International Journal of Advanced Statistics and Probability, 4 (1) (2016) 20-26

International Journal of Advanced Statistics and Probability

Website: www.sciencepubco.com/index.php/IJASP
doi: 10.14419/ijasp.v4i1.5735
Research paper

Maximum likelihood estimation based on type-i hybrid progressive censored competing risks data

S.K. Ashour [1], W.S. Abu El Azm [2] *

[1] *Department of Mathematical Statistics, Institute of Statistical Studies & Research, Cairo University, Egypt*
[2] *Department of Statistics, Faculty of Commerce, Zagazig University, Egypt*
**Corresponding author E-mail: Wael_sheta2003@yahoo.com*

Abstract

This paper is concerned with the estimators problems of the generalized Weibull distribution based on Type-I hybrid progressive censoring scheme (Type-I PHCS) in the presence of competing risks when the cause of failure of each item is known. Maximum likelihood estimates and the corresponding Fisher information matrix are obtained. We generalized Kundu and Joarder [7] results in the case of the exponential distribution while, the corresponding results in the case of the generalized exponential and Weibull distributions may be obtained as a special cases. A real data set is used to illustrate the theoretical results.

Keywords: *Competing Risks; Type-I Progressive Hybrid Censoring; Generalized Weibull Distributions; Generalized Exponential Distribution; Weibull Distributions; Exponential Distribution; Maximum Likelihood Estimation; Asymptotic Confidence Interval.*

1. Introduction

The two most common censoring schemes in life testing experiments are Type-I and Type-II censoring schemes and both censoring scheme have their own advantages whereas Type-II censoring scheme controls the efficiency of the test but time of the test is uncertain. The mixture of Type-I and Type-II censoring schemes, named as hybrid censoring scheme have been discussed in Epstein [4]. The hybrid censoring scheme is of two types namely Type-I hybrid and Type-II hybrid censoring scheme. Hybrid censored schemes have been introduced in the context of progressive censoring by Kundu and Joarder [7] and Childs et al. [2]. In Type-I progressive hybrid censoring scheme (Type-I PHCS) introduced by Kundu and Joarder [7] and Hashemi and Azar [5] in the presence of competing risks data and can be described as follows: Suppose n identical items are put to the test and the life time distributions of the n items are denotes by $X_1, X_2, ..., X_n$. The integer $r < n$ is fixed at the beginning of the experiment, and $(R_1, R_2, \ldots, R_r)$ are r per-fixed integers satisfying $R_1 + R_2 + \ldots + R_r + r = n$. The time point is also fixed beforehand. At the time of the first failure $x_{(1)}$, R_1 of remaining (the $n-1$ surviving) units are randomly removed. Similarly, at the time of the second failure $x_{(2)}$, R_2 of the $n - R_1 - 2$ surviving units are removed, and so on. Finally at the time of the r^{th} failure all $R_r = n - R_1 - ... - R_{r-1} - r$ surviving units are removed from the life-test. In this type, the experiment would terminate at the random time $T^* = min \ (X_{(r)}, T)$.

The main aim of this paper is to analyse the competing risk model when lifetimes have independent generalized Weibull distribution based on Type-I hybrid progressive censoring scheme (Type-I PHCS) when the cause of failure of each item is known. Maximum likelihood estimates and the corresponding Fisher information matrix are obtained. We also obtain the results in the case of the generalized exponential distribution.

The rest of this paper is organized as follows: In section 2, we introduce the model and the notation used throughout this paper. In section 3, we discuss the maximum likelihood estimation and asymptotic confidence interval in the case of the generalized Weibull distribution. We discuss the results in the case of the generalized exponential distribution in section 4. In section 5, confidence intervals and goodness of fit will be discussed. Finally, in section 6, a real data set is used to illustrate the theoretical.

2. Model description and notation

A competing risks model arise from situations in which a unit is exposed to several risks at the same time, but the eventual failure of the unit is due to only one of these risks, which is called the "cause of failure". Usually, competing risks data are frequently obtained from industrial reliability life testing, epidemiological and biomedical studies. Consider a life time experiment with n identical units, where its lifetimes are described by independent and identically distributed (i.i.d) random variables $X_1, X_2, ..., X_n$. Without loss of generality; assume that there are only two causes of failure. We have $X_i = min \{X_{1i}, X_{2i}\}$ for $i = 1, ..., n,$ where X_{ji}, $j = 1, 2,$ denotes the latent failure time of the i^{th} unit under the j^{th} cause of failure. We assume that the latent failure times X_{1i} and X_{2i} are independent, and the pairs (X_{1i}, X_{2i}) are i.i.d. Assume that the failure times follows the generalized Weibull distributions with probability density function $f_j(x)$ survival function $\bar{F}_j(x)$ have the form

$$f_j(x) = \delta_j \lambda_j \beta_j x^{\delta_j - 1} e^{-\lambda_j x^{\delta_j}} (1 - e^{-\lambda_j x^{\delta_j}})^{\beta_j - 1}$$

$$\bar{F}_j(x) = 1 - (1 - e^{-\lambda_j x^{\delta_j}})^{\beta_j}$$
$$x > 0; \quad \lambda_j, \delta_j > 0, \beta_j > 1, j = 1, 2. \tag{1}$$

Under Type-I progressive hybrid censoring scheme (Type-I PHCS) and in presence of competing risks data we have the following forms of observation:

I: $\left(X_{(1)}, c_1, R_1\right), \left(X_{(2)}, c_2, R_2\right), ..., \left(X_{(r)}, c_r, R_r\right) \quad if \;\; X_{(r)} \le T$;

Case

II: $\left(X_{(1)}, c_1, R_1\right), ..., \left(X_{(D)}, c_D, R_D\right), \left(T, R_D^*\right) \quad if \;\; X_{(r)} > T$.

