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Research paper

Least Squares Approximation Method for Estimation of Volterra Fractional Integro-differential equation using Hermite Polynomial as the basis functions

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Abstract

This research work utilizes the least squares approximation method to estimate approximate solutions for fractional-order integro-differential equations, with Hermite polynomials as basis functions. The process begins by assuming an approximate solution of degree N, which is then substituted into the fractional-order integro-differential equation under investigation. After evaluating the integral, the equation is rearranged to isolate one side, allowing the application of the least squares method. Three examples were solved using this approach. In Example 1, the numerical results for α=0.9 and α=0.8 were compared to the exact solution for α=1. In Examples 2 and 3, the results for α=1.9 and α=1.8 were compared to the exact solution for α=2. These comparisons showed favorable alignment with the exact solutions. The numerical results and graphical illustrations demonstrate the validity, competence, and accuracy of the proposed method.

Keywords: *Least square Approximation, Volterra Fractional Integro-differential equation, Guassian Elimination.*

1. Introduction

Recent research has focused on developing numerical methods for solving fractional differential and integro-differential equations (FDEs) using orthogonal polynomials as basis functions. Numerous studies have explored the efficacy and efficiency of these methods across various types of FDEs. The significance of numerical methods in solving fractional-order and multi-order fractional equations cannot be overstated, given their profound relevance to mathematicians, engineers, and physicists. Recent efforts have particularly concentrated on solving fractional integro-differential equations using polynomial functions, intentionally avoiding non-polynomial functions such as exponentials and transcendentals.

Yang and Hou (2013) introduced the Laplace decomposition method for solving nonlinear fractional integro-differential equations. By leveraging the Laplace transform and Adomian polynomials, this method efficiently handles nonlinear terms in Caputo fractional derivatives. Through illustrative examples, they demonstrated its efficiency and accuracy in providing both analytical and numerical solutions for various equations. The method's rapid convergence and formal behavior determination using Pade approximants make it applicable to a wide range of nonlinear fractional differential equations.

Ajisope et al. (2021) demonstrated the application of the Least Squares Method for solving Volterra fractional integro-differential equations based on constructed orthogonal polynomials. The results showed that the method is powerful when compared with exact solutions, highlighting the similarity between the exact and approximate solutions.

Taiwo and Adio (2014) discussed the solution of systems of higher-order integro-differential equations using the perturbed variational iteration method. The results obtained from this method closely match those from the conventional variational iteration method, underscoring the reliability of both approaches. Additionally, the method offers simplicity, ease of use, and straightforward programming.

Oyedepo and Abubakar (2016) conducted numerical studies on solving fractional integro-differential equations using the Least Squares Method and Bernstein Polynomials. The paper introduces two numerical methods that employ the least squares method with Bernstein polynomials, converting the equations into linear algebraic systems. Numerical results, presented in tables and graphs, demonstrate the method's high accuracy in providing solutions for such equations.

Uwaheren and Taiwo (2016) demonstrated the construction of orthogonal polynomials as basis functions for solving fractional-order integro-differential equations. Their method involves constructing orthogonal polynomials with a quadratic weight function within the interval [0 , 1] [0,1], simplifying the equations, collocating them at equally spaced interior points, solving the resulting system of algebraic

equations, and substituting the solutions back to obtain the approximate solution. The results show a high level of convergence to the exact solution.
Shoushan (2022) proposed using the Hermite polynomial and Least Squares Technique (LST) for solving integro-differential equations. The research introduced a technique that demonstrated accuracy and efficiency through three example problems.
Rawashdeh (2006) proposed a collocation method for solving fractional integro-differential equations using polynomial spline functions as basis functions to approximate the solution. Momani and Qaralleh (2006a,b) introduced an efficient technique for solving systems of fractional integro-differential equations using the Adomian Decomposition Method (ADM). However, Mittal and Nigam (2008) noted that constructing Adomian polynomials for nonlinear fractional-order integro-differential equations, as required by ADM, is demanding and cumbersome.
Hashim et al. (2009) applied the Homotopy Analysis Method (HAM) to address initial value problems of fractional order. This work aims to contribute to the existing literature by employing the least squares method with Hermite polynomials as basis functions to estimate the approximate solution for fractional-order integro-differential equations.

2. Definition of Relevant Terms

Differential Equation

A differential equation is an equation which relating one or more unknown functions and its derivatives of which the variables involved are dependent and independent variables. A typical example of differential equation is given as:

$$\frac{d^2y}{dx^2} + 5x^2\frac{dy}{dx} = 4x - 9 - 8y$$

Integral Equation

An integral equation is an equation in which an unknown function appears under one or more integration signs.Example of an integral equation is :

$$F(x) = \int_{g(x)}^{h(x)} K(x,t)y(t)dt$$

Where g(x) and h(x) are limits of integration ,which can either be variables ,constants or mixed. K(x,t) is called the Kernel of the integral which is the known function inside the integral sign.

Integro Differential Equation

Integro-differential equations are type of differential equation that involve both differential operators and integral operators. They typically take the standard form:

$$\sum_{i=0}^{n} a_i(x)u^{(i)}(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dt$$

Subject to the conditions

$$u^{(i)}(\alpha_i) = \beta_i, \;\; i = 0,1,2,\ldots,n-1$$

$$u(\alpha_0) = \beta_0, u'(\alpha_1) = \beta_1, u''(\alpha_2) = \beta_2, \ldots, u^{(n-1)}(\alpha_{n-1}) = \beta_{n-1}$$

Where $g(x)$ *and* $h(x)$ are limits of integration, λ is a constant parameter, $k(x,t)$ is a function of the two variables t *and* x called the Kernel of the integral sign which is the known function in the integral sign. The unknown function $u(x)$ to be determined appears both inside and outside the integral sign as well. The function $f(x)$ *and* $k(x,t)$are given in advance. It is to be noted that the limits of integration $g(x)$ *and* $h(x)$ can either be not fixed(variables), fixed(constants) and mixed (variable and constant) and

$$u^{(n)} = \frac{d^{(n)}u}{dx^{(n)}}$$

Orthogonal Function

Two different functions say $z_n(x)$ *and* $z_m(x)$ are said to be orthogonal if their inner product is zero when n is not equal to m.

$$\langle z_n(x), z_m(x)\rangle \equiv \int_a^b z_n(x), z_m(x)dx = 0$$

On the other hand, a third function $w(x) > 0$ exists, then:

$$\langle z_n(x), z_m(x)\rangle = \int_a^b w(x)z_n(x), z_m(x)dx = 0$$

Then we say that $y_n(x)$ *and* $y_m(x)$ are mutually orthogonal with respect to the weight function w(x). The construction of our polynomial actually followed the basic procedure for obtaining orthogonal polynomials but using a quadratic weight functions.

