logarithmic versus exponential ratio-type estimators

The efficiency conditions of the logarithmic ratio-type estimator over the exponential ratio-type estimator are as stated in the following theorem.
Theorem 4: The proposed logarithmic ratio-type estimator, L_R is more efficient than the exponential ratio-type estimator, t_R, as well as the customary ratio-type estimator, $\bar{y}_R$, in terms of having a smaller mean square error, if:

$$Ln(\bar{X}) > 2 \; [or\ \bar{X} > exp(2) \approx 7.389] \quad and \quad \frac{\beta_{yx}}{R} < \frac{1}{4}\left(\frac{2+Ln(\bar{X})}{Ln(\bar{X})}\right) \tag{39}$$

Proof: Using (10) and (16), and following similar procedure as in the proof of Theorem 3, the proposed logarithmic ratio-type estimator, L_R would be more efficient than the exponential ratio-type estimator, t_R, if:

$\Delta_2 = MSE(t_R) - MSE(L_R) > 0$

Or $(1-4k^2)R^2S_x^2 - 4(1-2k)RS_{yx} > 0$
Or $(1-2k)[(1+2k) - 4\beta_{yx}/R] > 0$
Or $(1-2k) > 0 \quad and \quad [(1+2k) - 4\beta_{yx}/R] > 0$
Or $k < \frac{1}{2} \quad and \quad \frac{\beta_{yx}}{R} < \frac{1}{4}(1+2k)$
Or $Ln(\bar{X}) > 2 \; [or\ \bar{X} > exp(2) \approx 7.389] \quad and \quad \frac{\beta_{yx}}{R} < \frac{1}{4}\left(\frac{2+Ln(\bar{X})}{Ln(\bar{X})}\right)$, since $k = 1/Ln(\bar{X})$. Hence the efficiency condition in (39), which implies that:

$\bar{X} > exp(2) \approx 7.389 > exp(1) \approx 2.718 \; or \; \bar{X} > exp(1) \approx 2.718$

and

$$\frac{\beta_{yx}}{R} < \frac{1}{4}\left(\frac{2+Ln(\bar{X})}{Ln(\bar{X})}\right) < \frac{1}{4}\left(\frac{2+Ln(\bar{X})}{Ln(\bar{X})}\right) + \frac{1}{4} = \frac{1}{2}\left(\frac{1+Ln(\bar{X})}{Ln(\bar{X})}\right)$$

Or $\bar{X} > exp(1) \approx 2.718 \; and \; \frac{\beta_{yx}}{R} < \frac{1}{2}\left(\frac{1+Ln(\bar{X})}{Ln(\bar{X})}\right)$ as stated in (38). Consequently, (39) implies (38), indicating that under (39), the proposed logarithmic ratio-type estimator, L_R, is more efficient than the exponential ratio-type estimator, t_R, as well as the customary ratio-type estimator, $\bar{y}_R$, in terms of having a smaller mean square error, as stated in the theorem. This completes the proof.

3.3. Logarithmic versus customary product-type estimators

The efficiency conditions of the logarithmic product-type estimator over the customary product-type estimator are as follows:

Theorem 5: The proposed logarithmic product-type estimator, L_P is more efficient than the customary product-type estimator, $\bar{y}_P$, in terms of having a smaller mean square error, if:

$$Ln(\bar{X}) > 1 \; [or\ \bar{X} > exp(1) \approx 2.718] \quad and \quad \frac{\beta_{yx}}{R} > -\frac{1}{2}\left(\frac{1+Ln(\bar{X})}{Ln(\bar{X})}\right) \tag{40}$$

Theorem 5 can be proved by using (6) and (32), and following similar procedure as in the proof of Theorem 3.

3.4. Logarithmic versus exponential product-type estimators

The efficiency conditions of the logarithmic product-type estimator over the exponential product-type estimator are as stated in the following theorem.