Following Kundu and Joarder [7] and Hashemi and Azar [5], the likelihood function for the present censoring schemes competing risks models when the cause of failure is known is given by

$$\text{Case I}: L(\theta) \alpha \prod_{i=1}^{r} \left(\begin{array}{l} \left[f_1(x_i).\bar{F}_2(x_i)\right]^{I(c_i=1)}. \\ \left[f_2(x_i).\bar{F}_1(x_i)\right]^{I(c_i=2)}. \\ \left[\bar{F}_1(x_i).\bar{F}_2(x_i)\right]^{R_i} \end{array} \right),$$

$$\text{Case II}: L(\theta) \alpha \left[\prod_{i=1}^{D} \begin{array}{l} \left[f_1(x_i).\bar{F}_2(x_i)\right]^{I(c_i=1)} \\ \left[f_2(x_i).\bar{F}_1(x_i)\right]^{I(c_i=2)} \\ \left[\bar{F}_1(x_i).\bar{F}_2(x_i)\right]^{R_i} \end{array} \right] \left[\bar{F}_1(T).\bar{F}_2(T)\right]^{R_D^*}.$$

Thus combined likelihood can be written as;

$$L(\theta) \alpha \left[\prod_{i=1}^{m} \begin{array}{l} \left[f_1(x_i).\bar{F}_2(x_i)\right]^{I(c_i=1)} \\ \left[f_2(x_i).\bar{F}_1(x_i)\right]^{I(c_i=2)} \\ \left[\bar{F}_1(x_i).\bar{F}_2(x_i)\right]^{R_i} \end{array} \right] \left[\bar{F}_1(T).\bar{F}_2(T)\right]^{R_D^*} \tag{2}$$

Where $x_i = x_{(i)}$ for simplicity of notation, $m = r$, $R_D^* = 0$ in case I and $m = D$ in case II. Further, we define

$$I_1(c_i = 1) = \begin{cases} 1, & c_i = 1 \\ 0 & else \end{cases} \quad \text{And} \quad I_2(c_i = 2) = \begin{cases} 1, & c_i = 2 \\ 0 & else \end{cases}$$

Thus, the random variables $r_1 = \sum_{i=1}^{r} I_1(c_i = 1)$ and $r_2 = \sum_{i=1}^{r} I_2(c_i = 2)$ describe the number of failures due to the first and the second cause of failures, respectively.

3. Results in the case of the generalized weibull distribution

Using likelihood function (2) and the generalized Weibull distribution (1), the likelihood function of the observed data can be written as

$$L \propto (\lambda_1 \delta_1 \beta_1)^{m_1} (\lambda_2 \delta_2 \beta_2)^{m_2} \prod_{i=1}^{m_1} x_i^{\delta_1 - 1} e^{-\lambda_1 x_i^{\delta_1}} u_{1i}^{\beta_1 - 1}$$
$$\left[1 - u_{2i}^{\beta_2}\right] \prod_{i=1}^{m_2} x_i^{\delta_2 - 1} e^{-\lambda_2 x_i^{\delta_2}} u_{2i}^{\beta_2 - 1} \left[1 - u_{1i}^{\beta_1}\right]$$

$$\prod_{i=1}^{m} \left[\left(1 - u_{1i}^{\beta_1}\right)\left(1 - u_{2i}^{\beta_2}\right)\right]^{R_i} \times \left[\left(1 - z_1^{\beta_1}\right)\left(1 - z_2^{\beta_2}\right)\right]^{R_D^*} \tag{3}$$

Where $L = L(\lambda_1, \lambda_2, \delta_1, \delta_2, \beta_1, \beta_2)$, $m_1 = r_1$, $m_2 = r_2$, $m = r$ in case I $m_1 = D_1$, $m_2 = D_2$, $m = D$ in case II, $u_{ji} = u_{ji}(\lambda_j, \delta_j) = 1 - e^{-\lambda_j x_i^{\delta_j}}$, $z_j = u_j(\lambda_j, \delta_j) = 1 - e^{-\lambda_j T^{\delta_j}}$ and $j = 1, 2$. Now, the log likelihood function is given by

$$\begin{aligned} \ln L \,\alpha\; & m_1(\ln\lambda_1 + \ln\beta_1 + \ln\delta_1) + m_2(\ln\lambda_2 + \ln\beta_2 + \ln\delta_2) + \\ & \sum_{i=1}^{m_1}\left((\delta_1 - 1)\ln(x_i) - \lambda_1 x_i^{\delta_1}\right) \\ & + \sum_{i=1}^{m_1}\left((\beta_1 - 1)\ln(u_{1i}) + \ln\left[1 - u_{2i}^{\beta_2}\right]\right) + \\ & \sum_{i=1}^{m_2}\left((\delta_2 - 1)\ln(x_i) - \lambda_2 x_i^{\delta_2} + (\beta_2 - 1)\ln(u_{2i})\right) \\ & + \sum_{i=1}^{m_2}\ln\left[1 - u_{1i}^{\beta_1}\right] + \sum_{i=1}^{m} R_i \ln\left[\left(1 - u_{1i}^{\beta_1}\right)\left(1 - u_{2i}^{\beta_2}\right)\right] \\ & + R_D^* \ln\left[\left(1 - z_1^{\beta_1}\right)\left(1 - z_2^{\beta_2}\right)\right] \end{aligned} \tag{4}$$

