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Research paper

Alpha power transformed quasi lindley distribution

Unyime Patrick Udoudo [1]*, Ette Harrison Etuk [2]

[1] *Department of Statistics, Akwa Ibom State Polytechnic, Ikoro Osurua, Akwa Ibom State, Nigeria*
[2] *Department of Mathematics, Rivers State University, Port Harcourt, Nigeria*
**Corresponding author E-mail: udoudogeno@gmail.com*

Abstract

In this study, we proposed and studied the alpha power transformed quasi Lindley distribution. The new model has three sub models, namely, Lindley, quasi Lindley and alpha power transformed Lindley distributions. The pdf, hazard rate function, quantile function, moments, Rényi entropy, stochastic ordering and distributions of order statistics were derived based on the new model. The maximum likelihood method of estimating the model parameters was considered. A simulation study was conducted to investigate the behavior of the maximum likelihood estimates. It was observed that the average bias and mean squared error decreased as the sample size increased. By analyzing a real data set, we illustrated the usefulness of the proposed distribution.

Keywords: *Alpha Power Transformation; Bathtub Shape; Goodness of Fit Statistics; Maximum Likelihood Method; Quantile Function; Quasi Lindley.*

1. Introduction

The choice of a distribution for a given data set is critical to any data analysis using a parametric method. Studies have revealed that the quality of the results obtained by analyzing the data depends on the goodness of fit of the assumed distribution. In practice, a researcher may not know the true distribution of the data. To fit a suitable continuous distribution to a continuous data set, it is necessary to examine the histogram of the data as well as descriptive statistics for the data, especially the coefficients of skewness and kurtosis. The coefficient of skewness indicates if the data require a symmetric, left-skewed or right-skewed distribution. The coefficient of kurtosis tells one which of the platykurtic, mesokurtic and leptokurtic distributions should be fitted to the data.
The quasi Lindley distribution introduced by Shanker and Mishra (2013) is among the continuous distributions that have been used to model lifetime data. Let g(x) and G(x) denote the pdf and cdf of a continuous random variable X. Then X follows a quasi Lindley distribution if

$$g(x)=\frac{\theta(\beta+\theta x)}{\beta+1}e^{-\theta x}, x>0, \beta>-1, \theta>0 \tag{1}$$

and

$$G(x)=1-\frac{(\beta+1+\theta x)}{\beta+1}e^{-\theta x}, x>0, \beta>-1, \theta>0. \tag{2}$$

If $\beta=\theta$, the resulting distribution is the Lindley distribution (Lindley, 1958). Empirical information on the potentiality of the quasi Lindley distribution is available in a number of articles (Shanker et al., 2016; Opone and Ekhosuehi, 2018).
Authors have extended the quasi Lindley distribution. Roozergar and Esfandiyari (2015) introduced the MacDonald quasi Lindley distribution. The exponentiated quasi Lindley distribution was proposed by Elbatal et al. (2016). The Weibull quasi Lindley distribution (Hassan et al., 2016) and Marshall-Olkin extended quasi Lindley distribution (Unyime and Etuk, 2019) are also among the existing generalizations of the quasi Lindley distributions.
Though the quasi Lindley and its generalizations above have proven to be appropriate for modeling several lifetime data, more generalizations of the quasi Lindley distribution may be needed to adequately model some lifetime data. Methods of deriving new distributions with high degree of flexibility have been developed in previous studies. Among these methods is the alpha power transformation due to Mahdavi and Kundu (2017). Consider a continuous random variable X with cdf and pdf given by G(x) and g(x) respectively. The corresponding alpha power transformed distribution has cdf (F(x)) and pdf (f(x)) such that

$$F(x)=\begin{cases}\dfrac{\alpha^{G(x)}-1}{\alpha-1}, & \alpha>0,\alpha\neq 1\\ G(x), & \alpha=1\end{cases} \tag{3}$$

and

$$f(x)=\begin{cases}\dfrac{g(x)\alpha^{G(x)}\log(\alpha)}{\alpha-1}, & \alpha>0,\alpha\neq 1\\ g(x), & \alpha=1\end{cases}. \tag{4}$$

In the alpha power family of distributions with cdf (3) and baseline cdf G(x), α incorporates skewness into the baseline distribution. The alpha power transformation approach to generating new distributions has been used to propose distributions such as alpha power exponential distribution (Mahdavi and Kundu, 2017), alpha power transformed Lindley distribution (Dey et al., 2019), alpha power inverse Weibull distribution (Basheer, 2019), alpha power Pareto distribution (Ihtisham et al., 2019) and alpha power transformed power Lindley distribution (Hassan et al., 2019). The main objective of this paper is to generalize the quasi Lindley distribution to obtain the alpha power transformed quasi Lindley distribution (APTQLD) using the alpha power transformation method.

2. The APTQL distribution

The notion of APTQLD is introduced in this section.

Definition: A random variable X is said to follow an APTQLD with parameters α, β and θ, if its cdf is

$$F(x)=\begin{cases}\dfrac{\alpha^{1-e^{-\theta x}\left(1+\frac{\theta x}{\beta+1}\right)}-1}{\alpha-1}, & x>0,\ \alpha>0,\ \alpha\neq 1\\ 1-e^{-\theta x}\left(1+\dfrac{\theta x}{\beta+1}\right), & x>0,\ \alpha=1\end{cases}. \tag{5}$$

The pdf of the APTQLD is of the form

$$f(x)=\begin{cases}\dfrac{\theta(\beta+\theta x)\log(\alpha)\alpha^{1-e^{-\theta x}\left(1+\frac{\theta x}{\beta+1}\right)}e^{-\theta x}}{(\alpha-1)(\beta+1)}, & x>0,\ \alpha>0,\ \alpha\neq 1,\ \beta>-1,\ \theta>0\\ \dfrac{\theta(\beta+\theta x)e^{-\theta x}}{(\beta+1)}, & x>0\ \alpha=1,\ \beta>-1,\ \theta>0\end{cases}. \tag{6}$$

In addition to the cdf and pdf in (5) and (6) respectively, we define the reliability function (R(x)) and hazard rate function (h(x)) of the APTQLD. Consequently,

$$R(x)=\begin{cases}\dfrac{\alpha-\alpha^{1-e^{-\theta x}\left(1+\frac{\theta x}{\beta+1}\right)}}{\alpha-1}, & x>0,\alpha>0,\alpha\neq 0,\beta>-1,\theta>0\\ e^{-\theta x}\left(1+\dfrac{\theta x}{\beta+1}\right), & x>0,\alpha=1,\beta>-1,\theta>0\end{cases} \tag{7}$$

and

$$h(x)=\begin{cases}\dfrac{\theta(\beta+\theta x)\log(\alpha)\alpha^{1-e^{-\theta x}\left(1+\frac{\theta x}{\beta+1}\right)}e^{-\theta x}}{(\beta+1)\left(\alpha-\alpha^{1-e^{-\theta x}\left(1+\frac{\theta x}{\beta+1}\right)}\right)}, & x>0,\ \alpha>0,\ \alpha\neq 1,\ \beta>-1,\ \theta>0\\ \dfrac{\theta(\beta+\theta x)}{(\beta+1)\left(1+\dfrac{\theta x}{\beta+1}\right)}, & x>0,\ \alpha=1,\ \beta>-1.\ \theta>0\end{cases}. \tag{8}$$

The graphical representation of the pdf of the APTQLD is given in Figure 2 for various values of the parameters of the distribution.