Theorem 6: The proposed logarithmic product-type estimator, L_P is more efficient than the exponential product-type estimator, t_P, as well as the customary product-type estimator, $\bar{y}_P$, in terms of having a smaller mean square error, if:

$$Ln(\bar{X}) > 2 \; [or\ \bar{X} > exp(2) \approx 7.389] \quad and \quad \frac{\beta_{yx}}{R} > -\frac{1}{4}\left(\frac{2+Ln(\bar{X})}{Ln(\bar{X})}\right) \tag{41}$$

Theorem 6 can be proved by using (12) and (32), and following similar procedure as in the proof of Theorem 4.

3.5. Logarithmic versus customary and exponential estimators

The proposed logarithmic ratio and product-type estimators are intended to particularly provide alternative efficient estimators when the efficiency conditions of the exponential estimators over the customary ratio and product-type estimators are not readily satisfied. For this to happen, it means that the exponential estimators, in the first place, would be less efficient than the customary estimators, which in turn would be less efficient than the proposed logarithmic estimators. In other words, we seek for situations where the customary estimators are more efficient than the exponential estimators but less efficient than the proposed logarithmic estimators. In this way, the proposed logarithmic estimators would be more efficient than both the customary and exponential estimators and serve as alternative efficient estimators over the customary estimators when the efficiency conditions of the exponential estimators over the customary estimators are not satisfied. The relevant results for the proposed logarithmic ratio and product-type estimators are respectively given in theorems 7 and 8.

Theorem 7: When the exponential ratio-type estimator, t_R is less efficient than the customary ratio-type estimator, $\bar{y}_R$, the proposed logarithmic ratio-type estimator, L_R would be more efficient than the customary ratio-type estimator, $\bar{y}_R$, as well as the exponential ratio-type estimator, t_R, in terms of having the smallest mean square error, if:

$$\frac{\beta_{yx}}{R} > \frac{3}{4} \;\; and \;\; 1 < Ln(\bar{X}) < 2 \; [or\ exp(1) \approx 2.718 < \bar{X} < exp(2) \approx 7.389] \tag{42}$$

Proof: Using (3) and (10), the customary ratio-type estimator, $\bar{y}_R$ would be more efficient than the exponential ratio-type estimator, t_R, if: $\Delta_3 = MSE(t_R) - MSE(\bar{y}_R) > 0$ Or $-\frac{3}{4}R^2S_x^2 + RS_{yx} > 0$ Or

$$\frac{\beta_{yx}}{R} > \frac{3}{4} \tag{43}$$

Using (43) in (38) gives $\bar{X} > exp(1) \approx 2.718$ *and* $\bar{X} < exp(2) \approx 7.389$ Or

$$1 < Ln(\bar{X}) < 2 \; [or\ exp(1) \approx 2.718 < \bar{X} < exp(2) \approx 7.389] \tag{44}$$

This completes the proof.

Theorem 8: When the exponential product-type estimator, t_P is less efficient than the customary product-type estimator, $\bar{y}_P$, the proposed logarithmic product-type estimator, L_P would be more efficient than the customary product-type estimator, $\bar{y}_P$, as well as the exponential product-type estimator, t_P, in terms of having the smallest mean square error, if:

$$\frac{\beta_{yx}}{R} < -\frac{3}{4} \;\; and \;\; 1 < Ln(\bar{X}) < 2 \; [or\ exp(1) \approx 2.718 < \bar{X} < exp(2) \approx 7.389] \tag{45}$$

Theorem 8 can be proved by following similar procedure as in the proof of theorem 7.