Fractional Integro Differential Equation

A fractional integro-differential equation is a differential equation that involves both fractional derivatives and integrals. In these equations, the unknown function is differentiated to a fractional order and integrated to a fractional order simultaneously. The general form of fractional order integro- differential equation considered in this paper is given as

$$D^{\alpha}y(x) = f(x) + \int_{0}^{b(x)} K(x,t)y(t)dt \quad 0 < x,t < 1$$

together with the following supplementary condition,$y(0) = \beta$ and D^{α} is in the Caputo sense of the differential integral functions andα is a parameter denoting the fractional order derivative of the function. A very important property of $D^{\alpha}f(t)$ is:

$$D^{\alpha}x^{n} = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{(n-\alpha)}$$

Hermite Polynomial

Hermite polynomials, denoted as $H_n(x)$, are a set of orthogonal polynomials defined by the recurrence relation:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

with initial conditions $H_0(x) = 1$ and $H_1(x) = 2x$.
They satisfy the Hermite differential equation :

$$\frac{d^2H}{dx^2} - 2x\frac{dH}{dx} + 2nH = 0$$

and have orthogonality with respect to the weight function e^{-x^2}. The first five terms of Hermite Polynomials are:

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

Least Square Method

Least Square is defined as a standard method employed to obtain numerical solutions of fractional order integro differential equations by using Chebyshev Polynomials ,Legendre Polynomials,Hermite Polynomials,Power series as basis functions.

Exact Solution

The true solution of a differential equation that satisfies the given initial and boundary conditions.

Approximate Solution

An approximate solution refers to a numerical value or set of values obtained through computational methods that approximate the true solution of a mathematical problem. Due to the complexity or intractability of many mathematical problems, it's often not possible to find exact solutions analytically. In such cases, numerical methods are employed to obtain approximations to the solutions.

Absolute Error

Absolute error is the absolute difference between the exact solution and the approximate solution at any given point in interval under consideration.

$$Absolute\ error = |Exact\ Value - Approximate\ Value|$$

3. PROBLEM CONSIDERED

In this research work, the α^{th} order Volterra-fractional integro -differential equation considered is of the form

$$D^{\alpha}y(x) = f(x) + \int_{a}^{b(x)} K(x,t)y(t)dt \tag{1}$$

Subject to the condition

$$y^{(i)}(\beta_i) = \beta_i; \ \ i = 0,1,2,3,\cdots, \tag{2}$$

are considered.

4. CONSTRUCTION OF HERMITE POLYNOMIALS USED IN THIS WORK:

Hermite Polynomials are a sequence of orthoghonal polynomials , they are denoted by $H_n(x)$ where n is a non-negative integer and x is a real variable. We can generate Hermite Polynomials using the Rodrigue's formular

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

They can also be generated using the recurrence relation

$$H_{n+1} = 2xH_n(x) - 2nHn - 1(x)$$

With initial conditions $H_0(x) = 1$ and $H_1(x) = 2x$ The first few Hermite polynomials valid in the interval $[-\infty, \infty]$ are given as:

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

Normalization Of Hermite Polynomials

To convert the Hermite polynomials from the standard interval $[-\infty, \infty]$ to $[0, 1]$,we substitute x with a normalize variable t ,where t ranges from $0to1$

$$t = \frac{x-a}{b-a}$$

$a = 0$ *and* $b = 1$ hence $\frac{x-0}{1-0} = x$ which Implies $t = x$
Hence the first few Hermite polynomials valid in the interval $[0, 1]$ are given as:

$$H_0(t) = 1$$

$$H_1(t) = 2t$$

$$H_2(t) = 4t^2 - 2$$

$$H_3(t) = 8t^3 - 12t$$

$$H_4(t) = 16t^4 - 48t^2 + 12$$

5. DESCRIPTION OF PROPOSED METHOD ON THE PROBLEM CONSIDERED:

Here, we considered the general form of Volterra fractional Integro-differential equation:

$$D^{\alpha}y(x) = f(x) + \int_a^{b(x)} K(x,t)y(t)dt \qquad (3)$$

Subject to the condition

$$y^{(i)}(\beta) = \beta_i; i = 0, 1, \cdots \qquad (4)$$

We assume an approximate solution of the form

$$y_N(x) = \sum_{i=0}^{N} a_i \, H_i(x) \qquad (5)$$

(5) can be expanded as

$$y_N(x) = a_0H_0(x) + a_1H_1(x) + a_2H_2(x) + a_3H_3(x) + \cdots + a_NH_N(x) \qquad (6)$$