It is clear from Figure 2 that the pdf of APTQLD can be nonincreasing, unimodal and right-skewed.

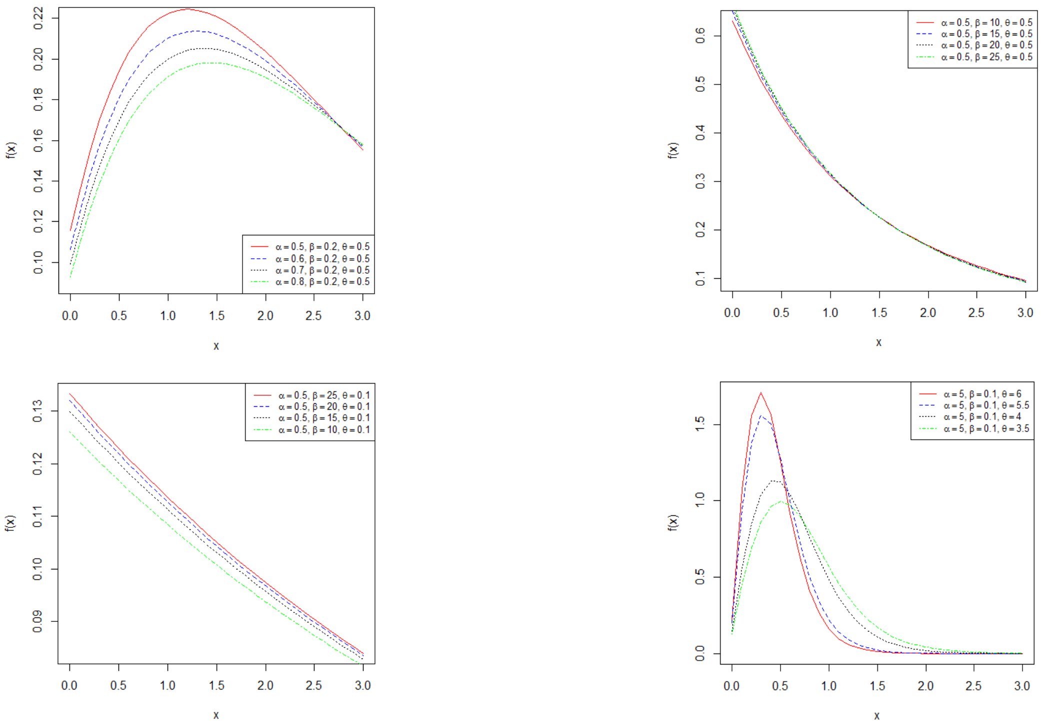

Fig. 1: Plots of the PDF of the APTQLD for Various Values of Its Parameters.

Figure 2 contains plots of the hazard rate function for some selected values of its parameters. This figure shows that the APTQLD has a very flexible hazard rate function. Specifically, the hazard rate function can an increasing function, a decreasing function or bathtub – shaped.

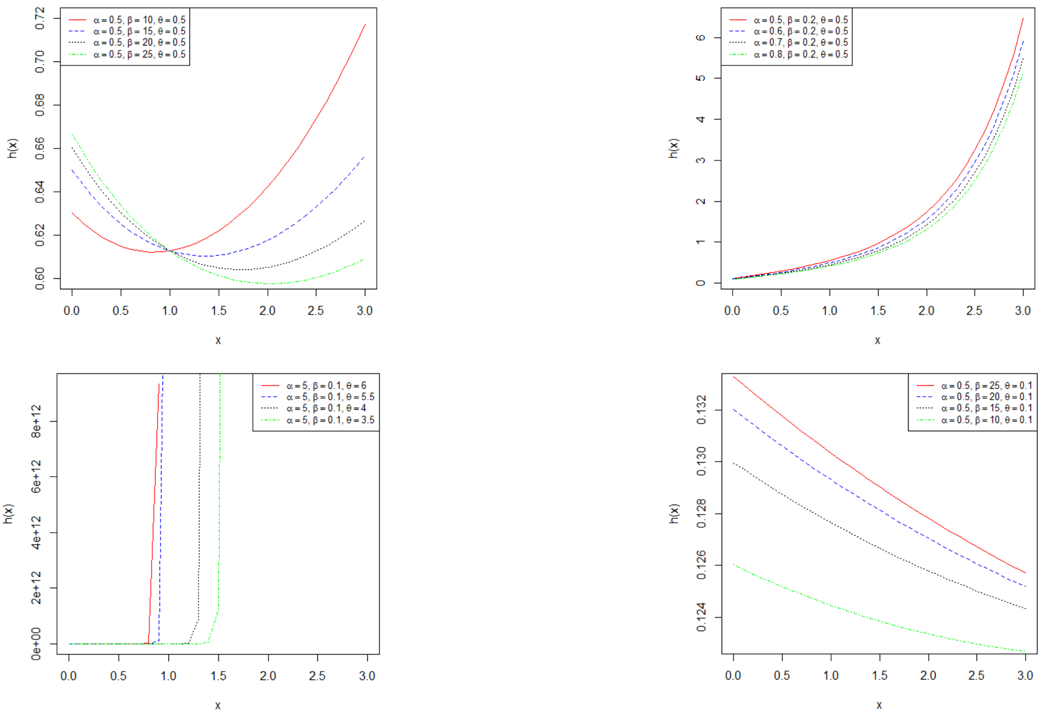

Fig. 2: Plots of the Hazard Rate Function of the APTQLD for Various Values of Its Parameters.

3. The statistical properties of APTQLD

The quantile function, moments, moment generating function, stochastic ordering and entropy are the key concepts discussed in this section.