The first derivations of (4) with respect to $\lambda_1, \lambda_2, \delta_1, \delta_2, \beta_1$ and β_2 are given, respectively, by

$$\frac{\partial \ln L}{\partial \lambda_1} = \frac{m_1}{\lambda_1} - \sum_{i=1}^{m_1} x_i^{\delta_1} + (\beta_1 - 1)\sum_{i=1}^{m_1} v_{1i} + \sum_{i=1}^{m_2} s_{1i} + \sum_{i=1}^{m} R_i s_{1i} + R_D^* s_1,$$

$$\frac{\partial \ln L}{\partial \lambda_2} = \frac{m_2}{\lambda_2} - \sum_{i=1}^{m_2} x_i^{\delta_2} + (\beta_2 - 1)\sum_{i=1}^{m_2} v_{2i} + \sum_{i=1}^{m_1} s_{2i} + \sum_{i=1}^{m} R_i s_{2i} + R_D^* s_2,$$

$$\begin{aligned} \frac{\partial \ln L}{\partial \delta_1} = & \frac{m_1}{\delta_1} + \sum_{i=1}^{m_1} \ln(x_i) - \lambda_1 \sum_{i=1}^{m_1} x_i^{\delta_1} \ln(x_i) \\ & + \lambda_1(\beta_1 - 1)\sum_{i=1}^{m_1} v_{1i} \ln(x_i) + \lambda_1 \sum_{i=1}^{m_2} s_{1i} \ln(x_i) \\ & + \lambda_1 \sum_{i=1}^{m} R_i s_{1i} \ln(x_i) + \lambda_1 R_D^* s_1 \ln(T), \end{aligned}$$

$$\begin{aligned} \frac{\partial \ln L}{\partial \delta_2} = & \frac{m_2}{\delta_2} + \sum_{i=1}^{m_2} \ln(x_i) - \lambda_2 \sum_{i=1}^{m_2} x_i^{\delta_2} \ln(x_i) \\ & + \lambda_2(\beta_2 - 1)\sum_{i=1}^{m_2} v_{2i} \ln(x_i) + \lambda_2 \sum_{i=1}^{m_1} s_{2i} \ln(x_i) \\ & + \lambda_2 \sum_{i=1}^{m} R_i s_{2i} \ln(x_i) + \lambda_2 R_D^* s_2 \ln(T), \end{aligned}$$

$$\begin{aligned} \frac{\partial \ln L}{\partial \beta_1} = & \frac{m_1}{\beta_1} + \sum_{i=1}^{m_1} \ln(u_{1i}) + \sum_{i=1}^{m_2} (1 - u_{1i}^{-\beta_1})^{-1} \ln(u_{1i}) + \sum_{i=1}^{m} R_i (1 - u_{1i}^{-\beta_1})^{-1} \ln(u_{1i}) \\ & + R_D^* (1 - z_1^{-\beta_1})^{-1} \ln(z_1), \end{aligned}$$

And

$$\frac{\partial \ln L}{\partial \beta_2} = \frac{m_2}{\beta_2} + \sum_{i=1}^{m_2} \ln(u_{2i}) + \sum_{i=1}^{m_1} (1-u_{2i}^{-\beta_2})^{-1} \ln(u_{2i})$$
$$+\sum_{i=1}^{m} R_i (1-u_{2i}^{-\beta_2})^{-1} \ln(u_{2i}) \quad (5)$$
$$+R_D^* (1-z_2^{-\beta_2})^{-1} \ln(z_2).$$

Where

$$v_{ji} = v_{ji}(\lambda_j, \delta_j) = (x_i^{\delta_j} e^{-\lambda_j x_i^{\delta_j}} / u_{ji}),$$
$$s_{ji} = s_{ji}(\lambda_j, \delta_j, \beta_j) = (\beta_j x_i^{\delta_j} e^{-\lambda_j x_i^{\delta_j}} u_{ji}^{\beta_j -1} / (u_{ji}^{\beta_j} - 1)),$$

$$s_j = s_j(\lambda_j, \delta_j, \beta_j) = (\beta_j T^{\delta_j} e^{-\lambda_j T^{\delta_j}} z_j^{\beta_j -1} / (z_j^{\beta_j} - 1)) \text{ And } j = 1, 2.$$

Equating the first derivations (5) to zero and solving for $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\delta}_1, \hat{\delta}_2, \hat{\beta}_1$ and $\hat{\beta}_2$ to obtain the MLE of the unknown parameters $\lambda_1, \lambda_2, \delta_1, \delta_2, \beta_1$ and β_2, we need numerical results and computer facilities.

The asymptotic variance-covariance matrix for $\lambda_1, \lambda_2, \delta_1, \delta_2, \beta_1$ and β_2 can be obtained by inverting the information matrix with the elements that are negative of the expected values of the second order derivatives of logarithms of the likelihood functions. Cohen [3] concluded that the approximate variance covariance matrix may be obtained by replacing expected values by their MLEs. Now the approximate sample information matrix associated with $\theta = (\lambda_1, \lambda_2, \delta_1, \delta_2, \beta_1, \beta_2)$ is defined as

$$I(\theta) \cong \begin{bmatrix} -\frac{\partial^2 \ln L}{\partial \lambda_1^2} & 0 & -\frac{\partial^2 \ln L}{\partial \lambda_1 \partial \delta_1} & 0 & -\frac{\partial^2 \ln L}{\partial \lambda_1 \partial \beta_1} & 0 \\ 0 & -\frac{\partial^2 \ln L}{\partial \lambda_2^2} & 0 & -\frac{\partial^2 \ln L}{\partial \lambda_2 \partial \delta_2} & 0 & -\frac{\partial^2 \ln L}{\partial \lambda_2 \partial \beta_2} \\ -\frac{\partial^2 \ln L}{\partial \delta_1 \partial \lambda_1} & 0 & -\frac{\partial^2 \ln L}{\partial \delta_1^2} & 0 & -\frac{\partial^2 \ln L}{\partial \delta_1 \partial \beta_1} & 0 \\ 0 & -\frac{\partial^2 \ln L}{\partial \delta_2 \partial \lambda_2} & 0 & -\frac{\partial^2 \ln L}{\partial \delta_2^2} & 0 & -\frac{\partial^2 \ln L}{\partial \delta_2 \partial \beta_2} \\ -\frac{\partial^2 \ln L}{\partial \beta_1 \partial \lambda_1} & 0 & -\frac{\partial^2 \ln L}{\partial \beta_1 \partial \delta_1} & 0 & -\frac{\partial^2 \ln L}{\partial \beta_1^2} & 0 \\ 0 & -\frac{\partial^2 \ln L}{\partial \beta_2 \partial \lambda_2} & 0 & -\frac{\partial^2 \ln L}{\partial \beta_2 \partial \delta_2} & 0 & -\frac{\partial^2 \ln L}{\partial \beta_2^2} \end{bmatrix}_{\hat{\lambda}_1, \hat{\lambda}_2, \hat{\delta}_1, \hat{\delta}_2, \hat{\beta}_1, \hat{\beta}_2}$$

