Transmutation of the two parameters Rayleigh distribution

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Abstract

In this study, transmuted two parameters Rayleigh distribution is proposed using quadratic rank transmutation map. This proposed distribution is more flexible and versatile than two parameters Rayleigh distribution. Some properties of the proposed distribution are derived such as moments, moment generating function, mean, variance, median, quantile function, reliability, and hazard function. The parameter estimation is approached through the method of least square estimation. The rth and joint order statistics are also derived for the proposed distribution. The application of proposed model illustrated and compared using real data.

Keywords: *Least Square Estimation; Moments; Order Statistics; Transmuted Two Parameters Rayleigh Distribution.*

1. Introduction

Rayleigh distribution is widely used for survival analysis. Therefore, its application appeared in the medical sciences (patients survival times after operation, period of recurrence after surgery), engineering (component life cycles), computer sciences (failure rates of any software system), marketing (lifetimes of market rates of components) and in social sciences (duration that graduate person remain unemployed). For more details about lifetime applications, we refer Lai [10].

Lord Rayleigh (1880) introduced the single scale parameter Rayleigh distribution. It became much popular in survival analysis as it has hazard rate function that is an increasing function. Two parameters Rayleigh distribution with location and scale parameters also present in literature and recently, Dey *et al*. [2] used the two parameters Rayleigh distribution. Some more properties of two parameters Rayleigh distribution are derived by Leemis and McQueston [6] and Khan *et al*. [5] . Abd-Elfattah *et al*. [1] studied Bayes estimator and Dyer and Whisenand [3] introduced best linear unbiased estimator of Rayleigh distribution.

Merovci [7] introduced the transmuted Rayleigh distribution that has the scale and shape parameters and it proposed by using quadratic rank transmutation map (QRTM). In this study, the transmuted two parameters Rayleigh distribution is introduced with location, scale and shape parameters.

Rest of the paper is organized as follows. In section 2, the probability density function (pdf) and cumulative distribution function (cdf) of the transmuted two parameters Rayleigh distribution are derived and also graphically presented. The statistical properties, rth moment, moments about the origin, mean, variance and moment generating function are derived in Section 3. Section 4 provides quantile function, random number generation and median of transmuted two parameters Rayleigh distribution. Section 5 is about hazard function and reliability function with their graphical presentation. Section 6 contains smallest, largest, rth and joint order distribution of the proposed distribution. Parameters estimation of the distribution is discussed in Section 7. To compare the transmuted two parameters Rayleigh distribution with its parent distribution a real data set is used in Section 8.

2. Transmuted two parameters Rayleigh distribution

A random variable X has a transmuted distribution if its satisfy the relationship that is given by Shaw and Buckley [8] which is known as quadratic rank transmutation map(QRTM)

$$F(x) = G(x)[(1 + \lambda) - \lambda G(x)], \qquad |\lambda| \leq 1 \qquad (2.1)$$

Which on differentiation yields

$$f(x) = g(x)[1 + \lambda - 2\lambda G(x)] \qquad (2.2)$$

Where λ is an additional parameter which is known as transmuted parameter and $g(x)$ and $G(x)$ are the pdf and cdf of base distribution.

The pdf of the two parameters Rayleigh distribution with location and scale parameters is

$$g(x; \alpha, \beta) = 2\beta(x - \alpha)e^{-\beta(x-\alpha)^2} \qquad x > \alpha, \beta > 0 \qquad (2.3)$$

Where β and α are scale and location parameters respectively.
And the corresponding cdf is defined as

$$G(x; \alpha, \beta) = 1 - e^{-\beta(x-\alpha)^2} \qquad x > \alpha \qquad (2.4)$$

By using (2.1) and (2.4), we obtained the cdf of the transmuted two parameters Rayleigh distribution

$$F_{TR}(x; \alpha, \beta, \lambda) = \left[1 - e^{-\beta(x-\alpha)^2}\right]\left[1 + \lambda e^{-\beta(x-\alpha)^2}\right] \qquad (2.5)$$

And by using (2.2),(2.3) and (2.4) or by differentiation (2.5) gives the corresponding pdf of transmuted two parameters Rayleigh distribution that is defined as

$$f_{TR}(x; \alpha, \beta, \lambda) = 2\beta(x - \alpha)e^{-\beta(x-\alpha)^2}\left[1 - \lambda + 2\lambda e^{-\beta(x-\alpha)^2}\right] \quad (2.6)$$

