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Research paper

Geometric interpretation of subjective probability: random numbers and objective conditions of coherence

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Abstract

In the domain of the logic of certainty we study the objective notions of the subjective probability with the clear aim of identifying their fundamental characteristics before the assignment, by the individual, of the probabilistic evaluation: probability is an additional and subjective notion that one applies within the range of possibility, thus giving rise to those gradations, more or less probable, that are meaningless in the logic of certainty. When we study the criteria for evaluations under conditions of uncertainty and their corresponding conditions of coherence we show an inevitable dichotomy between the subjective or psychological aspect of probability and the objective or logical or geometrical one. The affine properties are the basis of essential concepts of probability theory and only they make sense, being independent of the choice of a coordinate system; however, the importance of the metric properties appears in order to represent random numbers and analytical conditions of coherence.

Keywords: *Convex Hull; Linear Dependence; Logical Dependence; Metric; Scalar Product.*

1. Introduction

Putting the logical values true and false equal to the numbers 1 and 0, an event E is always a random number which can admit these two values called indicators of E. They are idempotent numbers because we have $1^n = 1$ and $0^n = 0$. Arithmetic and Boolean operations must be unified by applying arithmetic operations even to events and Boolean operations even to random numbers. For events, the arithmetic product is the same as the logical product $\wedge$, the arithmetic sum is the number of successes given by $Y = E_1 + \dots + E_n$ and complementation is negation, that is to say, $\overline{E} = 1 - E$. The logical sum $\vee$ can be expressed by an arithmetic formula like $A \vee B = 1 - (1 - A)\,(1 - B)$, where we must consider $A \vee B = \overline{(\overline{A} \wedge \overline{B})}$, with A and B which are random events. In the field of real numbers $\mathbb{R}$ we can make the definitions: $a \vee b = \max(a, b)$, $a \wedge b = \min(a, b)$, $\bar{a} = 1 - a$, where a and b are real numbers; then, in case a and b have as values 1 or 0, the logical product, the logical sum and the negation are recovered. Moreover, the notation for the probability of an event E and for the mathematical expectation or prevision of a random number X is unified; in fact, it is adopted P(E) for probability of E and P(X) for prevision of X, where P is linear, that is, additive and homogeneous. The affine properties are the basis of essential concepts of probability theory and only they make sense, being independent of the choice of a coordinate system; however, the importance of the metric properties appears in order to represent random numbers and analytical conditions of coherence (see [4], [5], [12]).

2. Logic of certainty

When a given individual, according to his state of information, defines a set more or less large of possible alternatives, of which one and only one is necessarily true, he finds himself into the domain of the logic of certainty. We denote by $\mathcal{S}$ the abstract space of alternatives and by $\mathcal{Q}$, subset of $\mathcal{S}$, the space of the only alternatives possible for a certain individual; in fact, it may be convenient to think of $\mathcal{Q}$ as embedded in a larger and more manageable space $\mathcal{S}$. However, his information as well as his knowledge could also allow him to eliminate a part of the alternatives that can be imagined, because he believes that they are impossible; vice versa, all the others will be possible. After all, a rather crude analysis can be made if all the possible alternatives are collected in order to obtain an unique and certain alternative. The possibility, unlike probability, has no gradations, thus the domain of the logic of certainty is objective; it is equally possible, for a given individual at a certain time, that the next FIFA world cup is won by a very weak national football team, that the next President of the Italian Republic is a woman, that the unemployment rate falls by three percentage points at the end of next year in Italy. Into the domain of the logic of certainty, only true and false exist as final and certain answers and certain and impossible and possible as options with regard to the temporary knowledge of any individual; in this domain we study the objective notions of the subjective probability with the clear aim of identifying their fundamental characteristics before the assignment, by the individual, of the probabilistic evaluation. Probability is an additional and subjective notion that one applies within the range of possibility, thus giving rise to those gradations, more or less probable, that are meaningless in the logic of certainty. The field of the logic of certainty is objective because the elements of $\mathcal{Q}$ do not depend on the individual's opinions but only on his degree of ignorance (see[10],[12]).