Table 1: Percentage Relative Efficiencies of Logarithmic (PRE3) and Exponential (PRE2) Estimators over Customary Ratio-type Estimator

SN	$\bar{X}$	$MSE(\bar{y}_R)$	$MSE(t_R)$	$MSE(L_R)$	PRE1	PRE2	PRE3
1	2.0	34935.74	8680.63	72855.09	100	402.5	48.0
2	2.5	22324.08	5539.23	26606.59	100	403.0	83.9
3	2.718	18873.93	4680.39	18877.86	100	403.3	100.0
4	3.0	15478.99	3835.63	12813.32	100	403.6	120.8
5	3.5	11355.03	2810.13	7216.07	100	404.1	157.4
6	4.0	8680.63	2145.64	4496.49	100	404.6	193.1
7	4.5	6848.60	1690.83	3007.71	100	405.0	227.7
8	5.0	5539.23	1366.05	2120.28	100	405.5	261.3
9	5.5	4571.25	1126.14	1556.37	100	405.9	293.7
10	6.0	3835.63	943.98	1179.74	100	406.3	325.1
11	6.5	3263.62	802.46	917.93	100	406.7	355.5
12	7.0	2810.13	690.35	729.88	100	407.1	385.0
13	7.389	2519.33	618.52	618.52	100	407.3	407.3
14	7.5	2444.57	600.06	591.07	100	407.4	413.6
15	8.0	2145.64	526.28	486.21	100	407.7	441.3
16	8.5	1898.10	465.24	405.40	100	408.0	468.2
17	9.0	1690.83	414.18	342.05	100	408.2	494.3
18	9.5	1515.56	371.04	291.62	100	408.5	519.7
19	10.0	1366.05	334.26	250.95	100	408.7	544.3
20	10.5	1237.48	302.67	217.76	100	408.9	568.3
21	11.0	1126.14	275.33	190.37	100	409.0	591.5
22	11.5	1029.09	251.53	167.57	100	409.1	614.1
23	12.0	943.98	230.67	148.41	100	409.2	636.0
24	12.5	868.95	212.29	132.20	100	409.3	657.3
25	13.0	802.46	196.02	118.37	100	409.4	677.9
26	13.5	743.27	181.55	106.50	100	409.4	697.9
27	14.0	690.35	168.63	96.25	100	409.4	717.2

4. Numerical illustration

The proposed logarithmic estimators, like many univariate estimators, are based on known value of the population mean ($\bar{X}$) of the auxiliary variable. The estimators, precisely, use information on the natural logarithm of the known population mean of the auxiliary variable. Theoretical results obtained in the present study indicate that efficiency conditions of logarithmic estimators over the customary ratio and product-type estimators are affected when $Ln(\bar{X}) > 1$ *or* $\bar{X} > exp(1) \approx 2.718$, while for the exponential ratio and product-type estimators, the efficiency conditions of the logarithmic estimators are affected when $Ln(\bar{X}) > 2$ *or* $\bar{X} > exp(2) \approx 7.389$. To verify the theoretical results using numerical illustration, consider the following two datasets for values of the known auxiliary variable population mean greater than or equal to 2.0. Dataset 1 involves positively correlated study and auxiliary variables for which ratio-type estimators are appropriate, while Dataset 2 involves negatively correlated study and auxiliary variables for which product-type estimators are appropriate.

Dataset 1: Positively correlated study/auxiliary variables ($\rho_{yx} = 0.26$)

$N = 31,\ n = 10,\ S_y^2 = 83.9806,\ \bar{Y} = 173.77,\ S_x^2 = 28.3793,\ S_{yx} = 19.5731, \bar{X} \geq 2.0$

Dataset 2: Negatively correlated study/auxiliary variables ($\rho_{yx} = -0.24$)

$N = 31,\ n = 10,\ S_y^2 = 83.9806,\ \bar{Y} = 173.77,\ S_x^2 = 68.7978,\ S_{yx} = -11.5575, \bar{X} \geq 2.0$

We use Dataset 1 (positively correlated study/auxiliary variables) for comparing the efficiencies of the ratio-type customary, exponential and proposed logarithmic estimators, while Dataset 2 (negatively correlated study/auxiliary variables) is for comparing the efficiencies of the product-type customary, exponential and proposed logarithmic estimators. Table 1 shows the percentage relative efficiencies of the logarithmic ratio-type estimator (PRE3) and exponential ratio-type estimator (PRE2) over the customary ratio-type estimator. Similarly, Table 2 shows the percentage relative efficiencies of the logarithmic product-type estimator (PRE3) and exponential product-type estimator (PRE2) over the customary product-type estimator.