The transmuted parameter λ has range -1 to +1 and if we take $\lambda = 0$ then this transmuted pdf and cdf reduces to the base distri-

bution. Various shapes of the cdf's and pdf's with different values of parameters are illustrated in Fig. 1 and Fig. 2 respectively.
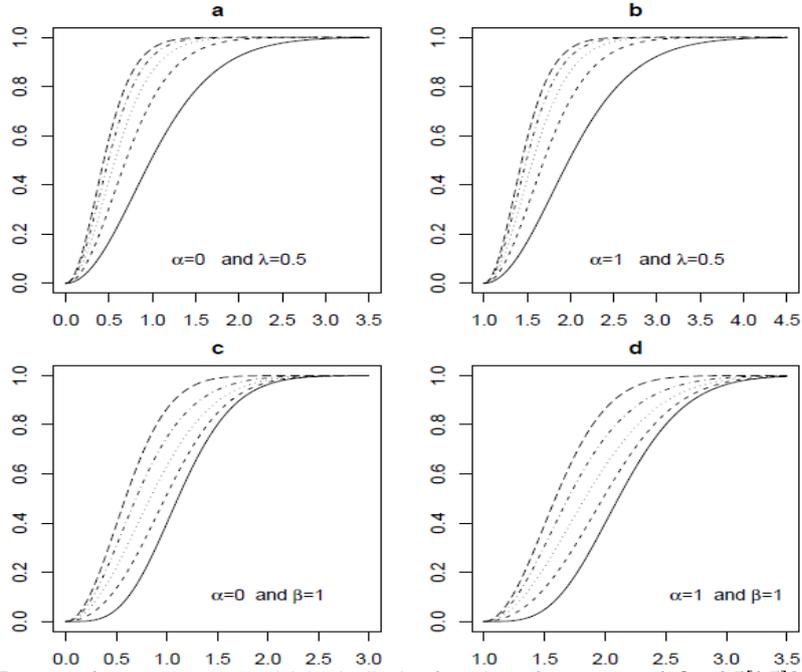

Fig. 1: The cdf's of various Transmuted two parameters Rayleigh Distribution for values of parameters: a) $\beta = 0.5[0.5]2.5$, b) $\beta = 0.5[0.5]2.5$, c) $\lambda = -1[0.5]1$, D) $\lambda = -1[0.5]1$ with Solid, Dashed, Dotted, Dotdash and Longdash lines respectively.

Fig. 2: The pdf's of Various Transmuted two parameters Rayleigh distribution for values of parameters: α) $\beta = 0.5[0.5]2.5$, b) $\beta = 0.5[0.5]2.5$, c) $\lambda = -1[0.5]1$, d) $\lambda = -1[0.5]1$ with Solid, Dashed, Dotted, Dotdash and Longdash lines respectively.
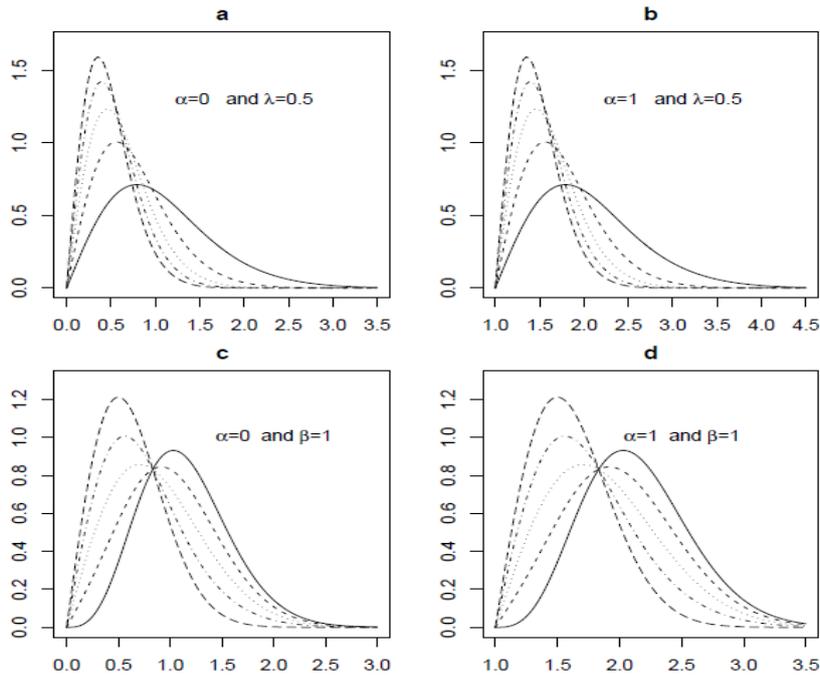
3. Moments

In this section, some statistical properties of transmuted two parameters Rayleigh distribution such as moments, mean, variance, the area under the curve and moment generating function are derived and discussed.