The elements of $I(\theta)$ will be as follows

$$\frac{\partial^2 \ln L}{\partial \lambda_j^2} = -\frac{m_j}{\lambda_j^2} - (\beta_j - 1) \sum_{i=1}^{m_j} v_{ji} u_{ji}^{-1} x_i^{\delta_j}$$
$$+ \sum_{i=1}^{m_{3-j}} x_i^{\delta_j} s_{ji} \begin{bmatrix} \left((\beta_j - 1) k_{ji} u_{ji}^{-1} - 1 \right) \\ -\beta_j u_{ji}^{\beta_j - 1} k_{ji} (u_{ji}^{\beta_j} - 1)^{-1} \end{bmatrix}$$
$$+ \sum_{i=1}^{m} R_i x_i^{\delta_j} s_{ji} \begin{bmatrix} \left((\beta_j - 1) k_{ji} u_{ji}^{-1} - 1 \right) \\ -\beta_j u_{ji}^{\beta_j - 1} k_{ji} (u_{ji}^{\beta_j} - 1)^{-1} \end{bmatrix}$$
$$+ R_D^* T^{\delta_j} s_j \begin{bmatrix} \left((\beta_j - 1) k_j u_j^{-1} - 1 \right) \\ -\beta_j u_j^{\beta_j - 1} k_j (u_j^{\beta_j} - 1)^{-1} \end{bmatrix},$$

$$\frac{\partial^2 \ln L}{\partial \delta_j^2} = -\frac{m_j}{\delta_j^2} - \lambda_j \sum_{i=1}^{m_j} x_i^{\delta_1} \ln(x_i)^2$$
$$+\lambda_j (\beta_j - 1) \sum_{i=1}^{m_1} x_i^{\delta_1} k_{ji} u_{ji}^{-2} \ln(x_i)^2 \left[u_{ji} - \lambda_j x_i^{\delta_j} \right]$$
$$+ \lambda_j \sum_{i=1}^{m_{3-j}} \ln(x_i)^2 s_{ji} \left[w_{ji} - \lambda_j s_{ji} \right]$$
$$+\lambda_j \sum_{i=1}^{m} R_i \ln(x_i)^2 s_{ji} \left[w_{ji} - \lambda_j s_{ji} \right]$$
$$+ \lambda_j R_D^* \ln(T)^2 s_j \left[w_j - \lambda_j s_j \right],$$

$$\frac{\partial^2 \ln L}{\partial \beta_j^2} = -\frac{m_j}{\beta_j^2} - \sum_{i=1}^{m_{3-j}} (1-u_{ji}^{-\beta_j})^{-2} u_{ji}^{-\beta_j} \ln(u_{ji})^2$$
$$-\sum_{i=1}^{m} R_i (1-u_{ji}^{-\beta_j})^{-2} u_{ji}^{-\beta_j} \ln(u_{ji})^2$$
$$- R_D^* (1-z_j^{-\beta_j})^{-2} z_j^{-\beta_j} \ln(z_j)^2,$$

$$\frac{\partial^2 \ln L}{\partial \lambda_j \partial \delta_j} = -\sum_{i=1}^{m_j} x_i^{\delta_1} \ln(x_i) + (\beta_j - 1) \sum_{i=1}^{m_j} x_i^{\delta_j} k_{ji} u_{ji}^{-2} \ln(x_i) \left[u_{ji} - \lambda_j . x_i^{\delta_j} \right]$$
$$+ \sum_{i=1}^{m_{3-j}} s_{ji} \ln(x_i) \left[w_{ji} + \lambda_j s_{ji} \right] + \sum_{i=1}^{m} R_i s_{ji} \ln(x_i) \left[w_{ji} + \lambda_j s_{ji} \right]$$
$$+ R_D^* s_j \ln(T) \left[w_j + \lambda_j s_j \right],$$

$$\frac{\partial^2 \ln L}{\partial \lambda_j \partial \beta_j} = \sum_{i=1}^{m_j} v_{ji} + \sum_{i=1}^{m_{3-j}} \beta_j^{-1} s_{ji} \left[1 - \beta_j \ln(u_{ji}) \left(u_{ji}^{\beta_j} - 1 \right)^{-1} \right]$$
$$+ \sum_{i=1}^{m} R_i \beta_j^{-1} s_{ji} \left[1 - \beta_j \ln(u_{ji}) \left(u_{ji}^{\beta_j} - 1 \right)^{-1} \right]$$
$$+ R_D^* \beta_j^{-1} s_j \left[1 - \beta_j \ln(z_j) \left(z_j^{\beta_j} - 1 \right)^{-1} \right],$$