3.1. The quantile function of the APTQLD

The quantile function of the APTQLD, denoted by Q(w), is obtained by inverting the cdf of the APTQLD as shown below:

$$Q(w)=F^{-1}(w),\ w\in(0,1).$$

Consequently,

$$\frac{\alpha^{1-e^{-\theta Q(w)}\left(1+\frac{\theta Q(w)}{\beta+1}\right)}-1}{\alpha-1}=w\,. \tag{9}$$

$$\left(1+\frac{\theta Q(w)}{\beta+1}\right)e^{-\theta Q(w)}=1-\frac{\log\left(w\left(\alpha-1\right)+1\right)}{\log\alpha}. \tag{10}$$

Let $l(w)=-\left(1+\frac{\theta Q(w)}{\beta+1}\right)$. Then, (9) can be written in the form

$$l(w)e^{l(w)(\beta+1)}=-\left(1-\frac{\log\left(w\left(\alpha-1\right)+1\right)}{\log\alpha}\right)e^{-(\beta+1)}. \tag{11}$$

Let $Z(w)=(\beta+1)l(w)$. Multiplying both sides of (11) by $(\beta+1)$ yields the equation:

$$Z(w)e^{Z(w)}=-(\beta+1)\left(1-\frac{\log\left(w\left(\alpha-1\right)+1\right)}{\log\alpha}\right)e^{-(\beta+1)}. \tag{12}$$

Solving for Q(w) in (12), we obtain

$$Q(w)=-\frac{\beta+1}{\theta}-\frac{1}{\theta}W_{-1}\left(-(\beta+1)\left(1-\frac{\log\left(w\left(\alpha-1\right)+1\right)}{\log\alpha}\right)e^{-(\beta+1)}\right),$$

Where $W_{-1}(.)$ is the negative branch of the Lambert W function. Q. E. D.

Notably, Q(0.25), Q(0.5) and Q(0.75) are the first quartile, median and third quartile of the APTQLD. Table 1 comprises the values of the first quartile (Q_1), second quartile (Q_2) and third quartile (Q_3) for selected values of the parameters of the APTQLD.

We can deduce from Table 1 that if the value of α increases and the values of β and θ are constant, the value of each of Q_1, Q_2 and Q_3 increases. For fixed values of α and and θ, the values of Q_1, Q_2 and Q_3 decrease as the value of β increases. Also, as the value of θ increases, the values of each of Q_1, Q_2 and Q_3 decrease provided the values of α and β are fixed.

Table 1: First Quartile (Q_1), Median (Q_2) and Third Quartile (Q_3) For Selected Values of the Parameters of the APTQLD

α	β	$\theta=0.5$			$\theta=1$		
		Q_1	Q_2	Q_3	Q_1	Q_2	Q_3
0.5	0.1	1.4290	2.6559	4.4917	0.7145	1.3279	2.2458
1.2	0.1	1.8369	3.3189	5.3852	0.9185	1.6595	2.6926
1.5	0.1	1.9609	3.5039	5.6125	0.9805	1.7520	2.8063
2	0.1	2.1323	3.7480	5.9008	1.0661	1.8740	2.9504
2.5	0.1	2.2734	3.9400	6.1193	1.1367	1.9698	3.0597
0.5	0.5	1.0067	2.1552	3.9459	0.5033	1.0776	1.9730
1.2	0.5	1.3795	2.7969	4.8279	0.6898	1.3984	2.4139
1.5	0.5	1.4951	2.9772	5.0528	0.7476	1.4886	2.5264
2	0.5	1.6561	3.2157	5.3383	0.8281	1.6079	2.6692
2.5	0.5	1.7898	3.4035	5.5549	0.8949	1.7017	2.7774
0.5	1.5	0.6846	1.6397	3.2743	0.3423	0.8198	1.6371
1.2	1.5	0.9813	2.1221	4.1095	0.4906	1.1061	2.0547
1.5	1.5	1.0763	2.3762	4.3244	0.5381	1.1881	2.1622
2	1.5	1.2107	2.5948	4.5981	0.6053	1.2974	2.2990
2.5	1.5	1.3238	2.7682	4.8062	0.6619	1.3841	2.4031

3.2. Moments and related concepts

The rth moment of an alpha power transformed quasi Lindley variable is

$$\mu_r'=E\left(X^r\right)=\frac{\theta\log\left(\alpha\right)}{\left(\alpha-1\right)\left(\beta+1\right)}\int_0^\infty\left(\beta+\theta x\right)x^r\alpha^{1-e^{-\theta x}\left(1+\frac{\theta x}{\beta+1}\right)}e^{-\theta x}dx.$$

Applying the power series expansion $\alpha^c=\sum_{i=0}^\infty\frac{\left(\log\alpha\right)^i c^i}{i!}$, we obtain

$$\mu_r'=\frac{\theta}{\left(\alpha-1\right)\left(\beta+1\right)}\sum_{i=0}^\infty\frac{\left(\log\alpha\right)^{i+1}}{i!}\int_0^\infty\left(\beta+\theta x\right)x^r\left(1-e^{-\theta x}\left(1+\frac{\theta x}{\beta+1}\right)\right)^i e^{-\theta x}dx.$$

$$=\frac{1}{\left(\alpha-1\right)}\sum_{i=0}^\infty\sum_{j=0}^i\sum_{k=0}^j\frac{\left(\log\alpha\right)^{i+1}\left(-1\right)^j\theta^{k+1}}{\left(\beta+1\right)^{k+1}i!}\binom{i}{j}\binom{j}{k}\int_0^\infty\left(\beta+\theta x\right)x^{k+r}e^{-\theta(j+1)x}dx\ . \tag{13}$$

After evaluating the integral in (13), the crude moment becomes

$$\mu_r' = \frac{1}{(\alpha-1)} \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j} M_{ijk}, \tag{14}$$

Where

$$M_{ijk} = \frac{(\log\alpha)^{i+1}(-1)^{j}\theta^{k+1}}{(\beta+1)^{k+1}\, i!}\binom{i}{j}\binom{j}{k}\left[\frac{\beta\Gamma(k+r+1)}{(\theta(j+1))^{k+r+1}} + \frac{\theta\Gamma(k+r+2)}{(\theta(j+1))^{k+r+2}}\right].$$

For the APTQLD, the rth central moment is

$$\mu_r = E(X-\mu)^r$$

$$= \sum_{i=0}^{r}(-1)^{i}\binom{r}{i}\mu\mu_{r-1}', \tag{15}$$

Where $\mu = \mu_1' = E(X)$.

The values of μ, variance (σ^2), coefficient of skewness (CS) and coefficient of kurtosis (CK) corresponding to some values of the parameters of the APTQLD are calculated and given in Table 2. Table 2 indicates that if β and θ are constant and we increase the value of α, the mean and variance of APTQLD will increase while the CV, CS and CK will decrease. If α and θ are fixed and we increase β, mean and variance will increase while CV, CS and CK will decrease.

