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Estimation of stress-strength parameter for two-parameter weibull distribution

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Abstract

In this paper, we consider the estimation of R= p(y<x) when x and y are two independent random variables from two-parameter weibull distribution with different scale parameters and the same shape parameter. Assuming that the common shape parameter is known, MLE, UMVUE and Bayes estimators of R are obtained. We also derive a confidence interval and shortest confidence interval for R based on MLE of R. Monte-Carlo simulation are performed to compare the different proposed methods.

Keywords: *Bayes estimator, Maximum likelihood estimator, shortest confidence interval, stress-strength model, uniformly minimum variance estimator*

1 Introduction

In this paper, we consider the problem of estimating the stress-strength parameter R=P(Y<X) when X and Y are two independent random variables from two-parameter Weibull distribution. R=P(Y<X) is arised when the random strength X exceeds the random stress Y and we are interested in calculating the probability of it. Because of that R=P(Y<X) is called the stress-strength parameter. Due to practical point of view of reliability stress-strength model many authors represented a lot of papers about the estimation of R=P(Y<X) for different distributions. Kundu and Gupta [4,9], Rezaei et al. [5], Panahi and Asadi [6], Krishnamoorthy et al. [12], Kundu and Raqab [13].
When the common shape parameter α is unknown, Kundu and Gupta [7] considered the estimation of R when $X{\sim}WE(\alpha,\lambda_1)$ and $Y{\sim}WE(\alpha,\lambda_2)$ are two independent Weibull distributions with different scale parameters. In this paper, we consider estimation of R when the common shape parameter α is known.
The layout of this paper is as follow: in section 2, we introduce the Weibull distribution. In section 3, we derive the estimation of R, in this section, the MLE , UMVUE and Bayes estimators of R are obtained. Simulation study for comparison between estimators are given in section 5.

2 Weibull distribution

Weibull is one of the most widely used distributions in reliability studies. It is often used as the lifetime distribution, because some failure models are described by their shape parameter. Therefore, the weibull distribution is important and has been studied extensively over the years.
A random variable X is said to have weibull distribution, if it´s probability density function (PDF) is given by

$$f(x;\alpha,\lambda)=\alpha\lambda x^{\alpha-1}e^{-\lambda x^{\alpha}}\ ,x>0 \tag{1}$$

The cumulative distribution function of weibull distribution (CDF) is defined by
$F(x)=1-e^{-\lambda x^{\alpha}}$,
where α is a shape parameter and λ is a scale parameter.

3 Estimation of R = P(Y<X)

3.1 MLE of R

Let X and Y are two independent Weibull random variables with parameters α ,and λ_1 , λ_2 respectively. Therefore

$$R = P(Y < X) = \int_0^{\infty}\int_0^{X} f(x)f(y)dydx = \frac{\lambda_2}{\lambda_1+\lambda_2} \tag{2}$$

Let $X_1, X_2, \dots, X_n$be a random sample from $WE(\alpha, \lambda_1)$, and $Y_1, Y_2, \dots, Y_m$be a random sample from $WE(\alpha, \lambda_2)$, then the log-likelihood function of observed data is

$$L(\alpha, \lambda_1, \lambda_2) = (m+n)ln\alpha - nln\lambda_1 - mln\lambda_2 + (\alpha - 1) \quad \times \left[\sum_{i=1}^{n} lnx_i + \sum_{j=1}^{m} lny_j\right] - \frac{1}{\lambda_1}\sum_{i=1}^{n} x_i^{\alpha} - \frac{1}{\lambda_2}\sum_{j=1}^{m} y_j^{\alpha}. \tag{3}$$

The MLE of λ_1, and λ_2, denoted by $\hat{\lambda}_1$ and $\hat{\lambda}_2$ respectively, can be obtained as follow

$$\frac{\partial L}{\partial \lambda_1} = -\frac{n}{\lambda_1} + \frac{1}{\lambda_1^2}\sum_{i=1}^{n} x_i^{\alpha} = 0 \tag{4}$$
$$\frac{\partial L}{\partial \lambda_2} = -\frac{m}{\lambda_2} + \frac{1}{\lambda_2^2}\sum_{j=1}^{m} y_j^{\alpha} = 0 \tag{5}$$