3. Events and random numbers

An event E is a statement which we do not know yet to be true or false; the event which is certain and the one which is impossible can be taken as a limit case. The statements of which we can say if they are true or false on the basis of an ascertainment well determined and always possible, at least conceptually, have objective meaning. Such objective statements are said propositions if one is thinking more in terms of the expressions in which they are formulated or, equally, events if one is thinking more in terms of the situations and circumstances to which their being true or false corresponds (see [5]). For any individual who does not know with certainty the value of a number X, which is random in a non-redundant usage for him, there are two or more than two, a finite or infinite number, possible values for X, where the set of these values is I(X): in any case, only one is the true value of each random number (see [12]).

Remark 1 Events are also questions whose wordings, unambiguous and exhaustive, have the aim of removing any opportunity to complain in case that a bet is based upon them: they admit two alternative answers, yes = 1 or no = 0, true = 1 or false = 0. Also the random numbers can be identified by questions whose wordings are indisputably clear and complete; unlike events, they contain two or more than two answers which consist only of numbers, only one of which is the one that actually occurs.

Remark 2 For the representation of random numbers it is useful to think of a set $\mathcal{S}$, whose subset $\mathcal{Q}$ is constituted by the only possible alternatives for a certain individual at a given time. Sometimes, $\mathcal{S}$ can coincide with a manifold less extensive of the linear space $\mathcal{A}$ in which $\mathcal{S}$ is contained: in case of two random numbers, $\mathcal{S}$ can coincide with a curve of the Cartesian plane $\mathcal{A}$, otherwise, if the numbers are three, $\mathcal{S}$ can coincide with a surface of the three-dimensional space $\mathcal{A}$. Then, the possible points of $\mathcal{Q}$ would be positioned on the curve of the Cartesian plane or on the surface of the three-dimensional space and such points may be all the points or a part or a few points of $\mathcal{S}$ according to the individual's knowledge at a given time and the existence of other restrictions and conditions. It can be $\mathcal{A} = \mathbb{R}^2$ or $\mathcal{A} = \mathbb{R}^3$ under one-to-one correspondence between the points of the two-dimensional or three-dimensional space and the ordered lists of two or three real numbers. If $\mathbb{R}^2$ and $\mathbb{R}^3$ are equipped with a scalar product positive-definite, they would be Euclidean spaces or metric spaces. However, since every vector space may be considered as an affine space over itself, $\mathcal{A}$ could also be an affine space and this, theoretically, would be the best thing by virtue of the fact that the affine properties are more general than the metric ones (see [12]).