Table 2: Some Descriptive Statistics for the APTQLD

α	β	θ	Mean	Variance	CV	CS	CK
0.5	0.1	0.5	3.3173	6.85872	0.789472	1.62234	6.96847
1.2	0.1	0.5	3.9555	8.23472	0.725476	1.37478	5.82714
1.5	0.1	0.5	4.1256	8.54472	0.708536	1.31931	5.61069
2	0.1	0.5	4.3471	8.91182	0.686726	1.25271	5.37045
2.5	0.1	0.5	4.5198	9.16781	0.669905	1.20539	5.21226
0.5	0.5	0.5	2.8509	6.44497	0.890489	1.73286	7.42634
1.2	0.5	0.5	3.4660	7.82674	0.807164	1.46007	6.11657
1.5	0.5	0.5	3.6306	8.14224	0.785948	1.39954	5.86955
2	0.5	0.5	3.8453	8.51847	0.759015	1.32715	5.59527
2.5	0.5	0.5	4.0128	8.78444	0.738600	1.27536	5.41355
0.5	1.5	0.5	2.3610	5.50648	0.993896	1.95141	8.55419
1.2	1.5	0.5	2.9213	6.82311	0.894159	1.63946	6.88475
1.5	1.5	0.5	3.0723	7.13177	0.869231	1.57109	5.68610
2	1.5	0.5	3.2698	7.50561	0.837861	1.48950	6.22092
2.5	1.5	0.5	3.4243	7.77457	0.814266	1.43109	5.98773

Table 2: Continued

α	β	θ	Mean	Variance	CV	CS	CK
0.5	0.1	1	1.6587	1.71451	0.789410	1.62267	6.96912
1.2	0.1	1	1.9777	2.05890	0.725534	1.37442	5.82651
1.5	0.1	1	2.0628	2.13616	0.708532	1.31939	5.61064
2	0.1	1	2.1736	2.22776	0.686681	1.25294	5.37086
2.5	0.1	1	2.2599	2.29195	0.669905	1.20539	5.21225
0.5	0.5	1	1.4254	1.61143	0.890573	1.73241	7.42530
1.2	0.5	1	1.7330	1.95671	0.807170	1.46000	6.11650
1.5	0.5	1	1.8153	2.03559	0.785953	1.39947	5.86949
2	0.5	1	1.9226	2.12981	0.759069	1.32690	5.59471
2.5	0.5	1	2.0064	2.19616	0.738609	1.27523	5.41356
0.5	1.5	1	1.1805	1.37662	0.993896	1.95141	8.55418
1.2	1.5	1	1.4607	1.70566	0.894097	1.63963	6.88531
1.5	1.5	1	1.5361	1.78310	0.869297	1.57087	6.57012
2	1.5	1	1.6349	1.87640	0.837861	1.48949	6.22095
2.5	1.5	1	1.7122	1.94347	0.814206	1.43134	5.98824

The moment generating function of the alpha power transformed quasi Lindley variable X is

$$M_x(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}E(X^r) \tag{16}$$

Applying (14) in (16) leads to

$$M_x(t) = \sum_{r,i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j} P_{ijkr}, \tag{17}$$

where

$$P_{ijkr} = \frac{(\log\alpha)^{i+1}(-1)^{j}\theta^{k+1}t^{r}}{(\alpha\text{-}1)(\beta+1)^{k+1}i!r!}\binom{i}{j}\binom{j}{k}\left[\frac{\beta\Gamma(k+r+1)}{(\theta(j+1))^{k+r+1}}+\frac{\theta\Gamma(k+r+2)}{(\theta(j+1))^{k+r+2}}\right].$$

Furthermore, the mth lower incomplete moment $(\varphi_m(t))$ of the APTQLD is

$$\varphi_m(t)=\frac{\theta\log(\alpha)}{(\alpha\text{-}1)(\beta+1)}\int_0^t(\beta+\theta x)x^m\alpha^{1\text{-}e^{-\theta x}\left(1+\frac{\theta x}{\beta+1}\right)}e^{-\theta x}dx.$$

$$=\frac{\theta}{(\alpha\text{-}1)(\beta+1)}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(\log\alpha)^{i+1}(-1)^{j}}{i!}\binom{i}{j}\int_0^t(\beta+\theta x)x^m\left(1+\frac{\theta x}{\beta+1}\right)^{j}e^{-\theta(j+1)x}dx \quad . \tag{18}$$

3.3. Probability weighted moments for the APTQLD

Let X be an alpha power transformed quasi Lindley (APTQL) variable with pdf f(x) and cdf F(x). If $r\geq 1$ and $s\geq 0$, the (s, r)th proability weighted moments of X is given by

$$\pi_{r,s}=E\left(X^rF^s(x)\right)$$

$$=\frac{\theta\log\alpha}{(\alpha-1)(\beta+1)}\int_0^\infty(\beta+\theta x)x^re^{-\theta x}\alpha^{1\text{-}\left(1+\frac{\theta x}{\beta+1}\right)e^{-\theta x}}\left(\alpha^{1\text{-}\left(1+\frac{\theta x}{\beta+1}\right)e^{-\theta x}}-1\right)^{s}dx.$$

Notably,

$$\left(\alpha^{1\text{-}\left(1+\frac{\theta x}{\beta+1}\right)e^{-\theta x}}-1\right)^{s}=\sum_{q=0}^{s}(\text{-}1)^{s\text{-}q}\binom{s}{q}\alpha^{q\left(1\text{-}\left(1+\frac{\theta x}{\beta+1}\right)e^{-\theta x}\right)}.$$

It follows that

$$\pi_{r,s}=\frac{\theta\log\alpha}{(\alpha-1)(\beta+1)}\sum_{q=0}^{s}(\text{-}1)^{s\text{-}q}\binom{s}{q}\int_0^\infty(\beta+\theta x)\alpha^{(q+1)\left(1\text{-}\left(1+\frac{\theta x}{\beta+1}\right)e^{-\theta x}\right)}x^re^{-\theta x}dx.$$

But

$$\alpha^{(q+1)\left(1\text{-}\left(1+\frac{\theta x}{\beta+1}\right)e^{-\theta x}\right)}=\sum_{m=0}^{\infty}\frac{(\log\alpha)^m(q+1)^m}{m!}\left(1\text{-}\left(1+\frac{\theta x}{\beta+1}\right)e^{-\theta x}\right)^m.$$

Thus,

$$\pi_{r,s}=\frac{\theta}{(\alpha\text{-}1)(\beta+1)}\sum_{q=0}^{s}\sum_{m=0}^{\infty}\frac{(\log\alpha)^{m+1}(q+1)^m(\text{-}1)^{s\text{-}q}}{m!}\binom{s}{q}\int_0^\infty(\beta+\theta x)\left(1\text{-}\left(1+\frac{\theta x}{\beta+1}\right)e^{-\theta x}\right)^m x^re^{-\theta x}dx.$$