Substituting (6) into (3), to get

$$D^{\alpha}\{a_0H_0(x) + a_1H_1(x) + a_2H_2(x) + \cdots + a_NH_N(x)\} = f(x) + \int_a^{b(x)} K(x,t)\{a_0H_0(t) + a_1H_1(t)$$

$$+a_2H_2(t) + \cdots + a_NH_N(t)\}dt \qquad (7)$$

Moving the RHS of (7) to the LHS, to get

$$D^{\alpha}\{a_0H_0(x) + a_1H_1(x) + a_2H_2(x) + \cdots + a_NH_N(x)\} - f(x) - \int_a^{b(x)} K(x,t)\{a_0H_0(t) + a_1H_1(t)$$

$$+a_2H_2(t)+\cdots+a_NH_N(t)\}dt=0 \tag{8}$$

Collecting like terms in (8) to get

$$\{D^{\alpha}H_0(x)-\int_a^{b(x)} K(x,t)H_0(t)dt\}a_0+\{D^{\alpha}H_1(x)-\int_a^{b(x)} K(x,t)H_1(t)dt\}a_1$$
$$+\{D^{\alpha}H_2(x)-\int_a^{b(x)} K(x,t)H_2(t)dt\}a_2+\cdots+\{D^{\alpha}H_N(x)-\int_a^{b(x)} K(x,t)H_N(t)dt\}a_N$$

$$-f(x)=0 \tag{9}$$

Here the Residue equation $R(x)$ of (9) is

$$R(x)=\{D^{\alpha}H_0(x)-\int_a^{b(x)} K(x,t)H_0(t)dt\}a_0+\{D^{\alpha}H_1(x)-\int_a^{b(x)} K(x,t)H_1(t)dt\}a_1$$
$$+\{D^{\alpha}H_2(x)-\int_a^{b(x)} K(x,t)H_2(t)dt\}a_2+\cdots+\{D^{\alpha}H_N(x)$$

$$-\int_a^{b(x)} K(x,t)H_N(t)dt\}a_N-f(x) \tag{10}$$

From the residual function, we generate our functional , $S(a_0,a_1,a_2,\cdots,a_N)$ so that

$$S(a_0,a_1,a_2,\cdots,a_N)=\int_0^1[R(x)]^2W(x)dx$$

Where w(x) is our weight function. In this work , we take our $w(x)=1$ for simplicity then

$$S(a_0,a_1,a_2,\cdots,a_N)=\int_0^1[\{D^{\alpha}H_0(x)-\int_a^{b(x)} K(x,t)H_0(t)dt\}a_0+\{D^{\alpha}H_1(x)-$$
$$\int_a^{b(x)} K(x,t)H_1(t)dt\}a_1+\{D^{\alpha}H_2(x)-\int_a^{b(x)} K(x,t)H_2(t)dt\}a_2$$

$$+\cdots+\{D^{\alpha}H_N(x)-\int_a^{b(x)} K(x,t)H_N(t)dt\}a_N-f(x)]^2dx \tag{11}$$

So,finding the values of $a_i, i=0,1,2,\cdots,N$, which minimize S is equivalent to finding the best approximate for the solution of the fractional integro-differential equation (3). The Minimum value of S is obtained by setting

$$\frac{\partial S}{\partial a_j}=0,\ \ j=0,1,\cdots,N \tag{12}$$

Using the condition in (12) we get $(N+1)$ system of equation
Hence,for $j=0$

$$\frac{\partial S}{\partial a_0}=2\int_0^1(D^{\alpha}H_0(x)-\int_a^{b(x)} K(x,t)H_0(t)dt)[\{D^{\alpha}H_0(x)$$
$$-\int_a^{b(x)} K(x,t)H_0(t)dt\}a_0+\{D^{\alpha}H_1(x)-\int_a^{b(x)} K(x,t)H_1(t)dt\}a_1+\{D^{\alpha}H_2(x)$$
$$-\int_a^{b(x)} K(x,t)H_2(t)dt\}a_2+\cdots+\{D^{\alpha}H_N(x)-\int_a^{b(x)} K(x,t)H_N(t)dt\}a_N$$

$$-f(x)]=0 \tag{13}$$

Hence,for $j=1$

$$\frac{\partial S}{\partial a_1}=2\int_0^1(D^{\alpha}H_1(x)-\int_a^{b(x)} K(x,t)H_1(t)dt)[\{D^{\alpha}H_0(x)-$$
$$\int_a^{b(x)} K(x,t)H_0(t)dt\}a_0+\{D^{\alpha}H_1(x)-\int_a^{b(x)} K(x,t)H_1(t)dt\}a_1$$
$$+\{D^{\alpha}H_2(x)-\int_a^{b(x)} K(x,t)H_2(t)dt\}a_2+\cdots+\{D^{\alpha}H_N(x)$$

$$-\int_a^{b(x)} K(x,t)H_N(t)dt\}a_N-f(x)]=0 \tag{14}$$

Hence,for $j = 2$

$$\frac{\partial S}{\partial a_2} = 2\int_0^1 (D^{\alpha} H_2(x) - \int_a^{b(x)} K(x,t)H_2(t)dt)[\{D^{\alpha} H_0(x)$$
$$-\int_a^{b(x)} K(x,t)H_0(t)dt\}a_0 + \{D^{\alpha} H_1(x) - \int_a^{b(x)} K(x,t)H_1(t)dt\}a_1 + \{D^{\alpha} H_2(x)$$
$$-\int_a^{b(x)} K(x,t)H_2(t)dt\}a_2 + \cdots + \{D^{\alpha} H_N(x) - \int_a^{b(x)} K(x,t)H_N(t)dt\}a_N$$

$$-f(x)] = 0 \tag{15}$$

⋮

Hence,for $j = N$

$$\frac{\partial S}{\partial a_0} = 2\int_0^1 (D^{\alpha} H_N(x) - \int_a^{b(x)} K(x,t)H_N(t)dt)[\{D^{\alpha} H_0(x)$$
$$-\int_a^{b(x)} K(x,t)H_0(t)dt\}a_0 + \{D^{\alpha} H_1(x) - \int_a^{b(x)} K(x,t)H_1(t)dt\}a_1 + \{D^{\alpha} H_2(x)$$
$$-\int_a^{b(x)} K(x,t)H_2(t)dt\}a_2 + \cdots + \{D^{\alpha} H_N(x) - \int_a^{b(x)} K(x,t)H_N(t)dt\}a_N$$

$$-f(x)] = 0 \tag{16}$$

Hence $(13),(14),(15)$ *and* (16)results into $(N+1)$ system of algebraic equation with $(N+1)$ unknown constants $(a_0, a_1, \cdots, a_N)$ which are then solved by Gaussian elimination to obtain the unknown constant. The unknown constant are then substituted back into the assumed approximate solution (6) and then simplified .