Remark 3 In $\mathcal{A} = \mathbb{R}^3$, with orthogonal and of unit length Cartesian coordinate system x_1, x_2, x_3, when we begin by considering three possible events E_i, i = 1, 2, 3, each of which is a particular random number, $\mathcal{S}$ turns out to be a cube with edges of unit length and vertices, which are elements of the set $\mathcal{Q}$ of the possible points, represented by eight ordered triplets (x_1, x_2, x_3) corresponding to the $8 = 2^3$ constituents, hypothetically, all possible; for each of these triplets is $x_i = 0$ or $x_i = 1$, with i = 1, 2, 3. The system $\mathcal{L}$ of linear combinations of fundamental events E_i consists of random numbers $X = u_1E_1 + u_2E_2 + u_3E_3$, with u_1, u_2, u_3 real coefficients, and it is a vector space; in particular, $\mathcal{L}$ is the dual vector space of $\mathcal{A}$, thus it is $\dim(\boldsymbol{\mathcal{L}}) = \dim(\boldsymbol{\mathcal{A}}) = 3$. The expression given by $u_1x_1 + u_2x_2 + u_3x_3$ shows that each random number of $\mathcal{L}$ is a scalar product of two vectors belonging to the two spaces $\mathcal{A}$ and $\mathcal{L}$ superposed. The components of the first vector, element of $\mathcal{A}$, are represented by the ordered triplet (x_1, x_2, x_3), the ones of the second vector, element of $\mathcal{L}$, are given by the ordered triplet (u_1, u_2, u_3). Therefore, in $\mathbb{R}^3$, by means of Cartesian coordinate system x_i which is superposed onto its dual, it is introduced a metric. Evidently, by virtue of the fact that $\mathcal{A}$ and $\mathcal{L}$ are two spaces for which the zero vector, thought of as point (0, 0, 0), has a meaning, it is possible to identify the components of each vector with the coordinates of the point corresponding to it. The possible values of each X can be, at most, equal in number to the one of the possible constituents and they are distributed over the planes $u_1x_1 + u_2x_2 + u_3x_3$ = constant, with x_i Cartesian coordinates in $\mathcal{A}$ and u_i Plücker coordinates in $\mathcal{L}$. If we consider $u_1 = u_2 = u_3 = 1$, it results $Y = E_1 + E_2 + E_3$, with Y number of successes; when we assume that all the vertices of the cube are possible, the possible values for the random number Y would be over 3 + 1 = 4 planes, for each of which it turns out to be $x_1 + x_2 + x_3$ = constant = 0, 1, 2, 3 according to the binomial coefficients $\binom{3}{0} = 1$, $\binom{3}{1} = 3$, $\binom{3}{2} = 3$, $\binom{3}{3} = 1$: in effect, there is an only way of obtaining zero successes in three events and three successes in three events, while there are three ways of obtaining, respectively, one success in three events and two successes in three events. Clearly, all the possible values for Y are not distinct. In general, if we consider $\dim(\mathcal{A}) = \dim(\mathcal{L}) = n$ (with n integer > 3), the prism having as vertices all the 2^n possible constituents would be a hypercube (see [12]).

By means of any finite number of events $E_1, \ldots, E_n$ we can obtain a partition, that is to say, a family of incompatible and exhaustive events for which it is certain that one and only one event actually occurs: for this purpose, we must consider the 2^n logical products $E_1' \wedge \ldots \wedge E_n'$, where each E_i' is either E_i or its complement $(1 - E_i)$. Some of the 2^n products may turn out to be impossible and do not have to be considered: those which remain are possible and are called the constituents $C_1, \ldots, C_s$ of the partition, with $s \leq 2^n$. They are also called the elementary cases or atoms; so, we will consider $C_1 \vee \ldots \vee C_s = C_1 + \ldots + C_s = 1$.

Remark 4 The random numbers X, in the linear space $\mathcal{L}$, may be obtained not only by linear combinations of determined events E_i, but also by linear combinations of fixed random numbers X_i. In general, all linear combinations of events or random numbers in $\mathcal{L}$ can be homogeneous or complete: if they are homogeneous we will have $X = \sum_i u_iX_i$ (with i = 1, …, n), while if they are complete we will have $X = \sum_i u_iX_i$ (i = 0, 1, …, n), with the number X_0 that is not random, since it is known that its true value is always just equal to 1, $X_0 \equiv 1$, so it turns out to be $u_0X_0 = u_0$; clearly, the essence of the linear combination does not change because x_0 is a variable only apparent, being continually $x_0 = 1$. However, in both cases, it results that X is linearly dependent on the X_i; furthermore, it is also logically dependent on the X_i, that is, $X = f(X_1, \ldots, X_n)$ or $X = f(X_0, \ldots, X_n)$, with the meaning of logical dependence which coincides with the concept of mathematical function. Cartesian coordinates of each point of $\mathcal{A}$, possible or not, are $(x_1, \ldots, x_n)$ or $(x_0, x_1, \ldots, x_n)$, where we have $x_0 = 1$; anyway, the true random numbers are n in both formulations. In the sum $u_1x_1 + \ldots + u_nx_n$ as well as in the sum $u_0 + u_1x_1 + \ldots + u_nx_n$ we always consider a scalar product of two vectors of two vector spaces superposed, $\mathcal{A}$ and $\mathcal{L}$, for which it follows $\dim(\mathcal{A}) = \dim(\mathcal{L}) = n$ (see [12]).

