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Research paper

Optimal resource allocation in network queuing system for healthcare delivery

Etoroabasi J. Isaac [1], Nse S. Udoh [2 *]

[1] *Research Scholars, Department of Statistics, Akwa Ibom State University, Ikot Akpaden, Mkpat Enin, Nigeria*
[2] *Senior Academics, Department of Statistics, University of Uyo, Nigeria*
**Corresponding author E-mail: nsesudoh@uniuyo.edu.ng*

Abstract

In healthcare settings, where patients' care involves multiple stages and various service providers, efficient resource allocation is critical. On this perspective, classical queuing theory offers insights into patients' flow dynamics and resource utilization, providing a foundation for understanding system performance. This research aims at effectively applying network queuing techniques to model and optimize resource allocation in network queuing systems for effective healthcare delivery. A network queuing system involving six departments that made up the healthcare service delivery at Immanuel General Hospital, Eket is formulated and analysed in this work to improve its operation, considering its arrival and departure rates as well as time spent at each node (waiting time). Also, simulation by bootstrap method was performed to test for the robustness as well as validation of the results. The distributions of inter arrival, service and waiting times of both observed and bootstrap data were modelled using the Easyfit (5.6) software at each phase to identify their respective distributions. Key queuing metrics such as resource utilization rates, which were obtained to be less than 1 in all cases, showed a stable system; the average number of patients in the queue; the average number of patients in the system and the average waiting times of patients in the queue and in the system for both observed and bootstrapped data were also obtained. Generally, the results provide insights for effective healthcare management system.

Keywords: *Healthcare Delivery; Queuing Network; Bootstrapping; Observed Data; Resource Utilization.*

1. Introduction

Queue theory, according to [1] is used to examine the phenomena of waiting, and being served. These phenomena occur when the demand for service surpasses the available capacity, [2]. Network queuing models have numerous practical applications in retail businesses, healthcare, telecommunications, manufacturing, road traffic, social justice systems and call centres, among others.

In healthcare system, patients often face waits of varying lengths from minutes to months to receive medical services. This leads to loss of time and energy as well as incur waiting time cost by patients. Therefore, estimating the model parameters in queuing models is essential to address particular service mechanism for effective service delivery and to add to queuing theory and its applications as addressed by [3]. Queuing analysis of telephone congestion was first performed in 1913 by a Danish Engineer, A.K Erlang. He started with the problem of the congestion of telephone traffic and later on, extended to business application and waiting lines in telecommunication, traffic, engineering, computing, the design of factories, shops, offices and hospitals [4]. However, deep survey of infinite-server queue in a random environment were given by [5], [6] and [7].

Further studies including [8] incorporated balking in queuing analysis. They considered a multi-server scenario due to delay in providing services. Moreso, [9] introduced the cellular automatic model to show how customers select service facility based on their experiences. Also, [10] examined queues and appointment systems in hospital out-patient departments, considering patient flow problem using queue theory to evaluate appointment systems and emergency bed allocations. In a later development, [11] investigated the effect of priority on the queue discipline, and [12] studied minimal cost service rate in priority queuing models for emergency cases in hospitals using pre-emptive priority queuing models, where the performance measures and waiting time cost for higher priority patients with stable cases were studied. The results were applied to obtain optimal service rate that minimizes the total cost of providing and waiting for service at the emergency consulting unit of the hospital. And [13] studied the waiting and service costs of a multi-server queuing system at National Health Insurance scheme (NHIS) unit of the General Hospital Minna, Niger state, Nigeria. The two conflicting costs of service and waiting for service were balanced and the optimal performance for the queuing system was determined.

Further application and analysis of queuing model abound the literature, for instance, M/G/1 queuing model in a two-phase random environment was investigated by [14] and [15], while [16] considered an unreliable M/M/1 retrial queuing system which arrival, service, failure, repair, and retrial rates are all subject to random environment. Then [17] analysed the M/G/1 queuing model and [18] dealt with a single server queue with Markov modulated service rates and impatient customers. Later, [19] studied a discrete-time Geo/G/1 queuing model with vacations in random environment and [20] applied the Erlang-A queuing model in staffing strategies for call centre operations with

uncertain arrival, service and abandonment rates, where the system rate followed Gamma distribution. Also, [21] discussed a G/M/1 queuing model in a multi-phase service environment with disasters and working breakdowns. Then, a performance evaluation of an M/M/1 queuing system with vacations operating in a multi-phase random environment was undertaken by [22] while [23] established the analysis of queues in a random environment with customers' impatience.
The use of Gumbel distribution to model single server queue process was undertaken by [24]. The inter-arrival time and service time were shown to follow Gumbel distribution. Also, the analysis of customers' impatience in a repairable retrial queue under postponed preventive actions was also considered by [25]. They analysed an unreliable retrial queue with persistent and impatient customers having different general service distributions, where the server is subject to active and passive breakdowns. A persistent customer whose service is interrupted enters the orbit, while an impatient one leaves the system. Two types of arbitrarily distributed maintenances were considered: preventive for improving system performances and preventing breakdowns, and corrective for restoring the service when a failure occurs. They stated that if a preventive maintenance occurs in a busy period, then it is postponed to an ulterior random date. A necessary and sufficient condition for the stability of the system were obtained as well as the joint probability distribution of the server-state and the number of customers in orbit in terms of Laplace and z-transforms. Some performance measures were derived and numerical results were obtained for a cost minimization problem.

2. Methodology

The healthcare service mechanism in Immanuel General Hospital Eket is represented by a queuing network consisting of six (6) departments (Phases) namely; the Records, the Clearance point, Nurses unit, Consulting Room, Accounting department, Pharmacy or Laboratory. Each department is regarded as a node or phase of the network system, where data on the arrival, waiting and service rates were obtained by method of direct observation and personal interview over four days. All the nodes (phases) under consideration have at least a server.

2.1. Description of the queuing system

The healthcare network queuing system at Immanuel Hospital Eket has s_{μ} multiple service channel with 6 identical stages in series, each with average service time of $\frac{1}{s\mu}$. The distribution of total service time of customers in the system is some joint distribution of time in all these stages. Customers arrive in a single queue and a set of them, whose number is based on the number of servers in each phase, enters the system to be served in the first phase before proceeding to the second phase, up to the sixth phase. The assumptions are that each set of customers is served in 6-phases set-by-set until the 6-phases have been completed. Moreso, the queue discipline is first-come-first served with infinite source. The healthcare network of queue phases in the hospital is represented in Figure 1.