Since

$$\left(1\text{-}\left(1+\frac{\theta x}{\beta+1}\right)e^{-\theta x}\right)^m=\sum_{j=0}^{m}\binom{m}{j}(\text{-}1)^j\left(1+\frac{\theta x}{\beta+1}\right)^je^{-\theta jx}$$

And

$$\left(1+\frac{\theta x}{\beta+1}\right)^j=\sum_{i=0}^{j}\binom{j}{i}\frac{\theta^ix^i}{(\beta+1)^i},$$

We have

$$\left(1\text{-}\left(1+\frac{\theta x}{\beta+1}\right)e^{-\theta x}\right)^m=\sum_{j=0}^{m}\sum_{i=0}^{j}\binom{m}{j}\binom{j}{i}(-1)^j\frac{\theta^ix^ie^{-\theta jx}}{(\beta+1)^i}.$$

Hence,

$$\pi_{r,s}=\sum_{q=0}^{s}\sum_{m=0}^{\infty}\sum_{j=0}^{m}\sum_{i=o}^{j}\frac{(\log\alpha)^{m+1}(q+1)^m(\text{-}1)^{s+j\text{-}q}\theta^{i+1}}{(\alpha\text{-}1)(\beta+1)m!}\binom{s}{q}\binom{m}{j}\binom{j}{i}\int_0^\infty(\beta+\theta x)x^{r+i}e^{-\theta(j+1)x}dx \;=\sum_{q=0}^{s}\sum_{m=0}^{\infty}\sum_{j=0}^{m}\sum_{i=o}^{j}T_{ijmq}, \tag{19}$$

Where

$$T_{ijmq}=\frac{(\log\alpha)^{m+1}(q+1)^m(-1)^{s+j-q}\theta^{i+1}}{(\alpha-1)(\beta+1)m!}\binom{s}{q}\binom{m}{j}\binom{j}{i}\left(\frac{\beta\Gamma(r+i+1)}{(\theta(j+1))^{r+i+1}}+\frac{\theta\Gamma(r+i+2)}{(\theta(j+1))^{r+i+2}}\right).$$

3.4. Rényi entropy for the APTQLD

The Rényi entropy for the APTQLD is of the form

$$I_R(x)=\frac{1}{1-v}\log\left(\int_0^\infty (f(x))^v\,dx\right), v>0, v\neq 1,$$

Where f(x) is the pdf of the APTQL variable.
Now,

$$(f(x))^v=\left(\frac{\theta\log(\alpha)}{(\alpha-1)(\beta+1)}\right)^v((\beta+\theta x))^v\,\alpha^{v\left(1-e^{-\theta x}\left(1+\frac{\theta x}{\beta+1}\right)\right)}e^{-\theta v x}$$

$$=\left(\frac{\theta\log(\alpha)}{(\alpha-1)(\beta+1)}\right)^v((\beta+\theta x))^v e^{-\theta v x}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(-1)^j(\log(\alpha))^i v^i}{i!}\binom{i}{j}\left(1+\frac{\theta x}{\beta+1}\right)^j e^{-\theta j x}$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\frac{(-1)^j(\log(\alpha))^{v+i}\theta^{v+k}v^i}{(\alpha-1)^v(\beta+1)^{v+k}i!}\binom{i}{j}\binom{j}{k}x^k((\beta+\theta x))^v e^{-\theta(v+j)x}.$$

Thus,

$$I_R(x)=\frac{1}{1-v}\log\left(=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\frac{(-1)^j(\log(\alpha))^{v+i}\theta^{v+k}v^i}{(\alpha-1)^v(\beta+1)^{v+k}i!}\binom{i}{j}\binom{j}{k}\int_0^\infty x^k((\beta+\theta x))^v e^{-\theta(v+j)x}dx\right), v>0, v\neq 1.$$

The above integral can only be evaluated numerically.

3.5. Stochastic Ordering for the APTQLD

Let X_1 and X_2 be two APTQL variables such that $X_1\sim APTQLD(\alpha_1,\beta_1,\theta_1)$ and $X_2\sim APTQLD(\alpha_2,\beta_2,\theta_2)$. Let the pdf, cdf, reliability function and hazard rate function of X_1 be $f_1(x,\alpha_1,\beta_1,\theta_1)$, $F_1(x,\alpha_1,\beta_1,\theta_1)$, $R_1(x,\alpha_1,\beta_1,\theta_1)$ and $h_1(x,\alpha_1,\beta_1,\theta_1)$ respectively. Given that $f_2(x,\alpha_2,\beta_2,\theta_2)$, $F_2(x,\alpha_2,\beta_2,\theta_2)$, $R_2(x,\alpha_2,\beta_2,\theta_2)$ and $h_2(x,\alpha_2,\beta_2,\theta_2)$ are respectively, the pdf, cdf, reliability function and hazard rate function for X_2, then X_1 is smaller than X_2 in:

a) stochastic order $(X_1\leq_{st}X_2)$ if $R_1(x,\alpha_1,\beta_1,\theta_1)\leq R_2(x,\alpha_2,\beta_2,\theta_2)$,
b) hazard rate order $(X_1\leq_{hr}X_2)$ if $\frac{h_1(x,\alpha_1,\beta_1,\theta_1)}{h_2(x,\alpha_2,\beta_2,\theta_2)}$ is decreasing in x;
c) mean residual life order $(X_1\leq_{mrl}X_2)$ if $E(X_1\text{-}t|X_1<t)\leq E(X_2\text{-}t|X_2<t)$;
d) likelihood ratio order $(X_1\leq_{lr}X_2)$ if $\frac{f_1(x,\alpha_1,\beta_1,\theta_1)}{f_2(x,\alpha_2,\beta_2,\theta_2)}$ is a decreasing function of x.

Notably, if $(X_1\leq_{lr}X_2)$, then $(X_1\leq_{hr}X_2)$, $(X_1\leq_{mrl}X_2)$ and $(X_1\leq_{st}X_2)$.

Theorem 2: *Provides the conditions for X_1 to be smaller than X_2 in the likelihood ratio order.*

Theorem 2: *Let* $X_1\sim APTQLD(\alpha_1,\beta_1,\theta_1)$ *and* $X_2\sim APTQLD(\alpha_2,\beta_2,\theta_2)$. *If* $\alpha_1<\alpha_2$, $\beta_1=\beta_2=\beta$ *and* $\theta_1=\theta_2=\theta$, *then* $X_1\leq_{lr}X_2$.