4. Logic of uncertainty

The subjectivistic conception of probability, through psychological analysis, vivifies notions that are mathematically correct but that is not sufficient to consider from the formal point of view, because the instrument really propulsive of scientific thinking is not classical logic or, in the specific instance, logic of certainty that, as such, involves no affective demonstration, no judgment by anyone, but is probability and probability calculus. Therefore, when we consider any problem concerning the assignment of probability among possible cases and how to define it and to express it quantitatively, we find ourselves into the field, personal and subjective, of logic of uncertainty, distinct and separate from that one of logic of certainty (see [3]). In fact, when we say that we are not satisfied of logic of certainty, we mean that we are not satisfied of agnostic and undifferentiated attitude towards uncertainty; for all those things which, not being known to us with certainty, are uncertain or possible, any individual feels a more or less strong propensity to expect that some cases possible are true rather than others, to believe that the answer to a given question is no rather than yes, to estimate that the unknown value of a certain quantity is small rather than large. Evidently, these attitudes express, in the domain of uncertainty, different degrees of subjective probability, each of which is assigned to one of the possible alternatives, identified by a given individual on the basis of his knowledge. So, finding oneself into the field of logic of prevision means to examine carefully desires or hopes that certain alternatives occur, anxieties and fears regarding the occurrence of unfavourable alternatives and to weigh up the pros and cons of each choice trying to reason about it in order to distribute, among all the possible alternatives and in the way which will appear most appropriate, one's own sensations of probability (see [6], [7], [8], [11]).

Remark 5 When a particular individual chooses to be guided only by the logic of certainty, after having distinguished a set more or less large of possible alternatives in the way which seems to him most effective, he has to stop because the question is closed: remaining within the logic of certainty, the only thing that he could make is a prophecy, that is to say, among the cases that he believes possible, he might venture to guess the alternative that, according to him, will occur, transforming in this way, but unreasonably, the uncertainty in illusory certainty (see [10], [12]).

Remark 6 The space of n random numbers coincides with the n-dimensional vector space $\mathcal{A}$ after the introduction of a coordinate system $x_1, \ldots, x_n$ in $\mathcal{A}$: by virtue of the fact that each event is a random number, a set of n possible events $E_1, \ldots, E_n$ is embedded in $\mathcal{A}$. From such a set, $s \leq 2^n$ constituents are originated: they are identified by particular ordered lists of n numbers expressed by $(x_1, \ldots, x_n)$, with $x_i = 0$ or $x_i = 1$, $i = 1, \ldots, n$, each of which is a possible point of $\mathcal{Q}$ contained in the vector space $\mathcal{A}$. Such considerations make clear, from the point of view of the logic of certainty, why the probability of an event is automatically incorporated in the prevision of a random number. In fact, going beyond the domain of the logic of certainty, we enter into the field of the logic of uncertainty and in case that X is a random number, P(X) is its prevision: if $I(X) = \{x_1, \ldots, x_n\}$, when we assign to each value x_i of X the probability p_i $(i = 1, \ldots, n)$, with $0 \leq p_i \leq 1$ and $\sum p_i = 1$, it turns out to be $P(X) = x_1 p_1 + \ldots + x_n p_n$. The prevision of X coincides with the probability of an event E when and only when X, admitting only two possible values, 1 and 0, is an event, thus prevision and probability are two different words that express the same concept extra-logical, subjective and personal (see [9]).

Remark 7 The die symmetry and the knowledge of an observed frequency are elements which any individual carefully examines to express his opinion from which is originated the only probability that, according to the subjectivistic conception, exists in any case: the subjective probability. It must be understood as the degree of belief of a certain individual in the occurrence of a specific event; anyway, probability of an event E is not an intrinsic characteristic of E because it depends on the information that the individual making the probabilistic evaluation has, so it is always subordinate to his present state of knowledge which can change for the possible attainment of new essential information and for the passage of time (see [8]).

Remark 8 A probabilistic evaluation, known over a set of whatever events, always expresses the opinion of a given individual, real or hypothetical; the only restriction admissible is that this opinion is coherent, consequently, if it is not coherent, it should be corrected by the individual in order to make it coherent (see [1], [2]).