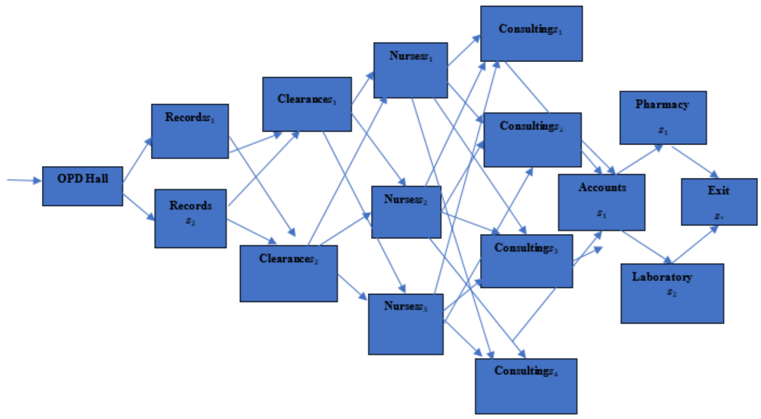

Fig. 1: Schematic Diagram of Queue Network Showing All Phases of Healthcare at Immanuel General Hospital, Eket.

The healthcare queuing network model of Immanuel General Hospital Eket represented in Figure 1 is made up of the following departments, the Records, the Clearance point, Nurses unit, Consulting Room, Accounting department, Pharmacy or Laboratory. Figure 1 shows that patients who come to the Hospital for healthcare services will commence by going first to the records unit to register and then proceed to the clearance point and then move to the nursing unit, from there, patients move to see the doctor at the consulting room, followed by payment in the accounts department either for drugs or laboratory test and then exit at the departure. Each department has a single queue and more than one service point.

2.2. Mathematical model of the queuing system

The Kendal's notation is adopted to represent the queuing model as follows;
$(G_a|G_s|k)$: $(FCFS|\infty|\infty)$, where;

1) Inter-arrival time distribution: G_a denotes a general probability distribution for inter-arrival times.
2) Service time distribution: G_s denotes a general probability distribution for service time.
3) Number of servers: k servers are available.
4) Service discipline: FCFS (First- Come- First- Serve). Customers are served in the order they arrive.
5) Population size: is assumed to be infinite (∞)
6) Queue capacity: is also infinite which means that there is no limit to the number of customers that can wait in the queue (∞).

2.3. Assumptions of queue model

The following assumptions are considered in this work;

1) Immanuel General Hospital healthcare delivery network queuing system is considered as an independent queuing system.

2) Queuing discipline is first come first served.
3) Each phase in the hospital has at least a server.
4) There is no limited capacity in the Hospital for the arriving patients and each service facility has unlimited waiting space.
5) Medical personnel are the servers in the Hospital.
6) All the medical service providers in the Hospital are working in full capacity.
7) Service rate is independent of queuing length.
8) Emergency cases with priority queuing discipline are not considered.

2.4. Bootstrapping

Bootstrapping is a special type of simulation that uses the existing data to generate new data. Bootstrapping involves resampling the data with replacement, meaning that each observation can be selected more than once. This was adopted to create multiple samples of the same size as the original data, but with different combinations of observations. Its purpose is to ascertain the accuracy and confidence of the estimates, as well as the robustness and sensitivity of the tests.

2.5. Selection criteria of probability distribution models for inter-arrival and service time distributions

2.5.1. Goodness-of-fit test

The goodness-of-fit test was used as a preliminary analysis to identify appropriate probability distributions for both the inter-arrival and service times using the Easyfit (5.6) software. The best-fit distributions with highest rank were obtained mostly from the chi-square-goodness-of fit test (and Kolmogorov Smirnov goodness-of-fit test where chi-squared goodness-of-fit test does not apply). The results of the tests are shown in Appendix 1.

2.5.2. The identified probability distributions for the inter-arrival and service rates

The following probability distributions were identified based on preliminary goodness-of-fit test using Easyfit (5.6) software as shown in Appendix 1:

a) General pareto distribution

The probability density function (PDF) of the Generalized Pareto Distribution (GPD) is given in (1) as:

$$f(X|\xi,\sigma) = \begin{cases} \frac{1}{\sigma}\left(1+\frac{\xi(x-\mu}{\sigma}\right)^{-\frac{1}{\xi-1}} & for\ x \geq \mu; if\ \xi \geq 0 \\ \end{cases} \quad (1)$$

The mean of the General Pareto Distribution is given by;

$$E(X) = \begin{cases} \mu + \frac{\sigma}{1-\xi},\ if\ \xi < 1 \\ undefined\ , otherwise \end{cases} \quad (2)$$

Where; μ = location parameter, σ = Scale parameter and ξ = Shape Parameter.

In this study, the diverse spectrum of tail behaviour contained in the Generalized Pareto Distribution (GPD) moving from heavy-tailed distributions to those with exponential and bounded upper tails are considered. Specifically, the assumption that the scale parameter $\sigma = 0$ is adopted. This particular choice simplifies the GPD, effectively transforming it into the exponential distribution. Notably, the exponential distribution boasts a constant hazard rate, rendering it exceptionally proficient in modelling events with memoryless characteristics such as queue.

b) Weibull distribution

The two parameter Weibull distribution is a continuous probability distribution often used to model lifetime data. It has two parameters: the shape parameter, λ and the scale parameter, k.

The probability density function (PDF) of the Weibull distribution is given in (3):

$$f(X|\lambda,k) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k}; X \geq 0, \lambda > 0\ and\ k > 0 \quad (3)$$

Where; $X \geq 0$ is the random variable, $\lambda > 0$ is the scale parameter and $k > 0$ is the shape parameter

The Mean of the Weibull distribution(μ) is given in (4):

$$E(X) = \lambda\Gamma\left(1+\frac{1}{k}\right), \text{where } \Gamma(.) \text{ is the gamma function} \quad (4)$$

c) Gamma distribution

The continuous gamma random variable X with shape and scale parameterization has density;

$$f(X) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{\alpha}{\beta}};\ 0 < X \leq \infty & ,\alpha > 0, \beta > 0 \\ 0 & elsewhere \end{cases} \quad (5)$$

Where; $\alpha > 0$ is the shape parameter and $\beta > 0$ is the scale parameter

The mean of the gamma distribution is given in (6):

$$\mu = E(X) = \alpha\beta \quad (6)$$

d) Generalized extreme values distribution

The Generalized Extreme Value (GEV) distribution is a meta-distribution containing the Weibull, Gumbel, and Fréchet families of extreme value distributions. It is used for modelling the distribution of extremes (maxima or minima) of stationary processes, such as the annual maximum wind speed, annual maximum truck weight on a bridge, and so on, without needing a priori decision on the tail behaviour. The GEV probability distribution function is given by:

$$f(X|\mu,\sigma,\xi) = \frac{1}{\sigma}\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi-1}} e^{\left\{-\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\right\}}; X \geq o, \mu > 0, \sigma > 0 \; and \; \xi = 0 \quad (7)$$

Where: X is the random variable, μ is the location parameter, σ is the scale parameter and ξ is the shape parameter.