Proof
The likelihood ratio is

$$\frac{f_1(x,\alpha_1,\beta_1,\theta_1)}{f_2(x,\alpha_2,\beta_2,\theta_2)}=\frac{\dfrac{\theta_1(\beta_1+\theta_1x)\log(\alpha_1)\alpha_1^{1-e^{-\theta_1x}\left(1+\frac{\theta_1x}{\beta_1+1}\right)}e^{-\theta_1x}}{(\alpha_1-1)(\beta_1+1)}}{\dfrac{\theta_2(\beta_2+\theta_2x)\log(\alpha_2)\alpha_2^{1-e^{-\theta_2x}\left(1+\frac{\theta_2x}{\beta_2+1}\right)}e^{-\theta_2x}}{(\alpha_2-1)(\beta_2+1)}}$$

$$=\frac{\theta_1(\beta_1+\theta_1x)(\alpha_2-1)(\beta_2+1)\log(\alpha_1)\alpha_1^{1-e^{-\theta_1x}\left(1+\frac{\theta_1x}{\beta_1+1}\right)}e^{-(\theta_1-\theta_2)x}}{(\alpha_1-1)(\beta_1+1)\theta_2(\beta_2+\theta_2x)\log(\alpha_2)\alpha_2^{1-e^{-\theta_2x}\left(1+\frac{\theta_2x}{\beta_2+1}\right)}}.$$

Consequently, the log of the likelihood ratio is

$$\log\left(\frac{f_1(x,\alpha_1,\beta_1,\theta_1)}{f_2(x,\alpha_2,\beta_2,\theta_2)}\right)=\log(\theta_1)+\log((\beta_1+\theta_1 x))+\log((\alpha_2-1))+\log((\beta_2+1))$$
$$+\log(\log(\alpha_1))+\left(1-e^{-\theta_1 x}\left(1+\frac{\theta_1 x}{\beta_1+1}\right)\right)\log(\alpha_1)-(\theta_1-\theta_2)x$$
$$-\log(\theta_2)-\log((\beta_2+\theta_2 x))-\log((\alpha_1-1))-\log((\beta_1+1))$$
$$-\log(\log(\alpha_2))-\left(1-e^{-\theta_2 x}\left(1+\frac{\theta_2 x}{\beta_2+1}\right)\right)\log(\alpha_2).$$

Differentiating the log-likelihood ratio, we get

$$\frac{d\log\left(\frac{f_1(x,\alpha_1,\beta_1,\theta_1)}{f_2(x,\alpha_2,\beta_2,\theta_2)}\right)}{dx}=\frac{\theta_1}{\beta_1+\theta_1 x}+\left(\theta_1 e^{-\theta_1 x}\left(1+\frac{\theta_1 x}{\beta_1+1}\right)-\frac{\theta_1}{\beta_1+1}e^{-\theta_1 x}\right)\log(\alpha_1)-(\theta_1-\theta_2)$$

$$-\frac{\theta_2}{\beta_2+\theta_2 x}-\left(\theta_2 e^{-\theta_2 x}\left(1+\frac{\theta_2 x}{\beta_2+1}\right)-\frac{\theta_2}{\beta_2+1}e^{-\theta_2 x}\right)\log(\alpha_2).$$

It is certain that for $\alpha_1<\alpha_2$, $\beta_1=\beta_2=\beta$ and $\theta_1=\theta_2=\theta$, $\frac{d\log\left(\frac{f_1(x,\alpha_1,\beta_1,\theta_1)}{f_2(x,\alpha_2,\beta_2,\theta_2)}\right)}{dx}<0$. Under these conditions, X_1 is stochastically smaller than X_2 in the likelihood ratio order.

3.6. Order Statistics for the APTQLD

Given a random sample $X_1, X_2, \ldots, X_n$ of size n from the APTQLD (α,β,θ) . Let $X_{(k)}$ be the kth order statistic. To find the pdf of $X_{(k)}$, we consider the formula:

$$f_{X(k)}(x)=\frac{n!}{(k-1)!(n-k)!}f(x)\left[F(x)\right]^{k-1}\left[1-F(x)\right]^{n-k},$$

Where f(x) and F(x) are the pdf and cdf of APTQL variable. Hence,

$$f_{X(k)}(x)=\frac{n!\theta(\beta+\theta x)\log(\alpha)\alpha^{1-e^{-\theta x}\left(1+\frac{\theta x}{\beta+1}\right)}e^{-\theta x}}{(k-1)!(n-k)!(\alpha-1)(\beta+1)}\left[\frac{\alpha^{1-e^{-\theta x}\left(1+\frac{\theta x}{\beta+1}\right)}-1}{\alpha-1}\right]^{k-1}\left[\frac{\alpha-\alpha^{1-e^{-\theta x}\left(1+\frac{\theta x}{\beta+1}\right)}}{\alpha-1}\right]^{n-k}.$$

If k=1, the pdf of the first order statistic becomes

$$f_{X(1)}(x)=\frac{n\theta(\beta+\theta x)\log(\alpha)\alpha^{1-e^{-\theta x}\left(1+\frac{\theta x}{\beta+1}\right)}e^{-\theta x}}{(\alpha-1)(\beta+1)}\left[\frac{\alpha-\alpha^{1-e^{-\theta x}\left(1+\frac{\theta x}{\beta+1}\right)}}{\alpha-1}\right]^{n-1}.$$

$$=\frac{n\alpha^{n-1}\theta(\beta+\theta x)\log(\alpha)e^{-\theta x}}{(\alpha-1)^n(\beta+1)}\sum_{i=0}^{n-1}\binom{n-1}{1}(-1)^i\alpha^{1-(i+1)e^{-\theta x}\left(1+\frac{\theta x}{\beta+1}\right)}.$$

It shall be noted that

$$\alpha^{1-(i+1)e^{-\theta x}\left(1+\frac{\theta x}{\beta+1}\right)}=\sum_{j=0}^{\infty}\frac{(\log(\alpha))^j}{j!}\left(1-(i+1)e^{-\theta x}\left(1+\frac{\theta x}{\beta+1}\right)\right)^j$$

$$=\sum_{j=0}^{\infty}\sum_{m=0}^{j}\sum_{q=0}^{m}\frac{(\log(\alpha))^j}{j!(\beta+1)^q}(-1)^m\binom{j}{m}\binom{m}{q}(i+1)^m\theta^q x^q e^{-m\theta x}.$$

It follows that

$$f_{X(1)}(x)=\frac{n\alpha^{n-1}(\beta+\theta x)}{(\alpha-1)^n}\sum_{i=0}^{n-1}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\sum_{q=0}^{m}(-1)^{i+m}(i+1)^m\binom{n-1}{i}\binom{j}{m}\binom{m}{q}\frac{\theta^{q+1}(\log(\alpha))^{j+1}}{j!(\beta+1)^{q+1}}x^q e^{-(m+1)\theta x}.$$