Theorem 3.1: *Let X is a random variable that has* $T_R(x; \beta, \alpha, \lambda)$ *with* $|\lambda| \leq 1$*, then* r*th moment* $E(x^r)$ *of transmuted two parameters Rayleigh distribution is*

$$\mu'_r = \sum_{k=0}^{r} \binom{r}{k} \alpha^{r-k} \left[(1-\lambda) \left\{ \frac{\Gamma\left(\frac{k}{2}+1\right)}{\beta^{\left(\frac{k}{2}\right)}} \right\} + \lambda \left\{ \frac{\Gamma\left(\frac{k}{2}+1\right)}{(2\beta)^{\left(\frac{k}{2}\right)}} \right\} \right] \quad (3.1)$$

Proof: The rth moment is given by

$$\mu'_r = E(X^r) = \int_{\alpha}^{\infty} x^r 2\beta(x-\alpha)e^{-\beta(x-\alpha)^2}\big[(1-\lambda) + 2\lambda e^{-\beta(x-\alpha)^2}\big]dx$$

For convenience substitute $x = \alpha + z$, then

$$\mu_r' = 2\beta \int_0^\infty (\alpha + z)^r z e^{-\beta z^2} [(1-\lambda) + 2\lambda e^{-\beta z^2}]dz$$

$$\mu_r' = 2\beta \int_0^\infty \sum_{k=0}^r \binom{r}{k} z^k \alpha^{r-k} z e^{-\beta z^2} [(1-\lambda) + 2\lambda e^{-\beta z^2}]dz$$

$$\mu_r' = 2\beta \sum_{k=0}^r \binom{r}{k} \alpha^{r-k} \int_0^\infty z^{k+1} e^{-\beta z^2} [(1-\lambda) + 2\lambda e^{-\beta z^2}]dz$$

Again for convenience we substitute $y = z^2$, then

$$\mu_r' = 2\beta \sum_{k=0}^r \binom{r}{k} \alpha^{r-k} \left[(1-\lambda) \int_0^\infty \frac{y^{\left(\frac{k+1}{2}\right)} e^{-\beta y}}{2\sqrt{y}} dy \right.$$
$$\left. + 2\lambda \int_0^\infty \frac{y^{\left(\frac{k+1}{2}\right)} e^{-2\beta y}}{2\sqrt{y}} dy \right]$$

$$\mu_r' = 2\beta \sum_{k=0}^r \binom{r}{k} \alpha^{r-k} \left[\frac{(1-\lambda)}{2} \int_0^\infty y^{\left(\frac{k}{2}\right)} e^{-\beta y} dy \right.$$
$$\left. + \lambda \int_0^\infty y^{\left(\frac{k}{2}\right)} e^{-2\beta y} dy \right]$$

$$\mu_r' = \sum_{k=0}^r \binom{r}{k} \alpha^{r-k} \left[(1-\lambda) \left\{ \frac{\Gamma\left(\frac{k}{2}+1\right)}{\beta^{\left(\frac{k}{2}\right)}} \right\} + \lambda \left\{ \frac{\Gamma\left(\frac{k}{2}+1\right)}{(2\beta)^{\left(\frac{k}{2}\right)}} \right\} \right]$$

By setting $r=1$ and $r=2$ in (3.1), we obtain mean, 1st and 2nd moments. The mean of transmuted two parameters Rayleigh distribution by using (3.1) can be defined as

$$Mean = E(X) = \alpha + \frac{\sqrt{\pi}}{2\beta}\left[(1-\lambda) + \frac{\lambda}{\sqrt{2}}\right] \qquad (3.2)$$

By using rth moment expression, 1st and 2nd moments about origin of transmuted Rayleigh distribution may be defined as
The 1st moment is given, by setting $r = 1$ in (3.1)

$$\mu_1' = \alpha + \frac{\sqrt{\pi}}{2\sqrt{\beta}}\left[(1-\lambda) + \frac{\lambda}{\sqrt{2}}\right] \qquad (3.3)$$

The 2nd moment about origin is given by setting $r = 2$ in (3.1)

$$\mu_2' = \alpha^2 + \frac{\alpha\sqrt{\pi}}{\sqrt{\beta}}\left[(1-\lambda) + \frac{\lambda}{\sqrt{2}}\right] + \frac{1}{\beta}\left[(1-\lambda) + \frac{\lambda}{2}\right] \qquad (3.4)$$

Coefficient of variation (CV), Skewness (Sk), and Kurtosis (Kr) of transmuted two parameters Rayleigh distribution can be derived by using following expressions through $\alpha_r' (r = 1,2,3,4)$

If $\mu_2 = \mu_2' - (\mu_1')^2$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 + 2(\mu_1')^3$$

Then

$$CV = \frac{\sigma}{\mu_1'}$$
$$Sk = \frac{\mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3}{\sigma^3}$$
$$Kr = \beta_2 = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 + 2(\mu_1')^3}{\sigma^4}$$

Theorem 3.2: *Let the random variable X follows transmuted two parameters Rayleigh distribution, and then its variance has the following form*

$$\sigma^2 = Var(X) = \frac{1}{\beta}\left[(1-\lambda) + \frac{\lambda}{2}\right] - \frac{\pi}{4\beta}\left[(1-\lambda) + \frac{\lambda}{2}\right]^2 \qquad (3.5)$$