From (4) and (5), we obtain

$$\hat{\lambda}_1 = \frac{n}{\sum_{i=1}^{n} x_i^{\alpha}} \tag{6}$$
$$\hat{\lambda}_2 = \frac{m}{\sum_{j=1}^{m} y_j^{\alpha}} \tag{7}$$

Then,

$$\hat{R} = \frac{\frac{m}{\sum_{j=1}^{m} y_j^{\alpha}}}{\frac{n}{\sum_{i=1}^{n} x_i^{\alpha}} + \frac{m}{\sum_{j=1}^{m} y_j^{\alpha}}} \tag{8}$$

It is easy to see that

$2\lambda_1 \sum_{i=1}^{n} x_i^{\alpha} \sim \chi^2_{(2n)}$ and $2\lambda_2 \sum_{j=1}^{m} y_j^{\alpha} \sim \chi^2_{(2m)}$

So

$$\hat{R} \sim \frac{1}{1+\frac{\lambda_1}{\lambda_2}w} \tag{9}$$

Where w has F distribution with $2m$ and $2n$ degrees of freedom. So the PDF of $\hat{R}$ can be abtian as follow

$$f_{\hat{R}}(r) = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)r^2}\left(\frac{m\lambda_2}{n\lambda_1}\right)^{m} \times \frac{\left(\frac{1-r}{r}\right)^{m-1}}{\left(1+\frac{m\lambda_2}{n\lambda_1}\left(\frac{1-r}{r}\right)\right)^{n+m}}, \tag{10}$$

Where $0 < r < 1$ and $\lambda_1, \lambda_2 > 0$.
Then $100(1-\gamma)\%$ the confidence interval is

$$\left[\frac{1}{1+(1/\hat{R}_{ML}-1)F_{2n,2m,1-\gamma/2}}, \frac{1}{1+(1/\hat{R}_{ML}-1)F_{2n,2m,\gamma/2}}\right] \tag{11}$$

We can also obtain the shortest confidence interval for R by using

$$Q = \frac{1-R}{R} \times \frac{\hat{\lambda}_2}{\hat{\lambda}_1} \tag{12}$$

As a pivotal quantity which it´s limiting distribution is F with 2m and 2n degrees of freedom. This confidence interval is as follow

$$[\frac{1}{1+(1/\hat{R}_{ML}-1)b}, \frac{1}{1+(1/\hat{R}_{ML}-1)a}] \tag{13}$$

We must choose a and b such that two below equations (14 and 15) are hold simultaneously

$$\int_a^b h_Q(t)dt = 1 - \alpha \tag{14}$$

$$\left(\left(\frac{1}{\hat{R}} - 1\right) a + 1\right)^2 h_Q(a) = \left(\left(\frac{1}{\hat{R}} - 1\right) b + 1\right)^2 h_Q(b) \tag{15}$$

where $h_Q(t)$ is a probability distribution function of F distribution with 2m and 2n degrees of freedom.

3.2 UMVUE of R

In this subsection we obtain the UMVUE of R using the Rao-Blackwell theorem and applying the result of Tong [1,2]. We know $(T_1, T_2) = (\sum_{i=1}^n x_i^\alpha, \sum_{j=1}^m y_j^\alpha)$ is a jointly sufficient statistic for (λ_1, λ_2). Let

$$\psi(V_1, W_1) = \begin{cases} 1 & V_1 > W_1 \\ 0 & V_1 < W_1 \end{cases} \tag{16}$$

Where $V_1 = x_1^\alpha$ and $W_1 = y_1^\alpha$. It is easily seen that $\psi(V_1, W_1)$ is an unbiased estimator of R. So the UMVUE of R, say $\hat{R}_{umvue}$ can be abtained as

$$\hat{R}_{umvue} = E[Z(V_1, W_1)|(T_1, T_2)] = \iint_{\mathcal{B}} f_{V_1|T_1=t_1}(v_1) f_{W_1|T_2=t_2}(w_1) dv_1 dw_1, \tag{17}$$

Where $\mathcal{B} = \{(v_1, w_1): v_1 > w_1, t_1 > v_1, t_2 > w_1\}$,

$$f_{V_1|T_1=t_1}(v_1) = (\mathrm{n}-1)\frac{(t_1 - v_1)^{n-2}}{t_1^{n-1}}, t_1 > v_1,$$

$$f_{W_1|T_2=t_2}(w_1) = (\mathrm{m}-1)\frac{(t_2 - w_1)^{m-2}}{t_2^{m-1}}, t_2 > w_1,$$