The mean of the generalized extreme value (GEV) distribution is given by:

$$E(X) = \mu + \sigma\left(\frac{\gamma}{\xi}\right) \quad (8)$$

Where $\gamma$is the Euler-Mascheroni constant ≈ 0.57721

The GEV distribution has three types based on the shape parameters

- When $\xi > 0$; it is type 1 GEV distribution, also known as the Gumbel distribution
- When $\xi < 0$; it is type 2 GEV distribution, also known as the Fréchet distribution.
- When $\xi = 0$; it is type 3 GEV distribution, also known as the Weibull distribution.

In this study, the extreme values in the dataset are better characterized by a Weibull distribution rather than a Gumbel or Fréchet distribution because the data set follows the Weibull type tail behaviour. Hence, we adopt the type 3 GEV distribution.

e) Logistic distribution

The probability density function (PDF) of the logistic distribution is given in (9):

$$f(X|\mu,s) = \frac{e^{-(x-\mu)/s}}{s\left(1+e^{-\frac{(x-\mu)}{s}}\right)^2}; X \geq 0, \mu \geq 0, s > 0 \quad (9)$$

Where; X is a random variable, μ is the location parameter and s is the scale parameter.

The mean of the logistic distribution is given by $E(X) = \mu$

f) Log-normal distribution

The probability density function (PDF) of the log-normal distribution is given in (10):

$$f(X|\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} e^{\left(-\frac{(ln(x)-\mu)^2}{2\sigma^2}\right)}; X > 0, \mu \geq 0, \sigma > 0 \quad (10)$$

Where; $X > 0$ is the random variable, μ is the mean of the natural logarithm of the distribution (location parameter) and σ is the standard deviation of the natural logarithm of the distribution (scale parameter).

The mean (expected value) of the Log-normal distribution is given in (11), which is calculated based on its parameters μ and σ.

$$E(X) = e^{\mu+\frac{\sigma^2}{2}} \quad (11)$$

g) Beta distribution

The Beta distribution is a continuous random variable model whose range is [0,1] and the probability distribution function (PDF) of the Beta Distribution is given in (12):

$$f(X,\alpha,\beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}; 0 \leq x \leq 1, \alpha > 0, \beta > 0 \quad (12)$$

The mean is given in (13), thus;

$$E(X) = \frac{\alpha}{\alpha+\beta} \quad (13)$$

Where α and β are the shape and scale parameters. Typically, $\alpha \; and \; \beta$ are > 0.

$B(\alpha,\beta)$ is the Beta function defined by $B(\alpha,\beta) = \frac{\Gamma\alpha\Gamma\beta}{\Gamma(\alpha+\beta)}$ where; $\Gamma(\alpha)$ is the gamma function defined as $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-t}dx$

h) Normal distribution

The probability density function (PDF) of the normal (or Gaussian) distribution is given in (14):

$$f(X) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} & -\infty < X < \infty \\ 0 & ; otherwise \end{cases}; \mu \geq 0, \sigma > 0 \quad (14)$$

Where; $X =$ the random variable, $\mu =$ location parameter and $\sigma^2 =$scale parameter.

The mean of the Normal distribution is given in (15) as:

$$E(X) = \mu = \frac{1}{n}\sum_{i=1}^{n} x_i \quad (15)$$

i) Exponential distribution

The PDF of the exponential distribution with parameter λ is given in (16) as:

$$f(X|\lambda) = \lambda e^{-\lambda x}\ for\ x\ \geq\ 0; x \geq 0, \lambda > 0 \quad (16)$$

Where; $X \geq 0$ is the random variable and $\lambda > 0$ is the rate parameter

The mean is given in (17) as:

$$E(X) = \frac{1}{\lambda} \quad (17)$$

2.6. Queuing models for inter-arrival and service times

The queuing models for each day with parameters of the identified probability distributions are presented as follows:

2.6.1. Queue models for day 1-4

Day 1: By using the Kendal's notations in section 2.2, the queue model for the six phases in day one is shown in Table 1.

Table 1: Identified Queue Model(S) for Day 1

Phases	Queuing model
1	$(Weibull|Weibull|2):(FCFS|\infty|\infty)$
2	$(Weibull|Weibull|2):(FCFS|\infty|\infty)$
3	$(Weibull|Weibull|3):(FCFS|\infty|\infty)$
4	$(Weibull|logistics|4):(FCFS|\infty|\infty)$
5	$(Weibull|Weibull|1):(FCFS|\infty|\infty)$
6	$(Weibull|Exponential|2):(FCFS|\infty|\infty)$

The notations in Table1 represents queue models with the following characteristics:

Inter-arrival time distribution follows the Weibull distribution for all six phases; service time distribution for phases 1, 2, 3 and 5 follows the Weibull distribution while phases 4 and 6 follow the logistic and exponential distributions respectively. Number of servers for phases 1, 2 and 6 is 2 while the number of servers for phase 3 is 3, phase 4 is 4 and phase 5 is 1; service discipline: FCFS (First-Come-First-Serve) which means that patients are served in the order they arrive; population size is assumed to be infinite and queue capacity is also infinite, which implies that there is no limit to the number of patients that can wait in the queue.

Similarly, the identified queue models for days 2- 4 are given as follows:

Day 2: Inter-arrival times distributions for all six phases follow the exponential distribution, while the service times distributions for phases 1 and 6 follow exponential distribution, phase 2 follows the Normal distribution, both phases 3 and 5 follow the Weibull distribution, while phase 4 and 6 follow the log-normal and Exponential distributions respectively. Number of servers for phases 1, 2 and 6 is 2 while the number of servers for phase 3 is 3, phase 4 is 4 and phase 5 is 1; service discipline is FCFS; population size is assumed to be infinite and Queue capacity is also infinite, which implies that there is no limit to the number of patients that can wait in the queue.