Proof. The Variance of transmuted two parameters Rayleigh distribution is given as

$$Var(X) = E(X^2) - [E(X)]^2$$

Then

$$Var(X) = \alpha^2 + \frac{\alpha\sqrt{\pi}}{\sqrt{\beta}}\left[(1-\lambda) + \frac{\lambda}{\sqrt{2}}\right] + \frac{1}{\beta}\left[(1-\lambda) + \frac{\lambda}{2}\right]$$
$$- \left[\alpha + \frac{\sqrt{\pi}}{2\sqrt{\beta}}\left[(1-\lambda) + \frac{\lambda}{\sqrt{2}}\right]\right]^2$$

$$Var(X) = \alpha^2 + \frac{\alpha\sqrt{\pi}}{\sqrt{\beta}}\left[(1-\lambda) + \frac{\lambda}{\sqrt{2}}\right] + \frac{1}{\beta}\left[(1-\lambda) + \frac{\lambda}{2}\right] - \alpha^2$$
$$- \frac{\pi}{4\beta}\left[(1-\lambda) + \frac{\lambda}{2}\right]^2 - \frac{\alpha\sqrt{\pi}}{\sqrt{\beta}}\left[(1-\lambda) + \frac{\lambda}{\sqrt{2}}\right]$$

$$Var(X) = \frac{1}{\beta}\left[(1-\lambda) + \frac{\lambda}{2}\right] - \frac{\pi}{4\beta}\left[(1-\lambda) + \frac{\lambda}{2}\right]^2$$

Theorem 3.3: *Let X has a transmuted two parameters Rayleigh distribution, and then the moment generating function (mgf) of transmuted two parameters Rayleigh distribution is*

$$M_X(t) = \sum_{r=\alpha}^\infty \sum_{k=0}^r \frac{t^r}{r!} \binom{r}{k} \alpha^{r-k} \left[(1-\lambda)\left\{\frac{\Gamma\left(\frac{k}{2}+1\right)}{\beta^{\left(\frac{k}{2}\right)}}\right\} + \lambda\left\{\frac{\Gamma\left(\frac{k}{2}+1\right)}{(2\beta)^{\left(\frac{k}{2}\right)}}\right\} \right] \qquad (3.6)$$

Proof: The mgf for X is given as

$$M_X(t) = E(e^{tX}) = \int_\alpha^\infty e^{tx} f_{TR}(x;\alpha,\beta,\lambda)dx$$

We know that

$$e^{tx} = 1 + tx + \frac{t^2 x^2}{2!} + \frac{t^3 x^3}{3!} + \cdots \frac{t^r x^r}{r!} + \cdots$$

Then

$$M_X(t) = \int_\alpha^\infty \left(1 + tx + \frac{t^2 x^2}{2!} + \frac{t^3 x^3}{3!} \right.$$
$$\left. + \cdots \frac{t^n x^n}{n!} + \ldots\ldots\right) f_{TR}(x;\alpha,\beta,\lambda)dx$$

$$M_X(t) = \sum_{r=\alpha}^\infty \frac{t^r E(X)^r}{r!}$$

$$M_X(t) = \sum_{r=\alpha}^\infty \frac{t^r}{r!} \left(\sum_{k=0}^r \binom{r}{k} \alpha^{r-k} \left[(1-\lambda)\left\{\frac{\Gamma\left(\frac{k}{2}+1\right)}{\beta^{\left(\frac{k}{2}\right)}}\right\} \right.\right.$$
$$\left.\left. + \lambda\left\{\frac{\Gamma\left(\frac{k}{2}+1\right)}{(2\beta)^{\left(\frac{k}{2}\right)}}\right\}\right]\right)$$

$$M_X(t) = \sum_{r=\alpha}^{\infty} \sum_{k=0}^{r} \frac{t^r}{r!} \binom{r}{k} \alpha^{r-k} \left[(1-\lambda) \left\{ \frac{\Gamma\left(\frac{k}{2}+1\right)}{\beta^{\left(\frac{k}{2}\right)}} \right\} + \lambda \left\{ \frac{\Gamma\left(\frac{k}{2}+1\right)}{(2\beta)^{\left(\frac{k}{2}\right)}} \right\} \right]$$

Theorem 3.4: *Suppose X has a transmuted two parameters Rayleigh distribution, then total area of transmuted two parameters Rayleigh distribution under the curve is unity.*

$$Area = \int_{\alpha}^{\infty} 2\beta(x-\alpha)e^{-\beta(x-\alpha)^2}\left[1-\lambda+2\lambda e^{-\beta(x-\alpha)^2}\right]dx = 1$$