Similarly, the pdf of the nth order statistic is

$$f_{X(n)}(x)=\frac{n\theta(\beta+\theta x)\log(\alpha)\alpha^{1-e^{-\theta x}\left(1+\frac{\theta x}{\beta+1}\right)}e^{-\theta x}}{(\alpha-1)(\beta+1)}\left[\frac{\alpha^{1-e^{-\theta x}\left(1+\frac{\theta x}{\beta+1}\right)}-1}{\alpha-1}\right]^{n-1}$$

$$=\frac{n(\beta+\theta x)}{(\alpha-1)^n}\sum_{i=0}^{n-1}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\sum_{q=0}^{m}(-1)^{i+m}(i+1)^j\binom{n-1}{i}\binom{j}{m}\binom{m}{q}\frac{\theta^{q+1}(\log(\alpha))^{j+1}}{j!(\beta+1)^{q+1}}x^q e^{-(m+1)\theta x}.$$

4. Maximum likelihood estimation of the parameters of APTQLD

Suppose that $X_1, X_2, \ldots, X_n$ represent a random sample from an APTQLD. The associated likelihood function is

$$L(x_1, x_2, \ldots, x_n \mid \alpha, \beta, \theta) = \prod_{i=1}^{n} f(x_i)$$

$$= \prod_{i=1}^{n}\left(\frac{\theta(\beta+\theta x_i)\log(\alpha)\alpha^{1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\beta+1}\right)}e^{-\theta x_i}}{(\alpha-1)(\beta+1)}\right)$$

$$= \left(\frac{\theta\log(\alpha)}{(\alpha-1)(\beta+1)}\right)^n\prod_{i=1}^{n}\left((\beta+\theta x_i)\alpha^{1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\beta+1}\right)}e^{-\theta x_i}\right).$$

The log-likelihood function can be written as

$$\ell = n\log(\theta)+n\log(\log(\alpha))-n\log(\alpha-1)-n\log(\beta+1)+\sum_{i=1}^{n}\log(\beta+\theta x_i)$$

$$+\log(\alpha)\sum_{i=1}^{n}\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\beta+1}\right)\right)-\theta\sum_{i=1}^{n}x_i\ .$$

The partial derivatives of ℓ with respect to α, β and θ are obtained below:

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha\log(\alpha)}-\frac{n}{\alpha-1}-\frac{n}{\alpha}+\sum_{i=1}^{n}\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\beta+1}\right)\right); \tag{20}$$

$$\frac{\partial\ell}{\partial\beta}=-\frac{n}{\beta+1}+\sum_{i=1}^{n}\left(\frac{1}{\beta+\theta x_i}\right)+\frac{\theta\log(\alpha)}{(\beta+1)^2}\sum_{i=1}^{n}x_i e^{-\theta x_i}\ ; \tag{21}$$

$$\frac{\partial\ell}{\partial\theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\left(\frac{x_i}{\beta+\theta x_i}\right)-\frac{\log(\alpha)}{\beta+1}\sum_{i=1}^{n}(\beta+\theta x_i)x_i e^{-\theta x_i}\ . \tag{22}$$

To find the maximum likelihood estimates of the parameters, we solve simultaneously the nonlinear equations obtained by equating each of the three partial derivatives to zero. Since it is practically impossible to solve the equations analytically, a suitable numerical approach may be used to find the solution.

5. A simulation study on APTQLD

Here, we carry out a simulation study to investigate the performance of the maximum likelihood method of estimation of the parameters of APTQLD. In this study simulation study, R programming language and the sample sizes n=50, 100, 150, 300 are considered. On the basis of two different sets of parameter values of APTQLD, namely,
Set 1 and Set 2, 1000 samples are simulated. In Set 1, $\alpha=0.5$, $\beta=0.1$ and $\theta=0.5$ while $\alpha=2$, $\beta=5$ and $\theta=10$ in Set 2. Table 4 contains the average estimate of each parameter, and the corresponding average bias and mean squared error. From Table 3, we deduce that the average bias and mean squared error corresponding to each of the average estimates decrease as the sample size increases.

Table 3: Simulation Results Based on APTQLD

n	Parameter	AE	AB	MSE	SE
	α=0.5	2.1578	1.6578	274.8168	16.5776
50	β=0.1	4.2984	4.1984	1762.66	41.9841
	θ=0.5	0.5302	0.0302	0.0909	0.3015
	α=0.5	0.9979	0.4979	24.7907	4.9790
100	β=0.1	0.2384	0.1384	1.9142	1.3835
	θ=0.5	0.5076	0.0076	0.0058	0.0762
	α=0.5	0.7891	0.2891	8.3602	2.8914
150	β=0.1	0.1391	0.0391	0.1527	0.3907
	θ=0.5	0.5041	0.0041	0.0017	0.0406

300	α=0.5	0.6471	0.1471	2.1634	1.4708
	β=0.1	0.1223	0.0228	0.0520	0.2280
	θ=0.5	0.4956	-0.0044	0.0020	0.0443

Table 3: Continued

n	Parameter	AE	AB	MSE	SE
50	α=2	2.4221	0.4221	17.8152	4.2208
	β=5	270.0512	265.0512	7025212	2650.512
	θ=10	10.2855	0.2855	8.1516	2.8551
100	α=2	2.1966	0.1966	3.8657	1.9661
	β=5	265.2313	260.2313	6772035	2602.313
	θ=10	10.1903	0.1903	3.6214	1.9030
150	α=2	1.9950	-0.0050	0.0025	0.0496
	β=5	170.9279	165.9279	2753208	1659.279
	θ=10	10.1428	0.1428	2.0403	1.4284
300	α=2	1.8321	-0.1679	2.8190	1.6790
	β=5	144.7801	139.7801	1953849	1397.801
	θ=10	9.9674	-0.0326	0.1066	0.3265