5. Criteria for the probabilistic evaluations

The criteria which may be used to reveal concretely P(X) or, in particular, P(E) according to the opinion of a given individual are two and entirely equivalent: they are based upon the identification of the practical consequences that a certain individual knows to accept and accepts when he expresses his evaluation of P(X) or P(E) and, if applied coherently, lead to the same $P(X) = \bar{x}$ in case that X is estimated or to the same P(E) = p in case that E is evaluated. If X is evaluated, both criteria consider the random magnitude $X - \bar{x}$, expressed by the difference between the real value X and the one chosen by a certain individual at his own will, $P(X) = \bar{x}$. The first criterion provides that, after the subjective choice of $\bar{x}$, the individual is obliged to accept any bet unilaterally determined by an opponent, whose gain is $c(X - \bar{x})$, with c any betting amount, positive or negative, determined equally by the opponent; in particular, if c = 1, the gain of the bet is $(X - \bar{x})$, while if we have $c = -1$, it is $(\bar{x} - X)$. On the contrary, the second criterion provides that, after choosing $\bar{x}$, the individual must suffer the penalty given by $(X - \bar{x})^2$, positively proportional to the square of the difference between X and $\bar{x}$ (see [12]). In particular, if an event E is evaluated, both criteria consider the magnitude $E - p$ given by the difference between the real value E, 1 or 0 according to whether E occurs or does not occur, and the one chosen by a certain individual according to his subjective opinion, P(E) = p. The first criterion provides that, after the choice of p by a given individual, he is obliged to accept any bet determined unilaterally by an opponent, whose gain is $c(E - p)$, where c is any betting amount, positive or negative, established by the opponent; in particular, if c = 1, the gain is $(E - p)$, while if $c = -1$, it is $(p - E)$. On the contrary, the second criterion provides that, after the choice of p, the individual must suffer the penalty $(E - p)^2$.

6. Necessary and sufficient conditions of coherence

The choice of P(X) or P(E), even if it is subjective, should not be contradictory and takes place within the set of coherent previsions of X or in that one of coherent probabilities of E; both the sets contain values objectively admissible which are independent of the personal views of any individual and also of the judgments about others' opinions. The necessary and sufficient conditions for coherence are two and completely equivalent, one for each evaluation criterion (see [12]).

Regarding the first definition of coherence, it is assumed that the individual who subjectively evaluates $P(X_i)$ or $P(E_i)$, with $i = 1, \ldots, n$, does not want to make bets on X_i or E_i that give him an inevitable loss, therefore a set of his previsions or probabilities is not intrinsically contradictory when and only when, among the linear combinations of bets that he is obliged to accept, there are not combinations with gains all uniformly negative. Analytically, this means that for the values of the random magnitude $Y = c_1(X_1 - \bar{x}_1) + \ldots + c_n(X_n - \bar{x}_n)$ or $Y = c_1(E_1 - p_1) + \ldots + c_n(E_n - p_n)$ must not be, objectively, that sup I(Y) is negative; conversely, inf I(Y) cannot be positive. Even if the bets are an infinite number, Y is always linear combination of a finite number of them.

Regarding the second definition of coherence, it is assumed that the individual who subjectively evaluates $P(X_i)$ or $P(E_i)$, with $i = 1, \ldots, n$, does not prefer a given penalty if he can choose another penalty certainly smaller, therefore a set of his previsions or probabilities is coherent when and only when he could not choose them in order to make his penalty certainly and uniformly smaller. Analytically, this means that there are not any evaluations $P^*(X_i)$ or $P^*(E_i)$ that replaced with the evaluations $P(X_i)$ or $P(E_i)$ chosen subjectively are such that for all the possible points, which are $(X_1, \ldots, X_n)$ or $(E_1, \ldots, E_n)$, the penalty expressed by $L^* = \sum_i (X_i - P^*(X_i))^2 \cdot (1/k_i)^2$ is uniformly smaller than the penalty $L = \sum_i (X_i - P(X_i))^2 \cdot (1/k_i)^2$ or the penalty $L^* = \sum_i (E_i - P^*(E_i))^2 \cdot (1/k_i)^2$ is uniformly smaller than $L = \sum_i (E_i - P(E_i))^2 \cdot (1/k_i)^2$, where we have $k_1, \ldots, k_n$ which are arbitrarily predetermined and homogeneous towards X_i or E_i.