Proof: The area under the curve is

$$Area = \int_{\alpha}^{\infty} 2\beta(x-\alpha)e^{-\beta(x-\alpha)^2}\left[1-\lambda+2\lambda e^{-\beta(x-\alpha)^2}\right]dx$$

For convenience we substitute $z = x - \alpha$, then

$$Area = 2\beta \int_{0}^{\infty} z\, e^{-\beta z^2}\left[(1-\lambda)+2\lambda e^{-\beta z^2}\right]dz$$

$$Area = 2\beta(1-\lambda) \int_{0}^{\infty} z\, e^{-\beta z^2} + 4\beta\lambda \int_{0}^{\infty} z\, e^{-2\beta z^2}dz$$

Again for convenience we substitute $z^2 = y$, then

$$Area = 2\beta(1-\lambda) \int_{0}^{\infty} \sqrt{y}\, e^{-\beta y} \frac{1}{2\sqrt{y}}dy$$
$$+ 4\beta\lambda \int_{0}^{\infty} \sqrt{y}\, e^{-2\beta y} \frac{1}{2\sqrt{y}}dy$$

$$Area = (1-\lambda)\beta \left|\frac{e^{-\beta y}}{-\beta}\right|_{0}^{\infty} + 2\lambda\beta \left|\frac{e^{-2\beta y}}{-2\beta}\right|_{0}^{\infty}$$

$$Area = -(1-\lambda)[e^{-\infty} - e^0] - \lambda[e^{-\infty} - e^0]$$

$$Area = (1-\lambda) + \lambda = 1$$

4. Quantile function Random number generation and Median

Hyndman and Fan [4] introduced the samples quantile function for every distribution in this shape

$$Q(q) = F^{-1}(q) = inf\{x : F(x) \geq q\} \qquad 0 < q < 1 \qquad (4.1)$$

Where $F(x)$ is the distribution function. Quantile function divides the ordered data into qth equal parts. The qth quantile function of transmuted Rayleigh distribution by using (4.1) and (2.5) is given as

$$Q(q) = x$$
$$= \alpha + \left[-\frac{1}{\beta}log\left[\frac{(\lambda-1)\pm\sqrt{\lambda^2+1+2\lambda-4\lambda q}}{2\lambda}\right]\right]^{\frac{1}{2}} \qquad (4.2)$$

The expression (4.2) can also be used for random number generation if we substitute u rather than q, where u follows standard uniform variate then random numbers are generated as

$$x = \alpha + \left[-\frac{1}{\beta}log\left[\frac{(\lambda-1)\pm\sqrt{\lambda^2+1+2\lambda-4\lambda u}}{2\lambda}\right]\right]^{\frac{1}{2}} \qquad (4.3)$$

Median is 50th percentiles, and then median of transmuted two parameters Rayleigh distribution by using (4.2) and if we substitute $q = 0.5$, then we can find median of transmuted Rayleigh distribution

$$x_{0.5} = \alpha + \left[-\frac{1}{\beta}log\left[\frac{(\lambda-1)\pm\sqrt{\lambda^2+1}}{2\lambda}\right]\right]^{\frac{1}{2}} \qquad (4.4)$$

5. Reliability analysis

The Reliability function $R(t)$ gives the failure of a system or gives the probability of surviving of an item at reach on t time. The reliability function of transmuted two parameters Rayleigh distribution is given by

$$R_{TR}(t) = P(T > t) = \int_{t}^{\infty} f(t)dt = 1 - F_{TR}(t)$$

$$R_{TR}(t) = e^{-\beta(t-\alpha)^2}\left(1 - \lambda\left(1 - e^{-\beta(t-\alpha)^2}\right)\right) \qquad (5.1)$$

With different values of parameters, we can get different patterns of reliability function of transmuted two parameters Rayleigh distribution. Fig. 3 illustrates the pattern of reliability function of transmuted two parameters Rayleigh distribution.

An important property of life phenomenon in life testing analysis is hazard rate function. It measures the failure rate and it depends on time. If hazard rate value will increase then the probability of failure will also increase. Hazard rate function by using (2.6) and (5.1) can be defined as