Table 1 indicates that the proposed logarithmic ratio-type estimator, L_R is more efficient than both the customary ($\bar{y}_R$) and exponential (t_R) ratio-type estimators for cases 14-27. It is easy to verify that (39) holds for these cases, that is, $\bar{X} > exp(2) \approx 7.389$ *and* $\frac{\beta_{yx}}{R} < \frac{1}{4}\left(\frac{2+Ln(\bar{X})}{Ln(\bar{X})}\right)$. For cases 4-12 of table 1, the proposed logarithmic ratio-type estimator, L_R is more efficient than the customary ratio-type estimator, $\bar{y}_R$ but less efficient the exponential ratio-type estimator, t_R when $exp(1) \approx 2.718 < \bar{X} < exp(2) \approx 7.389$ and $\frac{\beta_{yx}}{R} < \frac{1}{2}\left(\frac{1+Ln(\bar{X})}{Ln(\bar{X})}\right)$. Furthermore, the numerical illustration using Dataset 1, with results shown in table 1, confirms that the proposed logarithmic ratio-type estimator, L_R is as efficient as the customary ratio-type estimator, $\bar{y}_R$ when $\bar{X} = exp(1) \approx 2.718$ and also as efficient as the exponential ratio-type estimator, t_R when $\bar{X} = exp(2) \approx 7.389$.

Table 2: Percentage Relative Efficiencies of Logarithmic (PRE3) and Exponential (PRE2) Estimators ove Customary Ratio-type Estimator

SN	$\bar{X}$	$MSE(\bar{y}_P)$	$MSE(t_P)$	$MSE(L_P)$	PRE1	PRE2	PRE3
1	2.0	14373.46	3563.64	29997.16	100	403.3	47.9
2	2.5	9179.31	2271.90	10942.86	100	404.0	83.9
3	2.718	7758.74	1918.94	7760.35	100	404.3	100.0
4	3.0	6361.15	1571.89	5264.01	100	404.7	120.8
5	3.5	4663.91	1150.82	2961.30	100	405.3	157.5
6	4.0	3563.64	878.18	1843.37	100	405.8	193.3
7	4.5	2810.19	691.71	1231.92	100	406.3	228.1
8	5.0	2271.90	558.64	867.78	100	406.7	261.8
9	5.5	1874.09	460.43	636.61	100	407.0	294.4
10	6.0	1571.89	385.91	482.36	100	407.3	325.9
11	6.5	1336.99	328.05	375.25	100	407.6	356.3
12	7.0	1150.82	282.26	298.40	100	407.7	385.7
13	7.389	1031.48	252.93	252.94	100	407.8	407.8
14	7.5	1000.81	245.40	241.74	100	407.8	414.0
15	8.0	878.18	215.31	198.98	100	407.9	441.3
16	8.5	776.67	190.43	166.06	100	407.8	467.7
17	9.0	691.71	169.64	140.29	100	407.8	493.1
18	9.5	619.89	152.08	119.80	100	407.6	517.4
19	10.0	558.64	137.13	103.30	100	407.4	540.8
20	10.5	506.00	124.29	89.85	100	407.1	563.2
21	11.0	460.43	113.19	78.76	100	406.8	584.6
22	11.5	420.72	103.53	69.54	100	406.4	605.0
23	12.0	385.91	95.08	61.81	100	405.9	624.3
24	12.5	355.23	87.63	55.28	100	405.4	642.6
25	13.0	328.05	81.05	49.71	100	404.8	659.9
26	13.5	303.87	75.20	44.94	100	404.1	676.2
27	14.0	282.26	69.97	40.83	100	403.4	691.4