6. CONVERGENCE AND UNIQUENESS ANALYSIS FOR THE PROPOSED METHOD

The analysis of the Least Squares Method using Hermite polynomials for solving Volterra fractional integro-differential equations involves not only convergence but also the uniqueness of the solution. Below is a detailed explanation of both aspects.

1. **Approximation Properties of Hermite Polynomials:**
   As mentioned earlier, Hermite polynomials $H_i(x)$ form an orthogonal basis on the interval $[-\infty, \infty]$ with respect to the weight function $W(x) = e^{-x^2}$. The approximation of a smooth function $y(x)$ by a truncated series $y_N(x) = \sum_{i=0}^{N} a_i H_i(x)$ provides an increasingly accurate representation as N increases.
2. **Minimization of the Residual Function:**
   The residual function $R(x)$ for the approximate solution $y_N(x)$ is minimized using the Least Squares Method, leading to the best-fit coefficients a_j that minimize the residual.
3. **Error Analysis and Convergence:**
   The error in the approximation can be quantified by the L_2-norm of the residual function $R(x)$. As N increases, this error decreases, indicating convergence of the method.
4. **Uniqueness of the Solution:**
   - **Linear Independence of Hermite Polynomials:** Hermite polynomials $H_i(x)$ are linearly independent. This implies that the system of algebraic equations obtained from the minimization of the residual function will have a unique solution for the coefficients a_j.
   - **Non-Singularity of the System Matrix:** The system of equations generated by setting $\frac{\partial S}{\partial a_j} = 0$ for $j = 0, 1, \ldots, N$ can be written in matrix form as $\mathbf{Aa} = \mathbf{b}$, where $\mathbf{A}$ is a matrix derived from the differential operator and the integral kernel, $\mathbf{a}$ is the vector of coefficients, and $\mathbf{b}$ is the vector derived from the function $f(x)$.
     The uniqueness of the solution depends on the non-singularity of the matrix $\mathbf{A}$. If $\mathbf{A}$ is non-singular, the system has a unique solution, ensuring that the coefficients a_j are uniquely determined.
   - **Existence and Uniqueness Theorem for Integro-Differential Equations:** For many types of fractional integro-differential equations, existence and uniqueness theorems guarantee that a unique solution exists under certain conditions on the kernel $K(x,t)$ and the function $f(x)$. Since the Least Squares Method seeks to approximate this unique solution, the method inherits the uniqueness property.
5. **Stability and Robustness:**
   The stability of the solution is linked to the conditioning of the system matrix $\mathbf{A}$. A well-conditioned matrix ensures that small changes in the input (e.g., due to numerical errors) lead to small changes in the output, contributing to the robustness of the method.

7. NUMERICAL EXAMPLES:

Example 1

Consider the Volterra fractional order integro-differential equation of the form:

$$D^{\alpha}y(x) = 2+4x+8\int_0^x (x-t)y(t)dt \tag{17}$$

subject to the condition

$$y(0) = 1 \tag{18}$$

Solution

In this example, case $\alpha = 1,\ 0.9,\ 0.8$ shall be considered
For case $\alpha = 1$, (17) reduces to an integro-differential equation of the form:

$$y^{'}(x) = 2+4x+8\int_0^x (x-t)y(t)dt \tag{19}$$

subject to the condition

$$y(0) = 1 \tag{20}$$

Which has an exact solution of

$$y(x) = e^{2x} \tag{21}$$

Case $\alpha = 0.9$:
Subsituting $\alpha = 0.9$ into (17) to get

$$D^{0.9}y(x) = 2+4x+8\int_0^x (x-t)y(t)dt \tag{22}$$

Here an approximate solution of the form:

$$y_4(x) = \sum_{i=0}^{4} a_i H_i(x) \tag{23}$$

(23) can be expanded as

$$y_4(x) = a_0H_0(x)+a_1H_1(x)+a_2H_2(x)+a_3H_3(x)+a_4H_4(x) \tag{24}$$

Substituting the Hermite polynomial obtained in section (4) into (24) to get:

$$y_4(x) = a_0+2xa_1+a_2(4x^2-2)+a_3(8x^3-12x)+a_4(16x^4-48x^2+12) \tag{25}$$

Substituting (25) into (22) to get

$$D^{0.9}\{a_0+2xa_1+a_2(4x^2-2)+a_3(8x^3-12x)+a_4(16x^4-48x^2+12)\} = 2+4x$$
$$+8\int_0^x (x-t)\{a_0+2ta_1+a_2(4t^2-2)+a_3(8t^3-12t)$$
$$+a_4(16t^4-48t^2+12)\}dt \tag{26}$$

Moving the RHS of (26) to the LHS to get

$$D^{0.9}\{a_0+2xa_1+a_2(4x^2-2)+a_3(8x^3-12x)+a_4(16x^4-48x^2+12)\}-2-4x$$
$$-8\int_0^x (x-t)\{a_0+2ta_1+a_2(4t^2-2)+a_3(8t^3-12t)$$
$$+a_4(16t^4-48t^2+12)\}dt = 0 \tag{27}$$

Collecting like term in (27) to get

$$\{D^{0.9}-8\int_0^x (x-t)dt\}a_0+\{D^{0.9}\{2x\}-8\int_0^x 2t(x-t)dt\}a_1$$
$$+\{D^{0.9}\{4x^2-2\}-8\int_0^x (4t^2-2)(x-t)dt\}a_2$$
$$+\{D^{0.9}\{8x^3-12x\}-8\int_0^x (8t^3-12t)(x-t)dt\}a_3$$

$$+\{D^{0.9}\{16x^4-48x^2+12\}-8\int_0^x(16t^4-48t^2+12)(x-t)dt\}a_4-2-4x=0 \quad (28)$$