$$h(t) = \frac{f(t)}{R(t)}$$

$$x_{0.5} = \alpha + \left[-\frac{1}{\beta}log\left[\frac{(\lambda-1)\pm\sqrt{\lambda^2+1}}{2\lambda}\right]\right]^{\frac{1}{2}} \qquad (5.2)$$
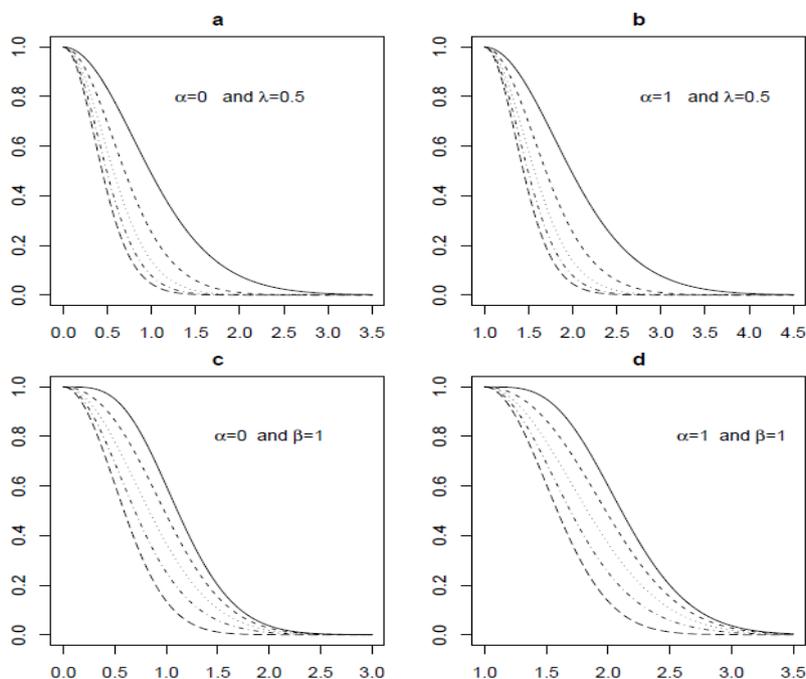
Fig. 3: The reliability function of various Transmuted two parameters Rayleigh distribution for values of parameters: $a)$ $\beta = 0.5[0.5]2.5$, $b)$ $\beta = 0.5[0.5]\ 2.5$, $c)$ $\lambda = -1[0.5]1$, $d)$ $\lambda = -1[0.5]1$ with Solid, Dashed, Dotted, Dotdash and Longdash Lines Respectively.

6. Order statistics of transmuted two parameters Rayleigh distribution

The distribution of smallest, largest, rth and joint order statistics are most important functions. Order statistics can be defined mathematically as, let $X_1, X_2, \ldots \ldots, X_n$ are random variables and can be ordered as $X_{(1)} \leq X_{(2)} \leq \cdots \ldots \leq X_{(n)}$ then the values $X_{(1)}, X_{(2)}, \ldots \ldots, X_{(n)}$ are ordered statistics of random variables.

The smallest order statistics of transmuted two parameters Rayleigh distribution is obtained as

$$f_{(1)}\big(x_{(1)}\big) = n\big[1 - F(x_{(1)})\big]^{n-1} f(x_{(1)})$$

$$f_{(1)}\big(x_{(1)}\big) = 2n\beta\big(x_{(1)} - \alpha\big)e^{-\beta(x_{(1)} - \alpha)^2}\left[1 - \lambda + 2\lambda e^{-\beta(x_{(1)} - \alpha)^2}\right]$$
$$\times \sum_{i=0}^{n-1}\sum_{j=0}^{i}\sum_{k=0}^{i}\binom{n-1}{i}\binom{i}{j}\binom{i}{k}(-1)^{i+j}(\lambda)^k\left[\left(e^{-\beta(x_{(1)} - \alpha)^2}\right)\right]^{j+k}$$

Let suppose that maximum values follow the transmuted two parameters Rayleigh distribution, and then the distribution of nth or maximum order statistics can be defined as

$$f_{(n)}\big(x_{(n)}\big) = n\big[F(x_{(n)})\big]^{n-1} f(x_{(n)})$$

$$f_{(n)}\big(x_{(n)}\big) = 2n\beta\left[\big(x_{(n)} - \alpha\big)e^{-\beta(x_{(n)} - \alpha)^2}\left[1 - \lambda + 2\lambda e^{-\beta(x_{(n)} - \alpha)^2}\right]\right]$$
$$\times \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\binom{n-1}{i}\binom{n-1}{j}(-1)^i(\lambda)^j\left[e^{-(i+j)\beta(x_{(n)} - \alpha)^2}\right]$$

Generally, the rth order statistics of transmuted two parameters Rayleigh distribution has specified pdf is given as

$$f_{(r)}\big(x_{(r)}\big) = \frac{n!}{(r-1)!\,(n-r)!}\big[F(x_{(r)})\big]^{r-1}\big[1 - F(x_{(r)})\big]^{n-r} f(x_{(r)})$$

$$f_{(r)}\big(x_{(r)}\big)$$
$$= \frac{2\beta n!}{(r-1)!\,(n-r)!}\big(x_{(r)} - \alpha\big)e^{-\beta(x_{(r)} - \alpha)^2}$$
$$\times \left[1 - \lambda + 2\lambda e^{-\beta(x_{(r)} - \alpha)^2}\right]$$
$$\times \sum_{i=0}^{r-1}\sum_{j=0}^{r-1}\sum_{k=0}^{n-r}\sum_{l=0}^{k}\sum_{m=0}^{k}\binom{r-1}{i}\binom{r-1}{j}\binom{n-r}{k}\binom{k}{l}\binom{k}{m}$$
$$\times (-1)^{i+k+l}(\lambda)^{j+l}\,e^{-\beta(i+j+l+m)(x_{(r)} - \alpha)^2}$$