And

$$\frac{\partial^2 \ln L}{\partial\delta_j \partial\beta_j} = \lambda_j \sum_{i=1}^{m_j} v_{ji} \ln(x_i)$$
$$+\lambda_j \sum_{i=1}^{m_{3-j}} \beta_j^{-1} s_{ji} \ln(x_i)\left[1-\beta_j \ln(u_{ji})\left(u_{ji}^{\beta_j}-1\right)^{-1}\right]$$
$$+\lambda_j \sum_{i=1}^{m} \beta_j^{-1} s_{ji} \ln(x_i)\left[1-\beta_j \ln(u_{ji})\left(u_{ji}^{\beta_j}-1\right)^{-1}\right]$$
$$+\lambda_j R_D^* \beta_j^{-1} s_j \left[1-\beta_j \ln(z_j)\left(z_j^{\beta_j}-1\right)^{-1}\right].$$

Where

$$k_{ji} = e^{-\lambda_j x_i^{\delta_j}}, \quad k_j = e^{-\lambda_j T^{\delta_j}}$$

$$w_{ji} = w_{ji}(\lambda_j, \beta_j, \delta_j) = (\lambda_j(\beta_j - 1)x_i^{\delta_j} k_{ji} u_{ji}^{-1} - \lambda_j x_i^{\delta_j} + 1),$$

$$w_j = w_j(\lambda_j, \beta_j, \delta_j) = (\lambda_j(\beta_j - 1)T^{\delta_j} k_j u_j^{-1} - \lambda_j T^{\delta_j} + 1) \text{ And } j = 1,2$$

In the case of generalized Weibull distribution, Sarhan et al. [10] introduced the relative risk rates, π_1 and π_2, due to cause 1 and 2, the relative risk due to cause 1 and cause 2 is defined as

$$\pi_1 = 1 - \lambda_1\delta_1\beta_1 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\beta_1 - 1}{i}\binom{\beta_2}{j}(-1)^{i+j}\int_0^{\infty} x^{\delta_1 - 1} e^{-\left[(i+1)\lambda_1 x^{\delta_1} + j\lambda_2 x^{\delta_2}\right]} dx \qquad (6)$$

And

$\pi_2 = 1 - \pi_1$.

As the integral in the right side of (6) has no analytical solution, we have to use a numerical technique to solve the integral. According to the invariance property of the MLE of the relative risk rates π_1 and π_2, can be obtained by replacing of $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\delta}_1, \hat{\delta}_2, \hat{\beta}_1$ and $\hat{\beta}_2$ in (6) respectively.

Special cases:

1) When $\delta_1 = \delta_2 = \beta_1 = \beta_2 = 1$, the MLE's of λ_1 and λ_2 and the relative risk rates π_1 and π_2, we obtain the corresponding results of the exponential distribution obtained by Kundu and Joarder [7], when the cause of failure is known, i.e., we generalized this results.
2) When $\beta_1 = \beta_2 = 1$, the MLE's of $\delta_1, \delta_2, \lambda_1$ and λ_2 and the relative risk rates π_1 and π_2, corresponds to the results of the Weibull distribution may be obtained.

4. Results in the case of the generalized exponential distribution

Using results in section (3), we obtain the present section results in the case of generalized exponential distribution. The pdf of the generalized exponential distribution is given by

$$f_j(x) = \lambda_j \beta_j e^{-\lambda_j x}(1 - e^{-\lambda_j x})^{\beta_j - 1}, \quad x > 0; \quad \lambda_j > 0, \beta_j > 1 \qquad (7)$$

And using the likelihood function (2) and (7), the likelihood function of the observed data ignoring the constant can be written as follows

$$L \propto (\lambda_1\beta_1)^{m_1}(\lambda_2\beta_2)^{m_2}\prod_{i=1}^{m_1} e^{-\lambda_1 x_i} u_{1i}^{\beta_1 - 1}\left[1 - u_{2i}^{\beta_2}\right]\prod_{i=1}^{m_2} x_i^{\beta_2 - 1} e^{-\lambda_2 x_i} u_{2i}^{\beta_2 - 1}\left[1 - u_{1i}^{\beta_1}\right]$$
$$\prod_{i=1}^{m}\left[\left(1 - u_{1i}^{\beta_1}\right)\left(1 - u_{2i}^{\beta_2}\right)\right]^{R_i} \times \left[\left(1 - z_1^{\beta_1}\right)\left(1 - z_2^{\beta_2}\right)\right]^{R_D^*}$$

Where $L = L(\lambda_1, \lambda_2, \beta_1, \beta_2)$, $m_1 = r_1, m_2 = r_2, m = r$ in case I $m_1 = D_1,\ m_2 = D_2,\ m = D$ in case II, $u_{ji} = u_{ji}(\lambda_j) = 1 - e^{-\lambda_j x_i}$, $z_j = z_j(\lambda_j) = 1 - e^{-\lambda_j T}$ and $j = 1,2$. Now, the log likelihood function is given by

$$\ln L \,\alpha\, m_1(\ln\lambda_1 + \ln\beta_1 + \ln\delta_1) + m_2(\ln\lambda_2 + \ln\beta_2 + \ln\delta_2)$$
$$+\sum_{i=1}^{m_1}\left((\delta_1 - 1)\ln(x_i) - \lambda_1 x_i^{\delta_1}\right)$$
$$+\sum_{i=1}^{m_1}\left((\beta_1 - 1)\ln(u_{1i}) + \ln\left[1 - u_{2i}^{\beta_2}\right]\right)$$
$$+\sum_{i=1}^{m_2}\left((\delta_2 - 1)\ln(x_i) - \lambda_2 x_i^{\delta_2} + (\beta_2 - 1)\ln(u_{2i})\right) \qquad (8)$$
$$+\sum_{i=1}^{m_2}\ln\left[1 - u_{1i}^{\beta_1}\right] + \sum_{i=1}^{m} R_i \ln\left[\left(1 - u_{1i}^{\beta_1}\right)\left(1 - u_{2i}^{\beta_2}\right)\right]$$
$$+R_D^* \ln\left[\left(1 - z_1^{\beta_1}\right)\left(1 - z_2^{\beta_2}\right)\right]$$