Table 2 indicates that the proposed logarithmic product-type estimator, L_P is more efficient than both the customary ($\bar{y}_P$) and exponential (t_P) product-type estimators for cases 14-27. It can be easily verified that (41) holds for these cases, that is, $\bar{X} > exp(2) \approx 7.389$ *and* $\frac{\beta_{yx}}{R} > -\frac{1}{4}\left(\frac{2+Ln(\bar{X})}{Ln(\bar{X})}\right)$. For cases 4-12 of table 2, the proposed logarithmic product-type estimator, L_P is more efficient than the customary product-type estimator, $\bar{y}_P$ but less efficient the exponential product-type estimator, t_P when $exp(1) \approx 2.718 < \bar{X} < exp(2) \approx 7.389$ and $\frac{\beta_{yx}}{R} > -\frac{1}{2}\left(\frac{1+Ln(\bar{X})}{Ln(\bar{X})}\right)$. Furthermore, the numerical illustration using Dataset 2, with results shown in table 2, confirms that the proposed logarithmic product-type estimator, L_P is as efficient as the customary product-type estimator, $\bar{y}_P$ when $\bar{X} = exp(1) \approx 2.718$ and also as efficient as the exponential product-type estimator, t_P when $\bar{X} = exp(2) \approx 7.389$.

5. Conclusion

The study introduced and discussed the use of logarithmic-type estimators of the population mean ($\bar{Y}$) of the study variable, y in simple random sampling scheme, using information on the natural logarithm of the known population mean ($Ln(\bar{X})$) of an auxiliary variable, x. Logarithmic ratio-type estimator was proposed for situations where the study and auxiliary variables are positively correlated, while logarithmic product-type estimator was proposed for situations where the study and auxiliary variables are negatively correlated. Properties of the proposed logarithmic estimators, including efficiency conditions over the existing customary and exponential ratio and product-type estimators, were obtained up to first order of approximations.

Both theoretical and numerical results revealed that part of the efficiency conditions for the proposed logarithmic estimators to perform better than existing customary ratio and product-type estimators is that the natural logarithm of $\bar{X}$ must be greater than unity, that is, $Ln(\bar{X}) > 1$ or $\bar{X} > exp(1) \approx 2.718$. Similarly for the proposed logarithmic estimators to be more efficient than the exponential ratio and product-type estimators, as well as the customary ratio and product-type estimators, part of the efficiency conditions is that $Ln(\bar{X}) > 2$ or $\bar{X} > exp(2) \approx 7.389$. Consequently, there is a high tendency for the proposed logarithmic estimators to be more efficient than existing customary and exponential ratio and product-type estimators when $Ln(\bar{X}) > 2$ or $\bar{X} > exp(2) \approx 7.389$. This is not to say that the proposed logarithmic estimators would be more efficient than the customary and exponential-type estimators once $Ln(\bar{X}) > 2$ or $\bar{X} > exp(2) \approx 7.389$. Additional conditions stated in (39) and (41) must also be satisfied.

The proposed logarithmic estimators are therefore recommended for consideration for use over the customary ratio and product-type estimators when $\bar{X} > exp(1) \approx 2.718$. Similarly, the proposed logarithmic estimators are recommended for consideration for use over the exponential ratio and product-type estimators when $\bar{X} > exp(2) \approx 7.389$. However, the efficiencies of the proposed logarithmic estimators over the customary ratio and product-type estimators are guaranteed only when (38) and (40) are respectively satisfied. Also, the efficiencies of the proposed logarithmic estimators over the exponential ratio and product-type estimators are guaranteed only when (39) and (41) are respectively satisfied.

Among the most important results of the study are (42) and (45), which established the conditions under which the proposed logarithmic estimators would serve as alternative efficient estimators when the existing exponential estimators are less efficient than the corresponding customary estimators. Under these conditions, the exponential estimators would be less efficient than the customary estimators, which, in turn, would be less efficient than the proposed logarithmic estimators. Finally, the study also revealed the conditions under which the proposed logarithmic estimators would be as efficient as the customary and exponential ratio and product-type estimators. The theoretical results were verified and confirmed using numerical illustrations with two datasets.

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