Evaluating $D^{0.9}$ in (28) using this relation

$$D^{\alpha}x^n=\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{(n-\alpha)}$$

And after simplification to get

$$\begin{aligned}&-2+8a_2x^2+16a_3x^3+32a_4x^4-48a_4x^2-4x^2a_0-2.66666666x^4c_2-2.666666668x^3c_1-3.2x^5c_3\\&+0.1051137006\frac{a_0}{x^{0.9}}+2.102274012a_1x^{0.1}+7.644632772a_2x^{1.1}\\&-0.2102274012\frac{a_2}{x^{0.9}}+21.84180792a_3x^{2.1}-12.61364407a_3x^{0.1}\\&+56.36595592a_4x^{3.1}-91.73559326c_4x^{1.1}+1.261364407\frac{a_4}{x^{0.9}}-4x\end{aligned}$$

$$-4.26666666\cdot a_4x^6=0 \quad (29)$$

Here the residue equation $R(x)$ of (29) is

$$\begin{aligned}R(x)=&-2+8a_2x^2+16a_3x^3+32a_4x^4-48a_4x^2-4x^2a_0-2.66666666x^4a_2-2.666666668x^3a_1-3.2x^5a_3\\&+0.1051137006\frac{a_0}{x^{0.9}}+2.102274012a_1x^{0.1}+7.644632772a_2x^{1.1}\\&-0.2102274012\frac{a_2}{x^{0.9}}+21.84180792a_3x^{2.1}-12.61364407a_3x^{0.1}\\&+56.36595592a_4x^{3.1}-91.73559326c_4x^{1.1}+1.261364407\frac{a_4}{x^{0.9}}-4x\end{aligned}$$

$$-4.26666666a_4x^6 \quad (30)$$

From the residual function, we generate our functional , $S(a_0,a_1,a_2,\cdots,a_N)$ so that

$$S(a_1,a_2,,a_3,a_4)=\int_0^1[R(x)]^2W(x)dx$$

where w(x) =1 is so that

$$\begin{aligned}S(a_0,a_1,a_2,,a_3,a_4)=&\int_0^1[-2+8a_2x^2+16a_3x^3+32a_4x^4-48a_4x^2-4x^2a_0-2.66666666x^4a_2\\&-2.666666668x^3a_1-3.2x^5a_3+0.1051137006\frac{a_0}{x^{0.9}}+2.102274012a_1x^{0.1}+7.644632772a_2x^{1.1}\\&-0.2102274012\frac{a_2}{x^{0.9}}+21.84180792a_3x^{2.1}-12.61364407a_3x^{0.1}\\&+56.36595592a_4x^{3.1}-91.73559326c_4x^{1.1}+1.261364407\frac{a_4}{x^{0.9}}-4x\end{aligned}$$

$$-4.26666666a_4x^6]^2dx \quad (31)$$

Hence,

$$\frac{\partial S}{\partial a_1}=-0.5906730344a_0+3.134308645a_2-24.88931098a_4$$

$$-1.072264938a_3+0.7618109194=0 \quad (32)$$

$$\frac{\partial S}{\partial a_2}=3.134308645a_1+63.03432259a_3-16.09416449a_0+61.56084972a_2$$

$$-354.3386933a_4-32.69663968=0 \quad (33)$$

$$\frac{\partial S}{\partial a_3}=-18.11711230a_0+63.03432259a_2-313.2450852a_4-1.072264938a_1$$

$$+95.10623670a_3-29.48168223=0 \quad (34)$$

$$\frac{\partial S}{\partial a_4} = -24.88931098a_1 - 313.2450852a_3 + 89.90647847a_0 + 2125.647352a_4$$

$$-354.3386933a_2 + 194.1442562 = 0 \tag{35}$$

From the boundary condition that is (18), to get

$$y(0) = 1 \to a_0 - 2a_2 + 12a_4 = 1 \tag{36}$$

Solving (32), (33), (34), (35), (36) using Guassian elimination method to get

$$a_0 = 6.581332015,$$
$$a_1 = -2.342454285,$$
$$a_2 = 4.107106771,$$
$$a_3 = -0.4621769608,$$
$$a_4 = 0.2194067940$$

The values of $a_i(i = 0(1)4)$ are then substituted into (25) and after simplification to get the required approximate solution

$$y_4(x) = 1.000000001 + 0.861214960x + 5.89690097x^2 - 3.697415686x^3 + 3.510508704x^4$$

Case $\alpha = 0.8$
Subsituting $\alpha = 0.8$ into (17) to get

$$D^{0.8}y(x) = 2 + 4x + 8\int_0^x (x-t)y(t)dt \tag{37}$$

Here an approximate solution of the form:

$$y_4(x) = \sum_{i=0}^{4} a_i H_i(x) \tag{38}$$

(4.22) can be expanded as

$$y_4(x) = a_0H_0(x) + a_1H_1(x) + a_2H_2(x) + a_3H_3(x) + a_4H_4(x) \tag{39}$$

Substituting the Hermite polynomial obtained in section (4) into (39) to get:

$$y_4(x) = a_0 + 2xa_1 + a_2(4x^2 - 2) + a_3(8x^3 - 12x) + a_4(16x^4 - 48x^2 + 12) \tag{40}$$