A prevision P is coherent if its use cannot lead to an inadmissible decision such that a different possible decision would have certainly led to better results, whatever happened. If the sets of possible values for X and for Y turn out to be $I(X) = \{x_1, \ldots, x_n\}$ and $I(Y) = \{y_1, \ldots, y_n\}$, when we assign the same weights p_i $(i = 1, \ldots, n)$, with $0 \leq p_i \leq 1$ and with $\sum p_i = 1$, to each x_i and y_i we will have $P(X + Y) = P(X) + P(Y)$, that is to say, P is additive. A prevision P of the random number X must satisfy the inequality $\inf I(X) \leq P(X) \leq \sup I(X)$, that is, P(X) must not be less than the lower bound of the set of possible values for X, inf I(X), nor greater than the upper bound, sup I(X). A prevision P of X must also be linear, that is to say, we will have P(aX) = aP(X), for every real number a. More generally, it is P(aX + bY + cZ + …) = aP(X) + bP(Y) + cP(Z) + …, with a, b, c, … whatever real numbers, for any finite number of summands. So, coherence reduces to linearity, which contains additivity property, and convexity (see [12]). Similarly, if E is an event, when we have $0 \leq P(E) \leq 1$, its evaluation is coherent; if $E_1, …, E_n$ are mutually exclusive events, their evaluations are coherent when we have $P(E_1 + … + E_n) = P(E_1) + … + P(E_n)$.

Remark 9 The first condition of coherence involves that a point P of $\mathcal{A}$, whose coordinates are $(P(X_1), …, P(X_n))$, is an admissible prevision if and only if no hyperplane separates it from the set $\mathcal{Q}$ of the possible points Q of $\mathcal{A}$: this characterizes the points of the convex hull, for which it is said that every linear equation between the numbers X_i, $c_1X_1 + … + c_nX_n = c$, must also apply to the previsions $P(X_i)$, $c_1P(X_1) + … + c_nP(X_n) = c$, as well as any inequation between them given by $c_1X_1 + … + c_nX_n \geq c$ must also be satisfied by the previsions $c_1P(X_1) + … + c_nP(X_n) \geq c$.

Example 1 Let $\mathcal{A}$ be the Cartesian plane defined by two perpendicular axes, whose points are in one-to-one correspondence with the set $\mathbb{R}^2$ of ordered lists of two real numbers having structure of two-dimensional vector space over the field $\mathbb{R}$. In $\mathcal{A}$ we consider the random number X_1, whose possible values are on the x-axis, and the random number X_2, whose possible values are on the y-axis: we will have, respectively, $I(X_1) = \{4, 6, 7, 11\}$ and $I(X_2) = \{3, 5, 6, 9, 10, 13\}$. The set $\mathcal{Q}$ of the possible points Q of $\mathcal{A}$, with Q having, in general, coordinates (x_1, x_2), consists of six ordered pairs of real numbers, (6, 3), (11, 5), (11, 9), (7, 13), (4, 10), (4, 6), each of which is a vertex of the closed polygonal chain which delimits the two-dimensional geometric shape which coincides with the set $\mathcal{P}$ of coherent previsions P having $(P(X_1), P(X_2))$ as coordinates. In $\mathcal{A}$ the straight line $1/4x_1 + 1/6x_2 - 2 = 0$ is a hyperplane which does not separate any coherent prevision P from the set $\mathcal{Q}$ of the possible points Q of $\mathcal{A}$: the points of $\mathcal{Q}$ with coordinates (6, 3) and (4, 6) satisfy the equation $1/4x_1 + 1/6x_2 - 2 = 0$, while for all the others is satisfied the inequation expressed by $1/4x_1 + 1/6x_2 - 2 \geq 0$. The point (9, 7) of $\mathcal{A}$ is a coherent prevision P, where we have $P(X_1) = 9$ and $P(X_2) = 7$, whose coordinates satisfy $1/4x_1 + 1/6x_2 - 2 \geq 0$; on the contrary, since the point (2, 2) of $\mathcal{A}$ is not an admissible prevision P, it is not element of the set $\mathcal{P}$ of coherent previsions P.