Day 3: Inter-arrival times distribution for all six phases follows the exponential distribution, while service time distribution for phase 1 follows the log-normal distribution, phase 2 follows the Weibull distribution, phase 3,4 and 5 follow the exponential distribution, while phase 6 follows the gamma distribution. Number of servers for phases 1, 2 and 6 is 2 while phase 3 has 3, phase 4 has 4 and phase 5 has 1. Service discipline is FCFS which means that patients are served in the order they arrive; population size is assumed to be infinite and queue capacity is also infinite.

Day 4: Inter-arrival time distribution for all six phases follows the gamma distribution, while the service time distribution for phases 1 and 3 follow exponential distribution, phase 2 follows the Beta distribution, phase 4 and 5 follow the Weibull distribution, while phase 6 follows the gamma distribution. Number of servers for phases 1, 2 and 6 is 2 while phase 3 is 3, phase 4 is 4 and phase 5 is 1. Service discipline is FCFS; population size is assumed to be infinite and Queue capacity is also infinite.

2.6.2. Performance measures of the queueing system

The performance measures of a multi-Server, single queue system with multi-phases for this work was adopted from [26]. Let k = number of phases and s = number of servers:

a) Arrival rate: $\lambda_i = \frac{1}{mean\ number\ of\ arrivals};\ i = 2, \ldots, 6$ (17)

b) Service rate: $\mu_i = \frac{1}{mean\ number\ of\ service\ times}; i = 1,2, \ldots, 6$ (18)

c) The Steady State: The steady state conditions are met if the average numbers of patients who come does not exceed the average number of patients who have been served. In other words, the traffic intensity, $\rho < 1$. For instance, if λ is the average number of patients arriving at the service point per certain time unit, μ is the average number of patients that has been served per unit of time and s is the number of service facilities (servers), then ρ – the traffic intensity or the utility factor is defined as:

$$\rho_i = \frac{\lambda_i}{s\mu_i} \quad (19)$$

a) Average number of patients in the system is denoted by: $E(n) = \frac{\rho}{(1-\rho)}$ (20)

b) Average queue length is denoted by: $E(m) = \frac{\rho^2}{(1-\rho)}$ (21)

c) Probability that there are at least k patients in the system is given by ρ^k (22)

d) Average waiting time: The average waiting time is the average time spent by a patient in the queue, denoted by:

e) $E(w) = \frac{\rho}{\mu(1-\rho)}$ (23)

f) Expected number of phases in the queue: $L_q(k) = \frac{L_s(k)}{s\mu} = \left(\frac{1+k}{2}\right)\left(\frac{\lambda}{s\mu(s\mu-\lambda)}\right)$ (24)

g) Expected number of patients in the queue: $L_s(k) = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{s\mu(s\mu-\lambda)}\right)$ (25)

h) Expected waiting time in the queue: $W_q = \frac{L_q}{\lambda} = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{s\mu(s\mu-\lambda)}\right)$ (26)

i) Expected waiting time of a patients in the system:

$$W_s = W_q + \frac{1}{s\mu} = \left(\frac{k+1}{2K}\right)\left(\frac{\lambda^2}{s\mu(s\mu-\lambda)}\right) + \frac{1}{s\mu} \quad (27)$$

j) Expected number of patients in the system:

$$L_s = L_q + \frac{\lambda}{s\mu} = \left(\frac{K+1}{2K}\right)\left(\frac{\lambda^2}{s\mu(s\mu-\lambda)}\right) + \frac{1}{s\mu} \text{ or } L_s = \lambda W_s \quad (28)$$

Equation (28) is also known as the Little's Formula.

3. Application

The summarized data for the Inter-arrival Times (I.A.T) and Service Times (S.T), the mean $E(t)$, the average arrival rates (λ) and the average service rates (μ) for the queuing system at the Immanuel General Hospital, Eket using the observed and the simulated (bootstrapped) data is shown in this section. The parameter estimates given by Easyfit (5.6) software was used to obtain the mean of the distributions.

3.1. Analysis of inter-arrival times for the observed data

Data on interarrival times in appendix 2: Table 1 was used to obtain the inter-arrival rate of the Weibull distribution having been identified in section 2.6.1. Hence, from (4);

$E(X) = 79.212\Gamma\left(1 + \frac{1}{0.94989}\right) = 79.212$, where $\lambda = 79.212; k = 0.94989$ and the average arrival rate for day 1 is,

$\lambda = \frac{1}{79.212} = 0.01262435.$

Similarly, the inter-arrival times for days 2 - 4 were obtained and are shown in Table 2.

Table 2: Summary of Inter-Arrival Times Distribution, Mean and Average Arrival Rate: Days 1- 4

Day	Inter-arrival time distribution	$Mean$	$average\ arrival\ rate(\lambda)$
1	Weibull	79.212	0.01262435
2	Exponential	90.00873	0.00873
3	Exponential	88.26	0.01133
4	Gamma	114.0706	0.008766501

3.2. Analysis of service times for the observed data

The service times for the observed data were computed for each day and each phase in a similar manner to the inter-arrival times. The results are summarized in Table 3.

Table 3: Summary of Service Times Distributions, Mean and Average Service Rate for All 6 Phases for Days 1- 4

Day	Phase	service time distribution	$Mean$	$average\ service\ rate(\mu)$
1	1(Records unit)	Weibull	90.978	0.01199
	2(clearance)	Weibull	51.14	0.0196
	3(out-patient)	Weibull	37.649	0.0266
	4(consulting unit)	Logistic	19.75	0.050632911
	5(pay point)	Weibull	57.234	0.01747
	6(Pharmacy or Lab.)	Normal	35.25	0.028368794
2	1(Records unit)	1(Records unit)	4.0274	0.2483
	2(clearance)	2(clearance)	3.267	0.268333915
	3(out-patient)	3(out-patient)	190.902	0.005238
	4(consulting unit)	4(consulting unit)	24.029241	0.041615962
	5(pay point)	5(pay point)	78.73	0.012701639
	6(Pharmacy or Lab.)	6(Pharmacy or Lab.)	39.37008	0.0254

3	1(Records unit)	Exponential	49.3583	0.02026
	2(clearance)	Weibull	52.829	0.017695
	3(out-patient)	Exponential	67.70481	0.01477
	4(consulting unit)	Exponential	29.6824	0.03369
	5(pay point)	Exponential	0.00417	239.8082
	6(Pharmacy or Lab.)	Gamma	30.73195669	0.032539418
4	1(Records unit)	Exponential	104.0583	0.00961
	2(clearance)	Beta	0.025003	39.99458
	3(out-patient)	Exponential	44.8833	0.02228
	4(consulting unit)	Weibull	38.966	0.025663399
	5(pay point)	Weibull	137.47	0.07274
	6(Pharmacy or Lab.)	Gamma	69.3083704	0.014428272

3.3. Performance measures of the observed data for the queuing models

The performance measures for the observed data for days 1- 4 of the inter-arrival times and phases 1- 6 for the service times are obtained using the queuing models. Microsoft Excel was used to aid the computation of the values. The number of phases is denoted by k and the number of servers by s:

3.3.1. Traffic intensity or server utilization (ρ_i)

From (19), the Traffic Intensity given by $\rho_i = \frac{\lambda_i}{s\mu_i}$ for all 6 phases for days 1- 4 are contained in Table 4.