The first derivations of (8) with respect to $\lambda_1, \lambda_2, \beta_1$ and β_2 are given, respectively, by

$$\frac{\partial \ln L}{\partial\lambda_1} = \frac{m_1}{\lambda_1} - \sum_{i=1}^{m_1} x_i + (\beta_1 - 1)\sum_{i=1}^{m_1} v_{1i} + \sum_{i=1}^{m_2} s_{1i} + \sum_{i=1}^{m} R_i s_{1i} + R_D^* s_1,$$

$$\frac{\partial \ln L}{\partial\lambda_2} = \frac{m_2}{\lambda_2} - \sum_{i=1}^{m_2} x_i + (\beta_2 - 1)\sum_{i=1}^{m_2} v_{2i} + \sum_{i=1}^{m_1} s_{2i} + \sum_{i=1}^{m} R_i s_{2i} + R_D^* s_2,$$

$$\frac{\partial \ln L}{\partial\beta_1} = \frac{m_1}{\beta_1} + \sum_{i=1}^{m_1}\ln(u_{1i}) + \sum_{i=1}^{m_2}(1 - u_{1i}^{-\beta_1})^{-1}\ln(u_{1i})$$
$$+\sum_{i=1}^{m} R_i (1 - u_{1i}^{-\beta_1})^{-1}\ln(u_{1i}) + R_D^*(1 - z_1^{-\beta_1})^{-1}\ln(z_1)$$

And

$$\frac{\partial \ln L}{\partial\beta_2} = \frac{m_2}{\beta_2} + \sum_{i=1}^{m_2}\ln(u_{2i}) + \sum_{i=1}^{m_1}(1 - u_{2i}^{-\beta_2})^{-1}\ln(u_{2i})$$
$$+\sum_{i=1}^{m} R_i (1 - u_{2i}^{-\beta_2})^{-1}\ln(u_{2i}) \qquad (9)$$
$$+R_D^*(1 - z_2^{-\beta_2})^{-1}\ln(z_2)$$

Where

$$v_{ji} = v_{ji}(\lambda_j) = (x_i e^{-\lambda_j x_i} / u_{ji}),$$

$$s_{ji} = s_{ji}(\lambda_j, \beta_j) = (\beta_j x_i e^{-\lambda_j x_i} u_{ji}^{\beta_j - 1} / (u_{ji}^{\beta_j} - 1)),$$

$s_j = s_j(\lambda_j, \beta_j) = (\beta_j T\ e^{-\lambda_j T}\ z_j^{\beta_j - 1} / (z_j^{\beta_j} - 1))$ And $j = 1, 2$.

Equating the first derivations (9) to zero and solving functions for $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\beta}_1$ and $\hat{\beta}_2$ to obtain the MLE of the unknown parameters $\lambda_1, \lambda_2, \beta_1$ and β_2, we need an illustrative examples and computer facilities. The Fisher information matrix associated with $\theta = (\lambda_1, \lambda_2, \beta_1, \beta_2)$ is defined as

$$I(\theta) \approx \begin{bmatrix} -\frac{\partial^2 \ln L}{\partial \lambda_1^2} & 0 & -\frac{\partial^2 \ln L}{\partial \lambda_1 \partial \beta_1} & 0 \\ 0 & -\frac{\partial^2 \ln L}{\partial \lambda_2^2} & 0 & -\frac{\partial^2 \ln L}{\partial \lambda_2 \partial \beta_2} \\ -\frac{\partial^2 \ln L}{\partial \lambda_1 \partial \beta_1} & 0 & -\frac{\partial^2 \ln L}{\partial \beta_1^2} & 0 \\ 0 & -\frac{\partial^2 \ln L}{\partial \lambda_2 \partial \beta_2} & 0 & -\frac{\partial^2 \ln L}{\partial \beta_2^2} \end{bmatrix}_{\hat{\lambda}_1, \hat{\lambda}_2, \hat{\beta}_1, \hat{\beta}_2}$$

The elements of $I(\theta)$ will be as follows

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \lambda_j^2} &= -\frac{m_j}{\lambda_j^2} - (\beta_j - 1)\sum_{i=1}^{m_j} v_{ji} u_{ji}^{-1} x_i \\ &+ \sum_{i=1}^{m_{3-j}} x_i s_{ji} \left[\left((\beta_j - 1) k_{ji} u_{ji}^{-1} - 1 \right) - \beta_j u_{ji}^{\beta_j - 1} k_{ji} (u_{ji}^{\beta_j} - 1)^{-1} \right] \\ &+ \sum_{i=1}^{m} R_i x_i s_{ji} \left[\left((\beta_j - 1) k_{ji} u_{ji}^{-1} - 1 \right) - \beta_j u_{ji}^{\beta_j - 1} k_{ji} (u_{ji}^{\beta_j} - 1)^{-1} \right] \\ &+ R_D^* T\ s_j \left[\left((\beta_j - 1) k_j u_j^{-1} - 1 \right) - \beta_j u_j^{\beta_j - 1} k_j (u_j^{\beta_j} - 1)^{-1} \right], \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \beta_j^2} &= -\frac{m_j}{\beta_j^2} - \sum_{i=1}^{m_{3-j}} (1 - u_{ji}^{-\beta_j})^{-2} u_{ji}^{-\beta_j} \ln(u_{ji})^2 \\ &- \sum_{i=1}^{m} R_i (1 - u_{ji}^{-\beta_j})^{-2} u_{ji}^{-\beta_j} \ln(u_{ji})^2 \\ &- R_D^* (1 - z_j^{-\beta_j})^{-2} z_j^{-\beta_j} \ln(z_j)^2, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \lambda_j \partial \beta_j} &= \sum_{i=1}^{m_j} v_{ji} + \sum_{i=1}^{m_{3-j}} \beta_j^{-1} s_{ji} \left[1 - \beta_j \ln(u_{ji}) \left(u_{ji}^{\beta_j} - 1 \right)^{-1} \right] \\ &+ \sum_{i=1}^{m} R_i \beta_j^{-1} s_{ji} \left[1 - \beta_j \ln(u_{ji}) \left(u_{ji}^{\beta_j} - 1 \right)^{-1} \right] + R_D^*\ \beta_j^{-1} s_j \left[1 - \beta_j \ln(z_j) \left(z_j^{\beta_j} - 1 \right)^{-1} \right] \end{aligned}$$