Substituting (40) into (37) to get

$$D^{0.8}\{a_0 + 2xa_1 + a_2(4x^2 - 2) + a_3(8x^3 - 12x) + a_4(16x^4 - 48x^2 + 12)\} = 2 + 4x$$
$$+8\int_0^x (x-t)\{a_0 + 2ta_1 + a_2(4t^2 - 2) + a_3(8t^3 - 12t)$$

$$+a_4(16t^4 - 48t^2 + 12)\}dt \tag{41}$$

Moving the RHS of (41) to the LHS to get

$$D^{0.8}\{a_0 + 2xa_1 + a_2(4x^2 - 2) + a_3(8x^3 - 12x) + a_4(16x^4 - 48x^2 + 12)\} - 2 - 4x$$
$$-8\int_0^x (x-t)\{a_0 + 2ta_1 + a_2(4t^2 - 2) + a_3(8t^3 - 12t)$$

$$+a_4(16t^4 - 48t^2 + 12)\}dt = 0 \tag{42}$$

Collecting like term in (42) to get

$$\{D^{0.8} - 8\int_0^x (x-t)dt\}a_0 + \{D^{0.8}\{2x\} - 8\int_0^x 2t(x-t)dt\}a_1$$
$$+\{D^{0.8}\{4x^2 - 2\} - 8\int_0^x (4t^2 - 2)(x-t)dt\}a_2$$
$$+\{D^{0.8}\{8x^3 - 12x\} - 8\int_0^x (8t^3 - 12t)(x-t)dt\}a_3$$

$$+\{D^{0.8}\{16x^4 - 48x^2 + 12\} - 8\int_0^x (16t^4 - 48t^2 + 12)(x-t)dt\}a_4 - 2 - 4x = 0 \tag{43}$$

Evaluating $D^{0.8}$ in (43) using this relation

$$D^{\alpha}x^{n}=\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{(n-\alpha)}$$

And after simplification to get

$$\begin{aligned}&\frac{0.2178248842\,a_0}{x^{0.8}}+2.178248842\,a_1\,x^{0.2}+7.260829472\,a_2\,x^{1.2}-\frac{0.4356497684\,a_2}{x^{0.8}}\\&+19.80226220\,a_3\,x^{2.2}-13.06949305\,a_3\,x^{0.2}+49.50565550\,a_4\,x^{3.2}-87.12995366\,a_4\,x^{1.2}\\&+\frac{2.613898610\,a_4}{x^{0.8}}-2-4x+21.33333334\,a_4\,x^{6}-(1.600000000\,(16x\,a_4-8\,a_3))\,x^{5}\\&-(2.000000000\,(8\,a_3\,x-4\,a_2+48\,a_4))\,x^{4}\\&-(2.666666666\,(x\,(4\,a_2-48\,a_4)-2\,a_1+12\,a_3))\,x^{3}\\&-(4.000000000\,(x\,(2\,a_1-12\,a_3)-a_0+2\,a_2-12\,a_4))\,x^{2}\\&-8x^{2}\,(a_0-2\,a_2+12\,a_4)=0\end{aligned}\tag{44}$$

Here the Residue equation $R(x)$ of (44) is

$$\begin{aligned}R(x)=&\frac{0.2178248842\,a_0}{x^{0.8}}+2.178248842\,a_1\,x^{0.2}+7.260829472\,a_2\,x^{1.2}-\frac{0.4356497684\,a_2}{x^{0.8}}\\&+19.80226220\,a_3\,x^{2.2}-13.06949305\,a_3\,x^{0.2}+49.50565550\,a_4\,x^{3.2}-87.12995366\,a_4\,x^{1.2}\\&+\frac{2.613898610\,a_4}{x^{0.8}}-2-4x+21.33333334\,a_4\,x^{6}-(1.600000000\,(16x\,a_4-8\,a_3))\,x^{5}\\&-(2.000000000\,(8\,a_3\,x-4\,a_2+48\,a_4))\,x^{4}\\&-(2.666666666\,(x\,(4\,a_2-48\,a_4)-2\,a_1+12\,a_3))\,x^{3}\\&-(4.000000000\,(x\,(2\,a_1-12\,a_3)-a_0+2\,a_2-12\,a_4))\,x^{2}\\&-8x^{2}\,(a_0-2\,a_2+12\,a_4)\end{aligned}\tag{45}$$

From the residual function, we generate our functional , $S(a_0,a_1,a_2,\cdots,a_N)$ so that

$$S(a_1,a_2,,a_3,a_4)=\int_0^1[R(x)]^2W(x)dx$$

Where w(x) =1 is so that

$$\begin{aligned}S(a_0,a_1,a_2,,a_3,a_4)=&\int_0^1[\frac{0.2178248842\,a_0}{x^{0.8}}+2.178248842\,a_1\,x^{0.2}+7.260829472\,a_2\,x^{1.2}\\&-\frac{0.4356497684\,a_2}{x^{0.8}}\\&+19.80226220\,a_3\,x^{2.2}-13.06949305\,a_3\,x^{0.2}+49.50565550\,a_4\,x^{3.2}-87.12995366\,a_4\,x^{1.2}\\&+\frac{2.613898610\,a_4}{x^{0.8}}-2-4x+21.33333334\,a_4\,x^{6}-(1.600000000\,(16x\,a_4-8\,a_3))\,x^{5}\\&-(2.000000000\,(8\,a_3\,x-4\,a_2+48\,a_4))\,x^{4}\\&-(2.666666666\,(x\,(4\,a_2-48\,a_4)-2\,a_1+12\,a_3))\,x^{3}\\&-(4.000000000\,(x\,(2\,a_1-12\,a_3)-a_0+2\,a_2-12\,a_4))\,x^{2}\\&-8x^{2}\,(a_0-2\,a_2+12\,a_4)]^2dx\end{aligned}\tag{46}$$

Hence,

$$\frac{\partial S}{\partial a_1}=-1.329228074a_3+.8109505676a_1-2.560485958-25.13490124a_4$$

$$+3.168125579a_2-0.5909480389a_0=0\tag{47}$$

$$\frac{\partial S}{\partial a_2}=-14.73266851a_0-328.9182586a_4+55.32602032a_2-31.49345064$$

$$+50.46430898a_3+3.168125579a_1=0\tag{48}$$

$$\frac{\partial S}{\partial a_3}=-1.329228074a_1+74.55475936a_3-23.79147983+50.46430898a_2$$

$$-256.6204860a_4-15.13272309a_0=0\tag{49}$$

$$\frac{\partial S}{\partial a_4} = 84.99756421a_0 - 328.9182586a_2 + 2025.898970a_4 + 194.0440665$$

$$-25.13490124a_1 - 256.6204860a_3 = 0 \quad (50)$$

From the boundary condition that is (18), to get

$$y(0) = 1 \rightarrow a_0 - 2a_2 + 12a_4 = 1 \quad (51)$$

Solving (47), (48), (49), (50), (51) using Guassian elimination method to get

$$a_0 = 9.828452703,$$

$$a_1 = -5.974684273,$$

$$a_2 = 6.460219451,$$

$$a_3 = -.9915155747,$$

$$a_4 = .3409988499$$

The values of $a_i(i = 0(1)4)$ are then substituted into (25) and after simplification to get the required approximate solution

$$y_4(x) = 1.000000000 - 0.05115033x + 9.47281663x^2 - 7.931968647x^3 + 5.455915704x^4$$