Table 4: Traffic Intensity for the 6 Phases for Days 1- 4

Phases	Day 1	Day 2	Day 3	Day 4
1	0.57427	0.00108	0.27962	0.45611
2	0.32280	0.22323	0.32015	0.00011
3	0.15843	0.10956	0.25570	0.13116
4	0.06233	0.05244	0.08408	0.08539
5	0.72254	0.68731	0.00004	0.12051
6	0.22250	0.17185	0.17409	0.30379

3.3.2. Expected number of patients in the queue ($L_s(k)$)

The expected number of patients in the queue for days 1- 4 for inter-arrival times and phases 1- 6 for service times are computed from (25) as shown in Table 5.

Table 5: Expected Number of Patients in the Queue:($L_s(k)$)

		Ls(k)		
Phases	Day 1	Day 2	Day 3	Day 4
1	35.79343	7.86000	5.58783	25.45236
2	7.11006	4.28645	7.76224	7.99E-07
3	1.37816	0.90072	4.52266	1.31743
4	0.19147	0.19395	0.39734	0.53059
5	86.9434	100.949	1.15E-07	1.09893
6	2.94229	2.38284	1.88946	8.82100
Total	134.35890	108.71290	20.15953	37.22032

Hence, the average number of patients in the queue network for days 1-4 respectively, is given by: $E_1(L_s(k)) \approx 135\ Patients$; $E_2(L_s(k)) \approx 109\ Patients$; $E_3(L_s(k)) \approx 21\ Patients$; $E_4(L_s(k)) \approx 38$ Patients.

3.3.3. Expected waiting time of patients in the queue (W_q)

The expected waiting time of patients in the queue for days 1-4 for inter-arrival times and phases 1-6 for service times are computed using (26) and the results are contained in Table 6.

Table 6: Expected Waiting Time of Patients in the Queue (W_q)

		Wq		
Phases	Day 1	Day 2	Day 3	Day 4
1	2835.26952	0.00900	493.18886	2903.36554
2	563.20210	491.00241	685.10471	9.1180E-05
3	109.16707	103.17532	399.17597	150.28040
4	15.16639	22.21631	35.06994	60.52527
5	6886.9638	11563.44748	1.01E-05	125.35549
6	233.06495	272.94841	166.76616	1006.21695
Total	10642.83	12452.80	1779.31	4245.74

Table 6 shows that the average waiting time of patients in the queue network for days 1- 4 is given as:
$E_1(W_q) \approx 10643\ minutes$; $E_2(W_q) \approx 1245\ minutes$; $E_3(W_q) \approx 1779\ minutes$; $E_4(W_q) \approx 4245$ minutes.

3.3.4. Expected waiting time of a patient in the system (W_s)

The expected waiting time of patients in the system for days 1- 4 for inter-arrival times and phases 1- 6 for service times are computed using (27) and the results are contained in Table 7.

Table 7: Expected Waiting Time for A Patient in the System (W_s)

		Ws		
Phases	Day 1	Day 2	Day 3	Day 4
1	2880.75852	0.13315	517.86804	2955.39467
2	588.77210	516.57241	713.36171	0.01259
3	121.71673	115.72498	421.74424	165.24150
4	20.10389	28.22362	42.490542	70.26676
5	6944.19783	11642.17748	0.0041800	139.10309
6	250.68995	292.63345	182.13214	1040.87113
Total	10806.24	12595.47	1877.60	4370.89

The results in Table 7 shows that the average waiting time of patients in the system for days 1-4 is given as: $E_1(W_s) \approx 10806\ minutes;\ E_2(W_s) \approx 12595\ minutes; E_3(W_s) \approx 1878\ minutes; E_4(W_s) \approx 4370$ minutes.

3.3.5. Expected number of patients in the system (L_s)

The expected number of patients in the system for days 1-4 for inter-arrival times and phases 1- 6 for service times are computed using (28) and are contained in Table 8.

Table 8: Expected Number of Patients in the System (L_s)

		Ls		
Phases	Day 1	Day 2	Day 3	Day 4
1	36.36770	0.00116	5.86744	25.90847
2	7.43286	4.50968	8.08239	0.00011
3	1.53659	1.01028	4.77836	1.44859
4	0.25379	0.24639	0.48142	0.61599
5	87.66598	101.63621	4.7E-05	1.21945
6	3.164798	2.55469	2.06355	9.12479
Total	136.42174	109.95840	21.27322	38.31741

The results in Table 8 shows that the average number of patients in the system for days 1- 4 using the observed data is given by: $E_1(L_s) \approx 137\ \ Patients\ ;\ E_2(L_s) \approx 110\ Patients; E_3(L_s) \approx 22\ patients; E_4(L_s) \approx 39$ patients.

3.4. Bootstrap simulation of the inter-arrival and service times

The observed data were resampled using the bootstrapped method to create many simulated samples using Microsoft Excel. As a result, multiple samples of the same sizes as the observed data, but with different combinations of observations were obtained. Their inter-arrival and service distributions, average inter-arrival and service rate and the estimates of their queuing measures were obtained and used to assess the accuracy and robustness of the original dataset. The results are presented as follows:

3.4.1. Inter-arrival time distributions of the simulated data for days 1 – 4

In a similar manner to the observed data and using the respective equations, the inter-arrival time distributions, estimates of the parameters, the calculated mean values and the average arrival rates for the simulated data are provided in Table 9.