The joint pdf of the transmuted two parameters Rayleigh distribution of $X_{(r)}$ and $X_{(s)}$, when $1 \leq r \leq s \leq n$ is defined as

$$f\big(x_{(r)}, x_{(s)}\big) = \frac{n!}{(r-1)!\,(s-r-1)!(n-s)!}[F(x_r)]^{r-1}f\big(x_{(r)}\big)$$
$$\times \big[F\big(x_{(s)}\big) - F\big(x_{(r)}\big)\big]^{s-r-1}f\big(x_{(s)}\big)\big[1 - F\big(x_{(s)}\big)\big]^{n-s}$$

$$= \frac{4\beta^2 n!}{(r-1)!\,(s-r-1)!\,(n-s)!}\big(x_{(r)} - \alpha\big)\big(x_{(s)} - \alpha\big)e^{-\beta(x_r - \alpha)^2}$$
$$\times e^{-\beta(x_{(s)} - \alpha)^2}\left[1 - \lambda + 2\lambda e^{-\beta(x_{(r)} - \alpha)^2}\right]\left[1 - \lambda + 2\lambda e^{-\beta(x_{(s)} - \alpha)^2}\right]$$
$$\times \sum_{i=0}^{r-1}\sum_{j=0}^{r-1}\sum_{k=0}^{s-r-1}\sum_{l=0}^{n-s}\sum_{m=0}^{k}\sum_{n=0}^{k}\sum_{p=0}^{s+l-r-k-1}\sum_{q=0}^{s+l-r-k-1}\binom{r-1}{i}\binom{r-1}{j}$$
$$\times \binom{s-r-1}{k}\binom{n-s}{l}\binom{k}{m}\binom{k}{n}\binom{s+l-r-k-1}{p}$$
$$\times \binom{s+l-r-k-1}{q}(-1)^{i+k+l+m+p}(\lambda)^{j+n+q}$$
$$\times e^{-\beta(i+j+m+n)(x_{(r)} - \alpha)^2}e^{-\beta(p+q)(x_{(s)} - \alpha)^2}$$

7. Parameter estimation

In this section, parameters estimation of transmuted two parameters Rayleigh distribution is discussed. The parameters estimate integrated through the least square estimation method. Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample of transmuted two parameters Rayleigh distribution with its cdf $F_{TR}(x)$ and suppose X_i denote the ordered sample where $i=1,\ 2,\ 3,\ldots n$. For n sample size, we have rank as

$$E\big(F(x_i)\big) = \frac{i}{n+1}$$

The least square estimation (LSE) can be obtained by minimizing the expression

$$Q(x; \alpha, \beta, \lambda) = \sum_{i=0}^{n} \left(F(x_i) - \frac{i}{n+1} \right)^2 \tag{7.1}$$

Now putting (2.5) in (7.1) we can write

$$Q(x; \alpha, \beta, \lambda) = \sum_{i=0}^{n} \left[\begin{array}{c} \left(1 - e^{-\beta(x-\alpha)^2}\right)\left(1 + \lambda e^{-\beta(x-\alpha)^2}\right) \\ - \dfrac{i}{n+1} \end{array} \right]^2 \tag{7.2}$$

To obtain the parameters estimates the (7.2) differentiated with respect to α, β and λ. Finally the following expression are obtained

$$\sum_{i=0}^{n} \left[\left(1 - e^{-\beta(t-\alpha)^2}\right)\left(1 + \lambda e^{-\beta(t-\alpha)^2}\right) - \frac{i}{n+1} \right]$$
$$\times \left[2\beta(x-\alpha)e^{-\beta(t-\alpha)^2}\left[1 - \lambda\left(1 + \lambda e^{-\beta(t-\alpha)^2}\right)\right] \right] = 0$$

$$\sum_{i=0}^{n} \left[\left(1 - e^{-\beta(t-\alpha)^2}\right)\left(1 + \lambda e^{-\beta(t-\alpha)^2}\right) - \frac{i}{n+1} \right]$$
$$\times (x-\alpha)^2 e^{-\beta(t-\alpha)^2}\left[1 - \lambda\left(1 - 2e^{-\beta(t-\alpha)^2}\right)\right] = 0$$

$$\sum_{i=0}^{n} \left[1 + \lambda e^{-\beta(t-\alpha)^2} - e^{-\beta(t-\alpha)^2} - \lambda e^{-2\beta(t-\alpha)^2} - \frac{i}{n+1} \right]$$
$$\times e^{-\beta(t-\alpha)^2}\left(1 - e^{-2\beta(t-\alpha)^2}\right) = 0$$

The perfect solution of the estimator for unknown parameters is not possible and then the estimates of parameters are derived by solving above three nonlinear equations simultaneously. This can be done by Newton-Raphson approach to numerically maximize the log likelihood function.