Therefore

$$\tilde{R}_{umvue} = \begin{cases} 1 - \sum_{i=0}^{m-1} (-1)^i \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+i)\Gamma(m-i)} \left(\frac{t_1}{t_2}\right)^i, & t_1 < t_2 \\ \sum_{j=1}^{n-1} (-1)^j \frac{\Gamma(n)\Gamma(m)}{\Gamma(n-j)\Gamma(m+j)} \left(\frac{t_2}{t_1}\right)^j, & t_1 > t_2 \end{cases} \tag{18}$$

3.3 Bayes estimator of R

In this subsection, we derive the Bayes estimator of R under SEL function . For this purpose, we assume independent Gamma priors on λ_1 and λ_2 , that is λ_1 and λ_2 follow $Gamma(a_1, b_1)$ and $Gamma(a_2, b_2)$ respectively. Then the posterior PDF´S of λ_1 and λ_2 is

$$\lambda_1|\alpha, data \sim Gamma(a_1 + n, b_1 + T_1) \tag{19}$$

$$\lambda_2|\alpha, data \sim Gamma(a_2 + m, b_2 + T_2) \tag{20}$$

Where $T_1 = \sum_{i=1}^n x_i^\alpha$ and $T_2 = \sum_{j=1}^m y_j^\alpha$. Note that, here α is known.
Since $\boldsymbol{\lambda_1}$ and $\boldsymbol{\lambda_2}$ are independent, using (19) and (20), the posterior PDF of R is

$$\pi(r|\boldsymbol{X}, \boldsymbol{Y}) = c(b_1 + T_1)^{n+a_1}(b_2 + T_2)^{m+a_2} \times \frac{r^{m+a_2-1}(1-r)^{n+a_1-1}}{[(1-r)(b_1+T_1) + r(b_2+T_2)]^{n+m+a_1+a_2}}, \quad 0 < r < 1. \tag{21}$$

Where $c = \frac{\Gamma(n+m+a_1+a_2)}{\Gamma(n+a_1)\Gamma(m+a_2)}$

Therefore, the Bayes estimator of R, $\hat{R}_{Bs}$, is the posterior mean which can be obtained as follow

$$\hat{R}_{Bs} = \int_0^1 r\pi(r|\boldsymbol{X}, \boldsymbol{Y})dr = \begin{cases} \left(\frac{\nu_2}{\nu_1}\right)^{\delta_2} \frac{\delta_2}{\delta_1+\delta_2} F_{2,1}\left(\delta_1 + \delta_2, \delta_2 + 1, \delta_1 + \delta_2 + 1; \frac{\nu_1 - \nu_2}{\nu_1}\right), & \nu_2 \le \nu_1 \\ \left(\frac{\nu_1}{\nu_2}\right)^{\delta_1} \frac{\delta_2}{\delta_1+\delta_2} F_{2,1}\left(\delta_1 + \delta_2, \delta_1, \delta_1 + \delta_2 + 1; \frac{\nu_2 - \nu_1}{\nu_2}\right), & \nu_2 > \nu_1 \end{cases} \tag{22}$$

Where $\delta_1 = n + a_1$, $\delta_2 = m + a_2$, $\nu_1 = b_1 + T_1$, $\nu_2 = b_2 + T_2$.

4 Simulation

In this section we use numerical results to compare the performance of the different estimators obtained in section 3 , with respect to their biases and mean square errors. For this purpose Monte Carlo simulation was used. The following steps were considered for obtaining these numerical results.
Step1: we generated $X_1, \dots, X_n$ from Weibull distribution for n=15, 25, $\lambda_1 = 4$ and $\alpha = 1.5$.
Step2: we generated $Y_1, \dots, Y_m$ from Weibull distribution for m=15, 25, 50 and λ_2 =2, 4, 6 and α =1.5.
Step3: we compute the MLE of R using Eq. (8).
Step4: Eq.(18) was used for computing the UMVUE of R.
Step5: To obtain Bayes estimator Eq.(22) was used when the values of the prior parameters being a_1 =1 , b_1 =2, a_2 =2 and b_2 =3.
Step6: We obtained the estimators based on N=2000 replications and calculate the MSE as follow