Remark 10 The vector space $\mathcal{A}$ is Euclidean when it is provided with a scalar product positive-definite: by virtue of the metric $\rho^2 = \sum_i (x_i/k_i)^2$, it results $L = (P - Q)^2$, that is to say, the penalty L coincides with the square of the distance between the prevision-point P and the outcome-point Q. Thus, regarding the second condition of coherence, the points of the convex hull also enjoy the property according to which P cannot be moved in such a way as to reduce its distance from all points Q.

Example 2 We consider, in the Euclidean space $\mathcal{A} = \mathbb{R}^2$, the random number X_1 and the random number X_2, whose possible values are, respectively, on the x-axis and on the y-axis; in particular, it is $I(X_1) = \{4, 6, 7, 11\}$ for the random number X_1 and $I(X_2) = \{3, 5, 6, 9, 10, 13\}$ for X_2. The set $\mathcal{Q}$ of the possible points Q of the space $\mathcal{A}$ consists of six ordered pairs of real numbers, (6, 3), (11, 5), (11, 9), (7, 13), (4, 10), (4, 6), each of which is a vertex of the closed polygonal chain which delimits the two-dimensional geometric shape which coincides with the set $\mathcal{P}$ of coherent previsions P having $(P(X_1), P(X_2))$ as coordinates. Evidently, these coordinates identify a point P of the set $\mathcal{P}$. Therefore, given the point P of $\mathcal{P}$ having coordinates (9, 7), where we have $P(X_1) = 9$ and $P(X_2) = 7$, its distance from each point Q of $\mathcal{Q}$ is every time calculated using the formula of the squared distance between two points, P and Q, of the two-dimensional space $\mathcal{A}$, thus, when we

7. Geometric interpretation of conditions of coherence

Given in $\mathcal{A}$ n random numbers $X_1, …, X_n$, with $\mathcal{A}$ n-dimensional vector space having coordinate system $x_1, …, x_n$, every prevision, coherent or not, of each random number X_i is always a point $(P(X_1), …, P(X_n))$ of $\mathcal{A}$. In this space, moreover, the coordinates of the points Q of the set $\mathcal{Q}$ of possible points are steadily identified by ordered lists $(x_1, …, x_n)$ of n real numbers, with x_1 that is a possible value of X_1, …, x_n that is a possible value of X_n. Thus, on the basis of the geometric interpretation of the necessary and sufficient conditions for coherence, the set $\mathcal{P}$ of coherent previsions P is the closed convex hull of the set $\mathcal{Q}$ of the possible points Q of $\mathcal{A}$

sum up the respective six numerical values, it turns out to be L = 141. Of course, if P is not element of $\mathcal{P}$, the penalty would be greater than the previous value of L.

The points which are admissible in terms of coherence could be obtained as barycentres of, at most, n + 1 points Q_j of $\mathcal{Q}$ in the n-dimensional space or they are adherent points of $\mathcal{Q}$, but not belonging to $\mathcal{Q}$.

Example 3 If $\mathcal{A}$ is the plane meant as the two-dimensional Euclidean space and if $\mathcal{Q}$ is the set of points of a circumference having, with respect to the full angle, rational angular distance from a point, chosen arbitrarily, of the circumference, then each point P of $\mathcal{P}$ admissible in terms of coherence is a barycentre, at most, of 3 = 2 + 1 points Q_j of $\mathcal{Q}$: three points Q_j are necessary to calculate the barycentre of triangles whose vertices are possible points of $\mathcal{Q}$ on the circumference, two points are necessary to calculate the barycentre of chords connecting rational points on the circumference, while only one point is necessary if it coincides with one of the possible points of $\mathcal{Q}$ on the circumference in which all the weights are concentrated. In other words, except that all the probabilities are concentrated at a unique point of $\mathcal{Q}$, each point inside the circle is inside triangles with vertices in $\mathcal{Q}$ or it is on chords connecting points of $\mathcal{Q}$. The points on the circumference for which the angular distance is not rational, although not belonging to the set $\mathcal{Q}$, are adherent points of $\mathcal{Q}$ and they are required in order to complete the closed convex hull: they are also admissible previsions in terms of coherence and there are points of $\mathcal{Q}$ in each of their neighbourhoods.