Where

$k_{ji} = e^{-\lambda_j x_i}$, $k_{ji} = e^{-\lambda_j T}$ And $j = 1, 2$.

In the case of generalized exponential distribution, Sarhan [9] introduced the relative risk rates, π_1 and π_2, due to cause 1 and 2, respectively. The corresponding π_1 and π_2 in the present case will be

$$\pi_1 = 1 - \lambda_1 . \beta_1 \sum_{i=0}^{\infty} \binom{\beta_2}{j} (-1)^i \int_0^{\infty} e^{-\lambda_1 x} \left(1 - e^{-\lambda_1 x}\right)^{\beta_1 - 1} e^{-i \lambda_2 x}\, dx$$

And

$$\pi_2 = 1 - \pi_1 \tag{10}$$

According to the invariance property of the MLE, the MLE of the relative risk rates π_1 and π_2, can be obtained by replacing the MLE of $\lambda_1, \lambda_2, \beta_1$ and β_2 in (10).

5. Confidence intervals and goodness of fit

In this section we derive the confidence intervals of the vector of the unknown parameters $\theta = (\lambda_1, \lambda_2, \delta_1, \delta_2, \beta_1, \beta_2)$. Based on the asymptotic distribution of the MLE of the parameters, it is known that

$$\left(\hat{\boldsymbol{\theta}} - \theta\right) \longrightarrow N_6\left(0, I^{-1}(\theta)\right)$$

Where $I(\theta)$ is the Fisher information matrix. The elements of matrix $I_{ij}(\theta)$ can be approximated by $I_{ij}(\hat{\theta})$, where

$$I_{ij}(\hat{\theta}) = -\left.\frac{\partial^2 \ln L}{\partial \theta_i \partial \theta_j}\right|_{\theta = \hat{\theta}}$$

And $\partial^2 \ln L / \partial \theta_i \partial \theta_j$ is the second derivations. The $100(1-\alpha)$ approximate confidence intervals of the vector of the unknown parameters $\theta = (\lambda_1, \lambda_2, \delta_1, \delta_2, \beta_1, \beta_2)$ can be obtained as follows

$$\hat{\theta}_j \pm z_{\alpha/2} \sqrt{\mathrm{Var}(\hat{\theta}_j)}\ ,\ j = 1, \ldots, 6\ .$$

Where $\mathrm{Var}(\hat{\theta}_j)$ is the elements on the main diagonal of $\boldsymbol{I}^{-1}(\hat{\boldsymbol{\theta}})$ and $z_{(\frac{\alpha}{2})}$ is the upper $(\frac{\alpha}{2})^{th}$ percentile of a standard normal distribution. The procedures discussed above can be easily modified to the case of ED, GED and WD.

Now, we investigate whether the generalized Weibull model can better fit a real data set rather than (1) exponential distribution model (ED), (2) generalized exponential distribution model (GED) and (3) Weibull distribution model (WD). We discuss the problem of testing goodness of fit of a competing risks model when the causes of failures follow generalized Weibull distributions against exponential distributions, generalized exponential distributions and Weibull distributions. The likelihood ratio test statistics will be used to this purpose. The null and alternative hypotheses are

First case: Testing the competing risks with exponential distribution against competing risks with generalized Weibull distribution:

$H_0 : \delta_1 = \delta_1 = \beta_1 = \beta_2 = 1$, the causes of death follow ED,

$H_1 : \delta_1 \neq \delta_2 \neq \beta_1 \neq \beta_2 \neq 1$, the causes of death follow GWD.

The log-likelihood ratio test statistic is

$$X_L = 2\left(L_{GWD} - L_{ED}\right)$$

Where L_{ED} and L_{GWD} are the log-likelihood functions under H_0 and H_1, respectively, after replacing the unknown parameters with their MLE. The test statistic is asymptotically distributed as a chi-squared distribution with 4 degrees of freedom.

Second case: Testing the competing risks with generalized exponential distribution against competing risks with generalized Weibull distribution:

$H_0 : \delta_1 = \delta_2 = 1$, the causes of death follow GED,

$H_1: \delta_1 \neq \delta_2 \neq 1$, the causes of death follow GWD.

The log-likelihood ratio test statistic is

$$X_L = 2(L_{GWD} - L_{GED})$$

Asymptotically, the test statistic is distributed as a chi-squared distribution with 2 degrees of freedom. Here, L_{GED} and L_{GWD} are the log-likelihood functions under H_0 and H_1, respectively, after replacing the unknown parameters with their MLE.
Third case: Testing the competing risks with Weibull distribution against competing risks with generalized Weibull distribution:

$H_0: \beta_1 = \beta_2 = 1$, the causes of death follow WD,

$H_1: \beta_1 \neq \beta_2 \neq 1$, the causes of death follow GWD.