8. Applications

In this section, an empirical study is performed using real data set to compare the fitting of the transmuted two parameters Rayleigh distribution with its parent distributions. This comparison is evaluated on the basis of Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Consistent Akaike Information Criterion used to check the fitting of the distributions.

The data set consists of the precipitation level (inches) which was recorded in Minneapolis-St. Paul area during the month of March over 30 years period. The length of this data set is based on 30 observations and given in Tierney [9].

Table 1: The Summary Statistics of Precipitation Level (Inches) During the Month of March.

Length	30
Average	1.6750
Minimum	0.3200
Maximum	4.7500
Q_1	0.9150
Median	1.4700
Q_3	2.8075
Standard Deviation	1.0006

To compare the transmuted two parameters Rayleigh distribution with base distribution we use criterions such as AIC, BIC and AICC. The better model or distributions have smaller AIC, AICC, and BIC.

$$AIC = 2k - 2L$$
$$AICC = AIC + \frac{2k(k+1)}{(n-k-1)}$$
$$BIC = 2L + k\log(n)$$

Where k is the numbers of parameters in any distribution and n is the sample size.

The results in Table 2 provide the least square estimation of transmuted two parameters Rayleigh, two parameters Rayleigh and one parameter Rayleigh distribution. This table also indicates the different information criterion values. In these results transmuted two parameters Rayleigh distribution has smallest AIC, AICC, and BIC as compared to one and two parameters Rayleigh distributions. This is employed that transmuted Rayleigh distribution is best fitted to precipitation data. Fig. 4 provides the graphical presentation of the empirical cdf compare with the transmuted two parameters Rayleigh distribution and its parent distribution. This figure also shows that transmuted two parameters Rayleigh distribution is better than its parent distribution.

Table 2: Least Square Estimation of Parameters and Goodness of Fit of the Transmuted Two Parameters Rayleigh Distribution and Related Distributions.

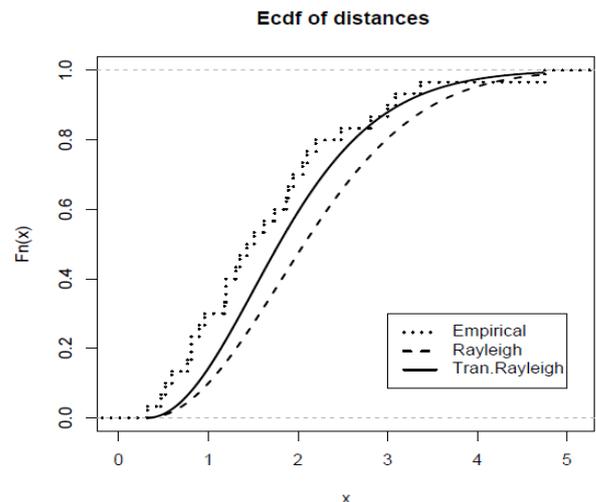| Model	Estimates	AIC	AICC	BIC
Transmuted two parameters Rayleigh	$\hat{\beta} = 0.2270$ $\hat{\alpha} = 0.32$ $\hat{\lambda} = 0.4803$	13.8225	14.2670	16.6249
Two parameters Rayleigh (λ=0)	$\hat{\hat{\beta}} = 9.6951$ $\hat{\alpha} = 0.32$	19.0685	19.2113	20.4697
One parameter Rayleigh (μ=0, λ=0)	$\hat{\beta} = 11.394$	20.9856	21.1284	22.3868

Fig. 4: Empirical Fitted Transmuted Two Parameters Rayleigh and Two Parameters Rayleigh cdf of the Precipitation Data Set.

9. Conclusion

The transmuted two parameters Rayleigh distribution introduced in this study. This generalized form of distribution has larger flexibility than base distribution. Plots of the density, distribution and reliability function have been sketched those shows the flexibility of proposed distribution. Some basic properties have been derived and the minimum, maximum, rth order and joint order statistics are also studied. Least square estimation method is employed for the estimation of the parameters of the proposed distribution. To check the significance of this generalized distribution three goodness of fit criterion are used to select the most appropriate model. The comparison is done between three distributions such as one parameter Rayleigh, two parameters Rayleigh and transmuted two parameters Rayleigh distribution. Application of generalized distribution on a real data set is done that shows comparatively transmuted two parameters Rayleigh distribution is more appropriate than other types of the Rayleigh distributions. In conclusion we can say this new proposed distribution will help better in survival data analysis, precipitation and wind data.

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