Table 9: Inter-Arrival Time Distributions, Mean and Average Arrival Rate Using Simulated Data for Days 1-4

Days	Distribution	Parameters estimate	Mean	Average arrival rate (λ)
1	Weibull	$\alpha = 1.024900; \beta = 96.87600$	96.87600	0.01032
2	Exponential	$\lambda = 0.00655$	152.67180	0.00655
3	Exponential	$\lambda = 0.01295$	77.22010	0.01295
4	Gamma	$\alpha = 0.83770; \beta = 115.95000$	101.68815	0.00983

3.4.2. Service time distributions of the simulated data, mean and average service rate for days 1-4 through the 6 phases

The Service time distributions in section 2.6.1, parameters estimate, means and the average service rate of the simulated data for days 1 – 4 through the 6 phases are displayed in Tables 10.

Table 10: Summary of Service Time Distribution of the Simulated Data, Mean and Average Service Rate for Days 1- 4 Through the 6 Phases

Day	Phase	Distributions	Parameters estimate	mean	Average service rate (μ)
1	1	Weibull	$\alpha = 1.60670; \beta = 35.27200$	21.34087	0.04686
	2	Weibull	$\alpha = 1.26890; \beta = 52.50400$	15.97624	0.06259
	3	Weibull	$\alpha = 0.21149; \beta = 12.01900$	13.30915	0.07514
	4	Logistics	$\mu = 20.495; \sigma = 10.81300$	20.49500	0.04879
	5	Weibull	$\alpha = 0.53215;\ \beta = 73.19200$	91.19200	0.01096
	6	Normal	$\mu = 29.595; \sigma = 30.58700$	29.59500	0.03379
2	1	Exponential	$\lambda = 0.02608$	38.34360	0.02608
	2	Normal	$\mu = 17.22582;\ \sigma = 19.31600$	17.22582	0.05805
	3	Weibull	$\alpha = 0.47342;\ \beta = 33.15100$	66.30200	0.01508
	4	Log-normal	$\mu = 2.85190;\ \alpha = 0.86049$	25.08127	0.03987
	5	Weibull	$\alpha = 1.41240;\ \beta = 90.21100$	90.21100	0.01109
	6	Exponential	$\lambda = 0.02592$	38.58030	0.02592
3	1	Exponential	$\lambda = 0.02102$	47.57370	0.02102
	2	Weibull	$\alpha = 1.52980; \beta = 52.82900$	52.82900	0.18929
	3	Exponential	$\lambda = 0.01857$	53.85030	0.01857
	4	Exponential	$\lambda = 0.03793$	26.36436	0.03793

	5	Exponential	$\lambda = 0.00453$	0.00453	220.75055
	6	Gamma	$\alpha = 1.61060; \beta = 24.54500$	39.53218	0.02529
4	1	Exponential	$\lambda = 0.00841$	118.90610	0.00841
	2	Beta	$\alpha = 0.01610;\ \beta = 0.64770$	4.89816	0.20416
	3	Exponential	$\lambda = 0.02191$	45.64126	0.02191
	4	Weibull	$\alpha = 0.21106; \beta = 20.40200$	170.20000	0.00580
	5	Weibull	$\alpha = 1.11120; \beta = 170.20000$	93.72800	0.01067
	6	Gamma	$\alpha = 1.29400; \beta = 45.88000$	59.36872	0.01684

3.5. Performance measures of the simulated data for the queuing models

The performance measures for the simulated data for days 1- 4 of the inter-arrival times and phases 1- 6 for the service times are obtained using the identified queuing models in section 2.6.1. Thus;

3.5.1. Traffic intensity or server utilization $(\rho_i{}^*)$ of the simulated data for days 1 – 4

Equation (19) was used to compute the traffic intensity of the simulated data for days 1- 4 in all 6 phases. The results are shown in Table 11.

Table 11: Traffic Intensity or Server Utilization of the Simulated Data for Days 1- 4

		ρi		
Phase	Day 1	Day 2	Day 3	Day 4
1	0.18205	0.12575	0.30804	0.58466
2	0.27099	0.05641	0.34207	0.02408
3	0.12177	0.14476	0.23245	0.14961
4	0.05289	0.04107	0.08535	0.05017
5	0.12387	0.59088	5.8E-05	0.16737
6	0.15275	0.12635	0.25597	0.29192

3.5.2. Expected number of patients in the queue using the simulated data; $L_s(k)^*$

The expected number of patients in the queue using the simulated data for days 1- 4 for inter-arrival times and phases 1-6 for service time computed from (25) are contained in Table 12.

Table 12: Expected Number of Patients in the Queue, $L_s(k)^*$

		Ls(k)*		
Phase	Day 1	Day 2	Day 3	Day 4
1	2.28967	1.60606	6.17704	0.00238
2	5.69232	0.30038	8.01106	0.00559
3	0.95414	2.18213	3.17114	0.00488
4	0.16691	0.15666	0.35879	0.00545
5	0.98968	76.0026	1.5E-07	0.00478
6	1.55619	1.62739	3.96678	0.00406
TOTAL	11.64891	81.87523	21.68482	0.027146

The results in Table 12 shows that the average number of patients in the queue network for days 1 – 4 is given as: $E_1(L_s(k)^*) \approx 12\ Patients;\ E_2(L_s(k)^*) \approx 82\ Patients; E_3(L_s(k)^*) \approx 22\ Patients\ ; E_4(L_s(k)^*) \approx 1$

3.5.3. Expected waiting time in the queue using the simulated data $(W_q{}^*)$

The expected waiting time of patients in the queue using the simulated data for days 1-4 for inter-arrival times and phases 1-6 for service time are computed using (26) as contained in Table 13.

Table 13: Expected Waiting Time in the Queue for the Simulated Data, $(W_q{}^*)$

		Wq*		
Phases	Day 1	Day 2	Day 3	Day 4
1	221.81397	29.05027	221.81387	0.24228
2	551.44915	45.39321	551.44915	0.56928
3	92.43310	323.72248	92.43310	0.49606
4	16.16933	23.74219	16.16933	0.55407
5	95.87615	9294.03918	95.87615	0.48569
6	150.75797	45.39322	150.75796	0.41305
TOTAL	1128.50	9761.34	1128.50	2.76

The results in Table 13 shows that the average waiting time of patients in the queue network for days 1-4 is given by: $E_1(W_q{}^*) \approx 1129\ minutes;\ E_2(W_q{}^*) \approx 9762\ minutes; E_3(W_q{}^*) \approx 1129\ minutes; E_4(W_q{}^*) \approx$ 3 minutes.