Example 2

Consider the Volterra fractional order integro-differential equation of the form:

$$D^{\alpha}y(x) = 1 + x + \int_0^x (x-t)y(t)dt \quad (52)$$

subject to the conditions

$$y(0) = 1 \quad (53)$$

$$y''(0) = 1 \quad (54)$$

Method of Solution

In this example, case $\alpha = 2,\ 1.9,\ 1.8$ shall be considered
For $\alpha = 2$, (52) reduces to an integro-differential equation of the form:

$$y''(x) = 1 + x + \int_0^x (x-t)y(t)dt \quad (55)$$

subject to the conditions

$$y(0) = 1 \quad (56)$$

$$y''(0) = 1 \quad (57)$$

Which has an exact solution of

$$y(x) = e^x \quad (58)$$

We shall estimate the approximate solution using our proposed method for case $\alpha = 1.9,\ 1.8$ following the same procedure as described in Example 1 we get the following approximate:

Case $\alpha = 1.9$:

$$y_4(x) = 1.000000000 + 1.000000000x + 1.617086058x^2 - .8294706504x^3 + 0.3829361560x^4$$

Case $\alpha = 1.8$:

$$y_4(x) = 1.000000001 + 1.000000000x + 2.561083006x^2 - 1.825138762x^3 + 0.7337109461x^4$$

7.1. Example 3

Consider the Volterra fractional order integro-differential equation of the form:

$$D^{\alpha}y(x) = 2 - 2x\sin(x) - \int_0^x (x-t)y(t)dt \tag{59}$$

subject to the conditions

$$y(0) = 0 \tag{60}$$

$$y''(0) = 0 \tag{61}$$

Method of Solution

In this example, case $\alpha = 2,\ 1.9,\ 1.8$ shall be considered
For $\alpha = 2$, (59) reduces to an integro-differential equation of the form:

$$y''(x) = 2 - 2x\sin(x) - \int_0^x (x-t)y(t)dt \tag{62}$$

subject to the conditions

$$y(0) = 0 \tag{63}$$

$$y'(0) = 0 \tag{64}$$

Which has an exact solution of

$$y(x) = x\sin(x) \tag{65}$$

We shall estimate the approximate solution using our proposed method for case $\alpha = 1.9,\ 1.8$ following the same procedure as described in Example 1 we get the following approximate:

Case $\alpha = 1.9$:

$$y_4(x) = -2.10^{-11} + 1.238750157x^2 - 0.2704426868x^3 - 0.07572950229x^4$$

Case $\alpha = 1.8$:

$$y_4(x) = -4.10^{-11} + 1.445837278x^2 - .4939066018x^3 - 0.02388311779x^4$$

8. TABLES AND GRAPHS

Example 1

x	Exact	0.9	0.8	$\lvert Exact - 0.9\rvert$	$\lvert 0.9 - 0.8\rvert$
0.0	1.0000000000	1.0000000010	1.0000000000	1.0000e-09	1.00000e-09
0.1	1.2214027580	1.1417441420	1.0822267560	7.9659e-02	5.95174e-02
0.2	1.4918246980	1.3841565210	1.3139563150	1.0767e-01	7.02002e-02
0.3	1.8221188000	1.7176904720	1.6672381610	1.0443e-01	5.04523e-02
0.4	2.2255409280	2.1412245590	2.1272159780	8.4316e-02	1.40086e-02
0.5	2.7182818280	2.6620625560	2.6921276440	5.6219e-02	3.00651e-02
0.6	3.3201169230	3.2959334660	3.3733052360	2.4183e-02	7.73718e-02
0.7	4.0551999670	4.0669915080	4.1951750330	1.1792e-02	1.28184e-01
0.8	4.9530324240	5.0078161240	5.1952575040	5.4784e-02	1.87441e-01
0.9	6.0496474640	6.1594119770	6.4241673220	1.0976e-01	2.64755e-01
1.0	7.3890560990	7.5712089490	7.9456133570	1.8215e-01	3.74404e-01