Remark 11 According to another interpretation, a prevision is a mixture of possibilities: every prevision-point P of $\mathcal{P}$ is admissible in terms of coherence when it is a barycentre of possible points Q_j of $\mathcal{Q}$, with non-negative weights, summing to 1; however, if all the weights are concentrated at a unique point Q_j, also the possible points turn out to be coherent previsions. If $\mathcal{P}_0$ is any set of coherent previsions, then its closed convex hull is also a set, $\mathcal{P}_1$, of coherent previsions: $\mathcal{P}_1$ consists of the mixtures of those in $\mathcal{P}_0$ (see [12]).

8. Conclusions

Each criterion for evaluations under conditions of uncertainty is a device or instrument for obtaining a measurement; it furnishes an operational definition of probability or prevision P and together with the corresponding conditions of coherence can be taken as a foundation for the entire theory of probability. When we study these criteria and their corresponding conditions of coherence we show an inevitable dichotomy between the subjective or psychological aspect of probability and the objective or logical or geometrical one. Although the affine properties have greater generality than the metric ones, the importance of the metric properties manifests itself even when it is necessary to interpret geometrically the necessary and sufficient conditions of coherence; analytically, the first definition of coherence is similar to the property of stable equilibrium of the barycentre, while the second property is similar to the property of minimum of the moment of inertia which characterizes the barycentre once again. Obviously, if the properties of the barycentre are not satisfied, the set of previsions of a given individual cannot be coherent. Given the probabilities of the possible values, finite in number, of X, its barycentre, which is P(X), can be expressed as a function of them; the prevision of X does not presuppose the introduction of the concept of continuous probability distribution that, extending to the general case the concept of mathematical expectation or mean value of X, requires the use of mathematical tools more advanced than necessary.

References

[1] B. de Finetti, Fondamenti logici del ragionamento probabilistico, Bollettino dell'Unione Matematica Italiana 9 (1930) 258-261.
[2] B. de Finetti, Problemi determinati e indeterminati nel calcolo delle probabilità, Rendiconti della R. Accademia Nazionale dei Lincei 12 (1930) 367-373.
[3] B. de Finetti, Probabilismo. Saggio critico sulla teoria delle probabilità e sul valore della scienza (Biblioteca di Filosofia, diretta da A. Aliotta), F. Perrella, Napoli-Città di Castello, 1931.
[4] B. de Finetti, Sul significato soggettivo della probabilità, Fundamenta Mathematicae 17 (1931) 298-329.
[5] B. de Finetti, La nozione di evento, Pubblicazioni delle Facoltà di Scienze e di Ingegneria dell'Università di Trieste Serie B 130 (1954) 170-174.
[6] B. de Finetti, La probabilità e le scienze sociali, L'industria 4 (1955) 467-495.
[7] B. de Finetti, F. Minisola, La matematica per le applicazioni economiche, Cremonese, Roma, 1961
[8] B. de Finetti, La decisione nell'incertezza, Scientia 88 (1963) 61-68.
[9] B. de Finetti, F. Emanuelli, Economia delle assicurazioni, Trattato Italiano di Economia, vol. XVI, UTET, Torino, 1967.
[10] B. de Finetti, L'adozione della concezione soggettivistica come condizione necessaria e sufficiente per dissipare secolari pseudoproblemi. In D. Fürst, G. Parenti, I fondamenti del calcolo delle probabilità. Atti della tavola rotonda tenuta a Poppi nei giorni 11-12 giugno 1966 57-94, Scuola di Statistica dell'Università, Firenze, 1967.
[11] B. de Finetti, Un matematico e l'economia, F. Angeli, Milano, 1969.
[12] B. de Finetti, Teoria delle probabilità, voll. I e II, Einaudi, Torino, 1970.