The log-likelihood ratio test statistic is

$$X_L = 2(L_{GWD} - L_{WD})$$

Asymptotically, the test statistic is distributed as a chi-squared distribution with 2 degrees of freedom. Here, L_{WD} and L_{GWD} are the log-likelihood functions under H_0 and H_1, respectively, after replacing the unknown parameters with their MLE. For comparison purposes between the candidate models, we can use two model criterion selection, the Akaike information criterion (AIC) (Akaike [1]) and Bayes information criterion (BIC) (Schwarz [8]) defined as

$$\text{AIC} = -2L + 2p$$

And

$$\text{BIC} = -2L + p.\ln(n) .$$

Where p is the number of parameters in the model, and L is the maximized value of the likelihood function for the model. As a model selection criterion, the researcher should choose the model that minimizes AIC and BIC.

6. Numerical illustration

In this section, we re-analyze one data set which was originally by Hoel [6] and later by Kundu and Joardar [7]. The data was obtained from a laboratory experiment in which male mice received a radiation dose of 300 roentgens at 5 to 6 weeks of age. The cause of death for each mouse was determined by reticulum cell sarcoma is considered as cause 1 and the other causes of death as cause 2. There were $n = 77$ observations remain in the analysis. Using the censoring scheme $m = 25$ and $R_1 = R_2 = ... = R_{24} = 2$ and $R_{25} = 4$, the progressive Type-II censored sample from the original data is given by

(40, 2), (42, 2), (62, 2), (163, 2), (179, 2), (206, 2), (222, 2), (228, 2), (252, 2), (259,2), (318, 1), (385, 2), (407, 2), (420, 2), (462, 2), (517, 2), (517, 2), (524, 2), (525, 1),(536, 1),(558, 1), (605, 1), (612, 1), (620,2), (621, 1).

The first component denotes the life time and the second component indicate the cause of failure.
Example 1: Considering $T = 700$, then $m = r = 25$, $m_1 = r_1 = 7$ and $m_2 = r_2 = 18$. From the above data, the MLEs of the unknown parameters , the corresponding approximate 95% two sided confidence intervals distributions and the relative risk due to cause one are obtained and given in table (1). The log-likelihood values (L), AIC, BIC, X_L and p-values shown in Table (2). All of the computations were performed using MATHCAD program version 2007.
Example 2 Now we use the same data, but use $T = 600$ instead of $T = 700$, while $m = D = 21$, $m_1 = D_1 = 4$ and $m_2 = D_2 = 17$. The results are reported in Tables (3) and (4).

Table 1: The MLE, Approximate 95% Two Sided Confidence Intervals of the Parameters and the Estimated Relative Risk Due Cause One in Each Model (ED, GED, WD and GWD).

Parameters	Model			
	ED	GED	WD	GWD
λ_1	0.0002417 (0.000063, 0.000421)	0.0049 (0.0023, 0.008)	0.00000001 (0, 0.0000000231747)	0.00000192 (0, 0.00002782)
λ_2	0.0006215 (0.000334, 0.000909)	0.0011 (0.0004, 0.002)	0.00003909 (0, 0.0001831256445)	0.00000085 (0, 0.0000398)
δ_1	--	--	2.59885355 (2.45663, 2.7410701)	2.10137652 (0.108559, 4.09419)
δ_2	--	--	1.45195511 (0.858016, 2.045893)	1.9692305 (0, 8.2601143 X_L 3)
β_1	--	28.2304 (0, 68.86)	--	4.81897126 (0, 11.98389969)
β_2	--	1.5537 (0.94, 2.167)	--	0.70162976 (0, 3.1593565 X_L 8)
Relative Risk	0.28	0.5605	0.4525	0.5558933

Table 2: The Log-Likelihood Values (L), AIC, BIC, and P-Values for the Compared Models.

Model	L	AIC	BIC	X_L	p-value
ED	-216.195	436.391	441.078	26.36	0.00002
GED	-203.736	415.471	424.846	1.442	0.486
WD	-207.272	422.544	431.919	8.402	0.014
GWD	-203.015	418.03	432.093	--	--

Table 3: The MLE, Approximate 95% Two Sided Confidence Intervals of the Parameters and the Estimated Relative Risk Due Cause One in Each Model (ED, GED, WD and GWD).

Parameters	Model			
	ED	GED	WD	GWD
λ_1	0.0001966 (0.000004, 0.000389)	0.00318 (0.0003, 0.00606)	0.00000001 (0, 0.000000031)	0.00000184 (0, 0.000034)
λ_1	0.0008355 (0.000438, 0.001233)	0.00104 (0.00037, 0.00172)	0.00005788 (0, 0.000267454)	0.00000081 (0, 0.00005947)
δ_1	--	--	2.51613 (2.264, 2.767905)	2.04481223 (0, 4.34409)
δ_2	--	--	1.38072 (0.795510, 1.96594)	1.95179778 (0, 11.711)
β_1	--	12.13171 (0, 31.78391)	--	3.57286717 (0, 14.852)
β_2	--	1.47291 (0.89922, 2.04659)	--	0.67473998 (0, 4.34584)
Relative Risk	0.19	0.5062	0.3828	0.5171969

Table 4: The Log-Likelihood Values (L), AIC, BIC, and P-Values for the Compared Models.

Model	L	AIC	BIC	X_L	p-value
ED	-184.294	372.587	377.275	14.704	0.00471
GED	-177.074	362.149	371.524	0.264	0.876341
WD	-178.354	364.708	374.083	2.824	0.244
GWD	-176.942	365.883	379.946	--	--

From Tables (1): (4), it is observed that T plays a major role in the estimation and for the construction of the corresponding confidence intervals. We also conclude that based on the values of X_L and the p-value, AIC and BIC, the GWD fits the data better than ED and WD, while the GED fits the data better than GWD. We have observed that the assumptions that the GED may be used to analyze this set of real data better that the ED, WD and GWD.

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