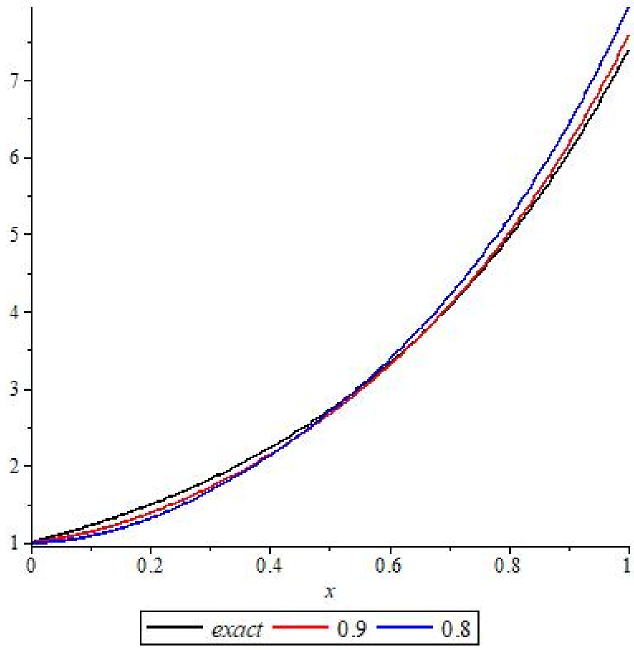

Figure 1: Graphical representation of Example 1 for case α= 1,0.9,0.8

Example 2

x	Exact	1.9	1.8	$\|Exact-1.9\|$	$\|1.9-1.8\|$
0.0	1.0000000000	1.0000000000	1.0000000010	0.0000e+00	1.00000e-09
0.1	1.1051709180	1.1153796840	1.1238590630	1.0209e-02	8.47938e-03
0.2	1.2214027580	1.2586603750	1.2890161490	3.7258e-02	3.03558e-02
0.3	1.3498588080	1.4262438200	1.4871617840	7.6385e-02	6.09180e-02
0.4	1.4918246980	1.6154508130	1.7117474010	1.2363e-01	9.62966e-02
0.5	1.6487212710	1.8245211930	1.9579853410	1.7580e-01	1.33464e-01
0.6	1.8221188000	2.0526138460	2.2228488490	2.3050e-01	1.70235e-01
0.7	2.0137527070	2.2998067060	2.5050720770	2.8605e-01	2.05265e-01
0.8	2.2255409280	2.5670967540	2.8051500820	3.4156e-01	2.38053e-01
0.9	2.4596031110	2.8564000150	3.1253388310	3.9680e-01	2.68939e-01
1.0	2.7182818280	3.1705515640	3.4696551910	4.5227e-01	2.99104e-01

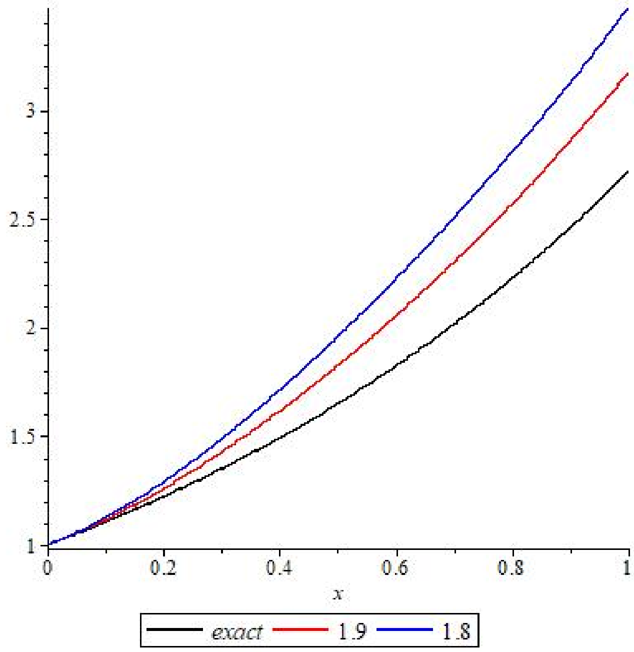

Figure 2: Graphical representation of Example 2 for case α= 2,1.9,1.8

Example 3

x	Exact	1.9	1.8	$\|Exact - 1.9\|$	$\|1.9 - 1.8\|$
0.0	0.0000000000	-0.0000000000	-0.0000001817	2.0000e-11	1.81689e-07
0.1	0.0099833417	0.0121094859	0.0139618962	2.1261e-03	1.85241e-03
0.2	0.0397338662	0.0472652976	0.0538438436	7.5314e-03	6.57855e-03
0.3	0.0886560620	0.1035721526	0.1165962417	1.4916e-02	1.30241e-02
0.4	0.1557673369	0.1789530178	0.1991123525	2.3186e-02	2.01593e-02
0.5	0.2397127693	0.2711491095	0.2982281177	3.1436e-02	2.70790e-02
0.6	0.3387854840	0.3777198927	0.4107221603	3.8934e-02	3.30023e-02
0.7	0.4509523810	0.4960430818	0.5333157835	4.5091e-02	3.72727e-02
0.8	0.5738848727	0.6233146408	0.6626729711	4.9430e-02	3.93583e-02
0.9	0.7049942186	0.7565487818	0.7954003867	5.1555e-02	3.88516e-02
1.0	0.8414709848	0.8925779679	0.9280473764	5.1107e-02	3.54694e-02

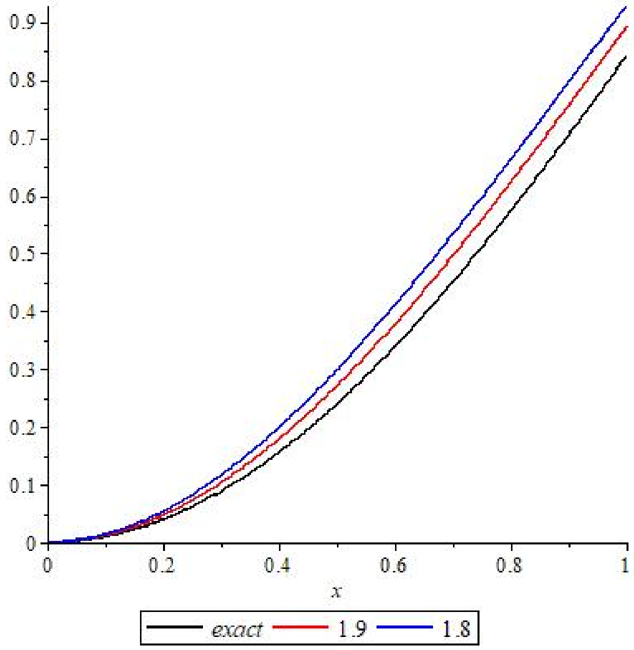

Figure 3: Graphical representation of Example 3 for case α= 2,1.9,1.8

9. CONCLUSION

The tables and graphs above illustrate that as the value of α decreases, the graph aligns more closely with the exact solution. However, the error increases as the value of α deviates from the exact solution. This suggests that the method is highly effective and accurate. In conclusion, the least squares method is a highly accurate approach for solving fractional-order integro-differential equations, particularly in scenarios where the values of α are unknown. Additionally, the Hermite polynomial significantly enhances the accuracy as a basis function in this method.

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