6. Application of APTQLD

In this section, we illustrate the usefulness of the APTQLD using a real data set originally presented in Maguire et al. (1952) and subsequently analyzed by Mahdavi and Kundu (2017). The data comprising intervals (in days) between 109 successive coal-mining disasters in Great Britain, for the period 1875-1951 are reported as follows:
1, 4, 4, 7, 11, 13, 15, 15, 17, 18, 19, 19, 20, 20, 22, 23, 28, 29, 31, 32, 36, 37, 47, 48, 49,50, 54, 54, 55, 59, 59, 61, 61, 66, 72, 72, 75, 78, 78, 81, 93, 96, 99, 108, 113, 114, 120, 120, 120,123, 124, 129, 131, 137, 145, 151, 156, 171, 176, 182, 188, 189, 195, 203, 208, 215, 217, 217, 217,224, 228, 233, 255, 271, 275, 275, 275, 286, 291, 312, 312, 312, 315, 326, 326, 329, 330, 336, 338,345, 348, 354, 361, 364, 369, 378, 390, 457, 467, 498, 517, 566, 644, 745, 871, 1312, 1357, 1613,1630.
The fits of APTQLD to the data are compared to the fits of five other distributions, namely, alpha power transformed Lindley distribution (APTLD)(Dey et al. 2019), alpha power transformed Power Lindley distribution (APTPLD)(Hassan et al., 2019), exponentiated quasi Lindley distribution (EQLD)(Elbatal et al., 2016), quasi Lindley distribution (QLD) and Lindley distribution (LD). We use the maximum likelihood procedure to estimate all the model parameters. All the numerical results in this section are obtained with the help of R software. Also, AIC, BIC, KS and W* are used to compare fits of the distributions. Notably, the distribution with smallest values of AIC, BIC, KS and W* is the most suitable distribution for the given data. With $f_1(x)$, $f_2(x)$, $f_3(x)$ and $f_4(x)$ representing the pdfs corresponding to APTLD, APTPLD, EQLD and LD respectively, we have

$$f_1(x)=\begin{cases}\dfrac{\theta^2(1+x)\log(\alpha)\alpha^{1-e^{-\theta x}\left(1+\frac{\theta x}{\theta+1}\right)}e^{-\theta x}}{(\alpha-1)(\theta+1)}, & x>0,\ \alpha>0,\ \alpha\neq 1,\ \theta>0\\ \dfrac{\theta^2(1+x)e^{-\theta x}}{(\theta+1)}, & x>0,\ \alpha=1,\ \theta>0\end{cases};$$

$$f_2(x)=\begin{cases}\dfrac{\beta\theta^2x^{\beta-1}(1+x^\beta)\log(\alpha)\alpha^{1-e^{-\theta x^\beta}\left(1+\frac{\theta x^\beta}{\theta+1}\right)}e^{-\theta x^\beta}}{(\alpha-1)(\theta+1)}, & x>0,\ \alpha>0,\ \alpha\neq 1,\ \beta>0,\ \theta>0\\ \dfrac{\beta\theta^2x^{\beta-1}(1+x^\beta)e^{-\theta x^\beta}}{(\theta+1)}, & x>0,\ \alpha=0,\ \beta>0,\ \theta>0.\end{cases};$$

$$f_3(x)=\frac{\alpha\theta(\beta+\theta x)}{\beta+1}e^{-\theta x}\left(1-\frac{\theta(\beta+1+\theta x)}{\beta+1}e^{-\theta x}\right)^{\alpha-1},\ x>0,\ \alpha>0,\ \beta>-1,\theta>0;$$

$$f_4(x)=\frac{\theta^2(1+x)e^{-\theta x}}{(\theta+1)},\ x>0,\ \theta>0.$$

Table 4 comprises the maximum likelihood estimates of the six distributions whose fits to the coal- mining data are being compared, their corresponding standard errors and values of the necessary goodness of fit statistics. For all the six distributions fitted to the coal-mining data, the results contained in Table 4 indicate that the APTQLD has the smallest AIC, BIC, KS and W* values. Consequently, the APTQLD provides the best fit to the data among all the six distributions.

Table 4: Maximum Likelihood Estimates of the Parameters (Standard Errors in Parentheses) of the Distributions Fitted to Coal Data and Values of Some Model Selection Statistics

Distribution	Parameter	Estimate	$-\ell$	AIC	BIC	KS	W*	A*
APTLD	α	0.0528 (0.0363)	723.0258	1450.052	1455.434	0.1755	0.8268	8.5544
	θ	0.0053 (0.0007)						
APTPLD	α	0.6206			1416.24	0.0696	0.0722	0.5062

		(1.4285)	701.0832	1408.166				
	β	0.6478 (0.0851)						
	θ	0.0591 (0.0531)						
APTQLD	α	0.0175 (0.0177)	700.455	1406.91	1414.984	0.0579	0.0690	0.4712
	β	1.0934 (0.7827)						
	θ	0.0023 (0.0002)						
EQLD	α	0.8251 (0.1071)	703.2251	1412.45	1420.524	0.0889	0.0962	0.8055
	β	3.2913 (2.5628)						
	θ	0.0046 (0.0006)						
LD	θ	0.0086 (0.0006)	734.2979	1470.596	1473.287	0.2102	1.3299	12.8063
QLD	β	747.4964 (5963.813)	703.3134	1410.627	1416.009	0.0788	0.1504	1.1246
	θ	0.0043 (0.0003)						

The estimated pdf and cdf plots in Figure 3 show that the APTQLD is an appropriate distribution for the coal-mining data set.

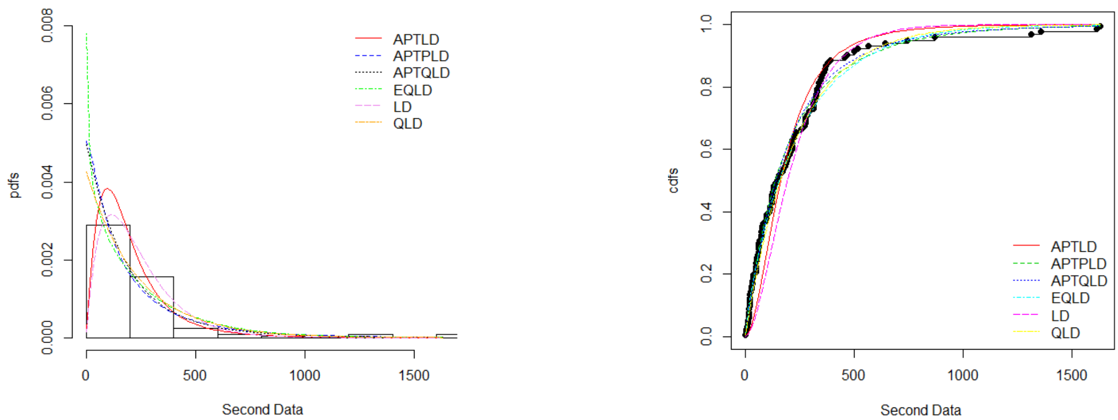

Fig. 3: The Estimated PDF and CDF Plots of the Six Distributions Fitted to the Coal-Mining Data.

7. Conclusion

We have proposed a three-parameter extension of the quasi Lindley distribution called the alpha power transformed quasi Lindley distribution. The widely studied and applied Lindley, quasi Lindley and alpha power transformed Lindley are all special cases of the new distribution. By making plots of the pdf and hazard rate functions of the distribution, we have been able to investigate the flexibility of the distribution. In particular, the hazard rate function can be bathtub shaped. Under certain conditions, the quantile function of the distribution can be reduced to those of its sub models. We demonstrated through model fitting that the APTQLD can outperform its submodels and other well-known continuous distribution.

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