3.5.4. Expected waiting time of a patient in the system using the simulated data $(W_s{}^*)$

The expected waiting time of patients in the system for days 1- 4 for inter-arrival times and phases1-6 for service time were obtained using (27) and are shown in Table 14.

Table 14: Expected Waiting Time for A Patient in the System, $(W_s{}^*)$

		Ws*		
Phase	Day 1	Day 2	Day 3	Day 4

1	239.44987	264.37103	500.77807	59.69531
2	577.70115	54.47305	645.02947	3.01837
3	104.22977	355.25021	262.82593	15.70981
4	21.29308	30.18747	34.29738	5.65457
5	107.87615	11693.66416	0.00454	17.50569
6	165.55547	267.74612	326.08112	30.09741
TOTAL	1216.11	12665.69	1769.02	131.68

Hence, the average waiting time of patients in the system for days 1- 4 in Table 14 is given as: $E_1(W_s{}^*) \approx 12167 minutes$; $E_2(W_s{}^*) \approx 12666\ minutes$; $E_3(W_s{}^*) \approx 1769\ minutes$; $E_4(W_s{}^*) \approx 132$ minutes.

3.5.5. Expected number of patients in the system $(L_s{}^*)$

The expected number of patients in the system of the simulated data for days 1- 4 for inter-arrival times and phases 1 - 6 for service time shown in Table 15 were obtained from (28).

Table 15: Expected Number of Patients in the System for the Simulated Data, $(L_s{}^*)$

	Ls*			
Phase	Day 1	Day 2	Day 3	Day 4
1	2.47172	1.73163	6.48508	0.58704
2	5.96331	0.35679	8.35313	0.02968
3	1.07591	2.32689	3.40359	0.15449
4	0.21979	0.19773	0.44415	0.05561
5	1.11355	76.59350	5.88E-05	0.17215
6	1.70894	1.75374	4.22275	0.29598
TOTAL	12.55322	82.96028	22.90876	1.29495

Hence, the results in Table 15 shows that the average number of patients in the system for days 1- 4 is given by:
$E_1(L_s{}^*) \approx 13$ Patients ; $E_2(L_s{}^*) \approx 83$ Patients; $E_3(L_s{}^*) \approx 23$ patients; $E_4(L_s{}^*) \approx 2$ Patients.

4. Discussion

Patients waiting time at Immanuel General Hospital, Eket in the six different units/departments (phases) of the hospital namely; Records unit, Clearance Unit, Nurses unit, Consulting unit, Account unit, Pharmacy/Laboratory unit followed different probability distributions for the inter-arrival and service times. The chi-square goodness-of-fit test and the Kolmogorov Smirnov test were used to identify the appropriate probability distributions as applicable. Easyfit (5.6) software was used to perform the goodness-of-fit test to identify the appropriate probability distributions and also obtain the parameters estimate of the distributions which were used to obtain the means of the inter-arrival and service times and the average inter-arrival and average service rates both for the observed and bootstrapped data.

4.1. Performance measures

The results of the performance measures on the observed and the simulated data for the identified queuing models is summarized in this section.

4.2. Traffic intensity or server utilization (ρ)

The results of the Traffic intensity (ρ) of the observed and simulated data for the queuing models in Tables 4 and 11 for all the six phases in the four days respectively are less than 1. This implies that ρ has met the steady state condition. Hence, the average number of patients that come does not exceed the average number that is served, implying a stable condition.

4.3. Expected number of patients in the queue

The results of the expected number of patients in the queue of the observed and simulated data for the queue models on Tables 5 and 12 respectively show that the totals of daily average number of patients in the queuing network for days 1- 4 are approximately; 135 patients, 109 patients, 21 Patients and 38 patients for the observed data and 12 patients, 82 patients, 22 patients and 1 patient for the simulated data. Daily details for the observed data on each phase are summarized as follows:

- The expected number of patients in the queue on the observed data for the classical queuing models on Table 4 shows that phase 5 (Account unit) has the highest average number of patients to be served for days 1 and 2 with a total of about 87 patients for day 1 and 101 customers for day 2. This high number suggests a bottleneck, which could be due to lengthy billing procedures, inadequate staffing, or inefficient processes. To address this, the hospital could consider adding more staff to the Account unit, streamlining the billing process, or adopting automated billing systems to reduce waiting times.

Also, the highest average number of patients (25) for Record unit (phase 1) was observed in day 4, suggesting a bottleneck in service delivery at this point in the hospital. Therefore, implementing electronic health records (EHR) systems, training staff on efficient data entry, and ensuring adequate staffing can help mitigate delays.

Generally, this variation suggests fluctuating patient inflow, potentially due to factors like scheduled procedures or outpatient visits. A flexible staffing model that adjusts to daily demand variations can improve patient's care and reduce waiting times.

The simulated data shows that the Clearance unit (phase 2) in day 3 with about 76 patients had the highest average number of patients to be served, constituting a bottleneck. This suggests that the Clearance unit needs sufficient staff and resources, particularly on days with high number of patients. This may involve scheduling more staff during peak times and ensuring the availability of necessary equipment and facilities. It was also observed that there was only one expected patient to be served in day 4. Hence, a day with little or no expected number of patients can be used for staff training, equipment maintenance, and other preparatory activities to ensure the hospital is ready for days with higher expected number of patients.

4.4. Expected waiting time of patients in the queue

The result of the expected waiting time of patients in the queue for the observed and simulated data are summarized as follows:
The results of the expected waiting time of customers in the queue for the observed data is given in Table 7. On days 1 and 2, the highest average waiting time of customers was in phase 5 (Account unit) which is approximately 6887 minutes and 11563 minutes respectively. Day 3 had its peak in phase 2 (Clearance unit) with approximately 6885 minutes. Also, for day 4, the highest average waiting time was in phase 1 (Records unit) with approximately 2903 minutes. On the whole, the total average waiting time of patients in the queue network for days 1-4 were given as 10643 minutes, 21245 minutes, 1779 minutes and 4245 minutes respectively. Suggested interventions such as increasing staffing levels, implementing automated billing systems, and optimizing billing processes are crucial for the Account unit; streamlining clearance processes and reducing administrative burdens can mitigate delays and help decrease waiting times in the Clearance unit and implementing electronic health records (EHR), staff training and ensuring adequate staffing and resource allocation can reduce waiting times in Record unit.
Also, the results of the expected waiting time of patients in the queue for the simulated data is given in Table 13. On days 1 and 3, the highest average waiting time of patients (551 minutes) was in phase 2 (Clearance unit). Account unit (phase 5) with 9294 minutes was the highest for day 2, while Clearance unit (Phase 2) was the highest with 0.57 minutes for day 4. On the whole, the total average waiting time of patients in the queue network for days 1 - 4 were 1129 minutes, 9761 minutes, 1128 minutes and 3 minutes respectively.
This result also points to potential issues on the efficiency of the Clearance unit. The Account unit with the highest average waiting time of 9294 minutes in day 2 indicates inefficiencies or insufficient resources in that unit. Addressing this bottleneck requires interventions, such as increase staffing, implementing automated billing systems, and reviewing the entire billing process to identify and eliminate inefficiencies.