$$\mathrm{MSE} = \frac{1}{2000}\sum_{i=1}^{2000}(\hat{R}_i - R)^2$$

where $\hat{R}$ is an estimator of R. All the results are given in table1.
From table 1, it is clear that when m=n and n, m increase then the MSEs decrease. We observed that for fixed n as m increase MSEs decrease and also for fixed m as n increase the MSEs decrease.
It is also observed that MSEs of the Bayes estimators of R are smaller than the MSEs of the MLEs and UMVUEs. Also MLEs has MSEs less than the UMVUEs. So the performance of Bayes estimators are better than MLEs and the performance of the MLEs are better than UMVUEs.

5 Conclusion

In this paper, we treated different point estimations of R=P(Y<X) under Weibull distribution. The MLE and UMVUE of R were obtained that had closed forms. We obtained the Bayes estimator of R with respect to the square error loss function based on gamma priors exactly. We also represented a confidence interval and shortest confidence interval for R based on MLE of R. At last, we performed simulation study for comparing between three estimators. As the Bayes estimators has smallest MSEs among all estimators so we recommend to use the Bayesian estimators in practice.

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Table 1: The bias and MSE values of the MLE, UMVUE and Bayes Estimator

n.m	λ_1, λ_2	R	MLE (bias, MSE)	UMVUE (bias, MSE)	Bayes (bias, MSE)
15,15	2	0.33	0.3391 (-0.005807,0.006590)	0.3343 (-0.001005,0.006903)	0.3471 (-0.013754,0.002942)
	4 4	0.5	0.4998 (0.000185,0.007856)	0.4998 (0.000193,0.008391)	0.4456 (0.054329,0.002639)
	6	0.6	0.5972 (0.002770,0.007518)	0.6003 (-0.000347,0.007974)	0.4941 (0.105864,0.002390)
15,25	2	0.33	0.3339 (-0.000572, 0.005263)	0.3329 (0.000399, 0.0055)	0.3684 (-0.035082, 0.002764)
	4 4	0.5	0.4971 (0.002883, 0.006497)	0.5004 (-0.000374, 0.006851)	0.4892 (0.010779, 0.002673)
	6	0.6	0.5946 (0.005385, 0.006611)	0.6002 (-0.000247, 0.006907)	0.5504 (0.049579, 0.002527)
15,50	2	0.33	0.3291 (0.004197, 0.004270)	0.3310 (0.002309, 0.004459)	0.3860 (0.002309, 0.002437)
	4 4	0.5	0.4937 (0.006317, 0.005285)	0.4994 (0.000619, 0.005522)	0.5285 (-0.028557, 0.002371)
	6	0.6	0.5936 (0.006413, 0.005126)	0.6012 (-0.001168, 0.005290)	0.6040 (-0.004028, 0.002136)
25,15	2	0.33	0.3417 (-0.008403, 0.005269)	0.3349 (-0.001603, 0.005431)	0.3231 (0.010199, 0.002497)
	4 4	0.5	0.5023 (-0.002271, 0.006802)	0.4990 (0.000999, 0.007170)	0.4186 (0.081364, 0.002475)
	6	0.6	0.5982 (0.001824, 0.005961)	0.5975 (0.002511, 0.006276)	0.4658 (0.134229, 0.002090)
25,25	2	0.33	0.3363 (-0.002937, 0.003928)	0.3333 (-0.000014, 0.004037)	0.3434 (-0.010062, 0.002361)
	4 4	0.5	0.4976 (0.002433, 0.004773)	0.4975 (0.002483, 0.004967)	0.4606 (0.039436, 0.002317)
	6	0.6	0.5990 (0.000997, 0.004622)	0.6009 (-0.000906, 0.004789)	0.5244 (0.075580, 0.002184)
25,50	2	0.33	0.3348 (-0.001505, 0.003038)	0.3349 (-0.001528, 0.003120)	0.3629 (-0.029585, 0.002065)
	4 4	0.5	0.4986 (0.001404, 0.003676)	0.5011 (-0.001075, 0.003787)	0.5031 (-0.003138, 0.002124)
	6	0.6	0.5959 (0.004067, 0.003449)	0.5997 (0.000273, 0.003530)	0.5777 (0.022303, 0.001918)