4.5. Expected waiting time for a patient in the system

The result of the expected waiting time for a patient in the system for the observed and simulated data is summarized as follows:
Analysis of expected waiting time in the system using observed data provides crucial insights into the efficiency and effectiveness of various phases in the hospital operations. Table 7 highlights the expected waiting times across the 6 phases in the 4 days. The Account unit (phase 5) causes the highest expected waiting time for patients in the system with approximately 6944 minutes and 11642 minutes for days 1 and 2 respectively, while the Clearance unit with approximately 713 had the highest for day 3 and the Record unit with approximately 2955 minutes had the highest for day 4. The total average waiting time for a customer in the system was approximately 10806 minutes for day 1, 12595 minutes for day 2, 1878 minutes for day 3 and 4370 minutes for day 4.
For days 1 and 2, the Account unit shows extremely high waiting times of 6944 minutes and 11642 minutes, respectively. Hence, the high waiting times in Account, Clearance and Record units indicate significant inefficiencies in the billing process, clearance and record-keeping process. This could be ameliorated possibly through digitization and better workflow management.
Similarly, Table 14 details the expected waiting times across the 6 phases of days 1-4 for the simulated data. The total average waiting time for a customer in the system was approximately 1216 minutes for day 1, 12666 minutes for day 2, 12665 minutes for day 3 and 131 minutes for day 4. The analysis reveals the following highlights:
Clearance unit (phase 2) had the highest expected waiting time of approximately 578 minutes for a patient in the system in day 1 and 645 minutes in day 3, while the Account unit (phase 5) with approximately 11694 minutes in day 2 had the highest expected waiting time in the system. Clearance unit (phase 2) had the highest with approximately 645 minutes for day 3 and Records unit had the highest with approximately 60 minutes in day 4.
These results indicate inefficiencies in processing clearance, which could be due to insufficient staffing, complex procedures, or bottlenecks in workflow. Optimizing clearance processes by revising procedures, increasing staffing levels and possibly automation of certain tasks can help reduce these waiting times.

4.6. Expected number of patients in the system

The result of the expected number of patients in the system for the observed and simulated data is summarized as follows:
Analysis of expected number of patients in the system based on observed data in Table 8 provides insights into the distribution and flow of patients across different phases for the 4 days with total average number of patients in the system as approximately 136 patients, 110 patients, 21 patients and 8 patients for days 1,2,3 and 4 respectively.
Again, the Account unit had the highest expected number of customers in the system for days 1 and 2 with approximately 88 patients and 102 patients respectively, while Records unit had its peak with approximately 26 patients on both days 3 and 4, indicating bottleneck.
However, a reduction in the total average number of patients in days 2 and 3 suggests variability in patient load. Understanding the causes of this variability such as specific days with higher appointments or external factors can help in better planning and resource allocation.
Similarly, analysis of expected number of patients in the system using simulated data as shown in Table 15 provides an understanding of patient distribution across different phases showing total average number of patients in the system as approximately 13 patients, 83 patients, 23 patients and 1 patient for days 1,2,3 and 4 respectively.

5. Conclusion

The results obtained in this work show that the inter-arrival and service times distributions for all the phases in the four days follow different probability distributions both for the observed and bootstrapped data as against the fixated use of assumed specific distributions to denote the inter-arrival and service times. These distributions were used to obtain the means, the queuing parameters and other performance measures of the queuing system. Furthermore, the phases with highest waiting time in queue and system, and highest expected number of patients in queue and system are red flags to congestion, pointing at inefficient resource allocation and operations and/or insufficient resources. The hospital may need to adopt a dynamic resource allocation strategy to manage varying patient loads effectively. This includes having flexible staffing plans and ensuring that critical units are not overwhelmed during peak times. Days with high patient numbers suggest need for additional resources, while days with low-patient turnout might provide opportunity for training and maintenance for improve performance. Hence, this work provides insights to potential operational efficiencies and areas for improvement in healthcare delivery with a view to optimizing resource allocation in hospital management as a whole.

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Appendix 1

Goodness-of-fit test for inter-arrival and service times distributions.

Goodness of fit test for inter-arrivals			
Days	Test statistic		Best fit distribution
1	Kolmogorov Smirnov		Weibull
2	Kolmogorov Smirnov		General Pareto
3	Kolmogorov Smirnov		General Pareto
4	Kolmogorov Smirnov		Gamma
Goodness of fit test for service times			
DAY1			
Phases		Test statistic	Best fit distribution
1		Chi-square	General Extreme values
2		Chi-square	General Extreme values
3		Chi-square	General Extreme values
4		Chi-square	Logistics
5		Kolmogorov	Weibull 3 parameter
6		Chi-square	General Pareto
DAY2			
Phases		Test statistic	Best fit distribution

1	Chi-square	General Pareto
2	Chi-square	Normal
3	Chi-square	Weibull
4	Chi-square	Log-normal
5	Kolmogorov	Weibull
6	Chi-square	General Pareto
DAY3		
Phases	Test statistic	Best fit distribution
1	Chi-square	Log-normal
2	Chi-square	General Extreme values
3	Chi-square	General Pareto
4	Chi-square	General Pareto
5	Kolmogorov	Exponential
6	Chi-square	Gamma
DAY4		
Phases	Test statistic	Best fit distribution
1	Chi-square	General Pareto
2	Chi-square	Beta
3	Chi-square	General Pareto
4	Chi-square	Weibull
5	Kolmogorov	General Extreme values
6	Chi-square	Gamma

Appendix 2

Table showing inter-arrival times for Day 1

Table 1: Inter-Arrival Times for Day 1

Inter-arrival Times (Mins)	F	X	f(x)
1-5	11	3	33
6-10	2	8	16
11-15	1	13	13
16-20	4	18	72
21-25	6	23	138
26-30	2	28	56
31-35	10	33	330
36-40	4	38	152