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Research paper

Characterization for Gompertz distribution based on general progressively type-II right censored order statistics

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Abstract

In this article, we establish recurrence relations for single and product moments based on general progressively Type-II right censored order statistics (GPTIICOS). Characterization for Gompertz distribution (GD) using relation between probability density function and distribution function is obtained. Moreover recurrence relations of single and product moments based on GPTIICOS are also used to characterize the distribution. Further, the results are specialized to the progressively Type-II right censored order statistics (PTIICOS).

Keywords: *Characterization; Gompertz Distribution; General Progressively Type-II Right Censored Order Statistics; Recurrence Relations.*

1. Introduction

In failure data analysis, it is common that some individuals cannot be observed for the full failure times. GPTIICOS is a useful and more general scheme in which a specific fraction of individuals at risk may be removed from the study at each of several ordered failure times. Progressively censored samples have been considered, among others, as solved by Davis and Feldstein [7], Aggarwala, and Balakrishnan [3], Balakrishnan et al. [6] and Guilbaud [9].
Mohie El-Din et al. [13] derived recurrence relations for expectations of functions of order statistics for doubly truncated distributions and their applications. Aggarwala and Balakrishnan [3] derived recurrence relations for single and product moments of PTIICOS from exponential and truncated exponential distributions. Abd El-Aty and Marwa Mohie El-Din [1] derived recurrence relations for single and double moments of generalized order statistics from the inverted linear exponential distribution and any continuous function. Mohie El-Din and Kotb [12] derived recurrence relation for product moments and characterization of generalized order statistics based on a general class of doubly truncated Marshall-Olkin extended distributions. Mohie El-Din et al.[11] discussed estimation for parameters of Feller-Pareto distribution from PTIICOS and some characterizations .Abd El-Hamid and Al-Hussaini [2] derived Inference and optimal design based on step-partially accelerated life tests for the generalized Pareto distribution under progressive Type-I censoring .Mohie El-Din et al. [14] derived characterization for linear failure rate distribution using recurrence relations of single and product moments based on general progressively Type-II right censored order statistics.
This scheme of censoring was generalized by Balakrishnan and Sandhu [5] as follows: at time $X_0 \equiv 0$, n units are placed on test; the first r failure times, $X_1, \dots, X_r$, are not observed; at time $X_i + 0$, where X_i is the i^{th} ordered failure time $(i = r+1, \dots, m-1)$, R_i units are removed from the test randomly, so prior to the $(i+1)^{th}$ failure there are $n_i = n - i - \sum_{j=r+1}^{i} R_j$ units on test; finally, at the time of the m^{th} failure, X_m, the experiment is terminated, i.e., the remaining R_m units are removed from the test. The R_i's, m and r are prespecified integers which must satisfy the conditions $0 \le r < m \le n,\ 0 \le R_i \le n_{i-1}\ for\ i = r+1, \dots, m-1$ with $n_r = n - r$ and $R_m = n_{m-1} - 1$, (see Arturo and Fernandez [4]).
If the failure times are based on an absolutely continuous distribution function (cdf) F with probability density function (pdf) f, the joint probability density function based on GPTIICOS failure times $X_{r+1:m:n}, X_{r+2:m:n}, \dots, X_{m:m:n}$, is given by

$$f_{X_{r+1:m:n},X_{r+2:m:n},\dots,X_{m:m:n}}(x_{r+1},\dots,x_m) = K_{(n,m-1)}[F(x_{r+1},\theta)]^r \times$$

$$\prod_{i=r+1}^{m} f(x_i,\theta)[1-F(x_i,\theta)]^{R_i},\ x_{r+1} < x_{r+2} < \dots < x_m, \quad (1.1)$$

where,

$$K_{(n,m-1)} = \frac{n!}{r!\,(n-r)!}\left(\prod_{j=r}^{m-1} n_j\right), \quad n_i = n - i - \sum_{j=r+1}^{i} R_j\,,$$
$$i = r+1, \dots, m-1.$$

Through out this paper, we introduce recurrence relations among single and product moments based on GPTIICOS. Also characterization for GD using recurrence relations of single and product moments based on GPTIICOS, are obtained.
Let

$$X_{r+1:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)} < X_{r+2:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)} < \dots < X_{m:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)}$$

Be the m ordered observed failure times in a sample of size $(n - r)$ under GPTIICOS scheme from the GD with probability density function (pdf) given by

$$f(x,\alpha,\beta) = \alpha e^{\beta x - \frac{\alpha}{\beta}(e^{\beta x}-1)}, \alpha > 0, \beta > 0,\ \ x \ge 0$$
(1.2)

And the corresponding cumulative distribution function(cdf) is given by

$$F(x,\alpha,\beta) = 1 - e^{-\frac{\alpha}{\beta}(e^{\beta x}-1)}, \alpha > 0, \beta > 0,\ x \geq 0. \quad (1.3)$$

The GD was introduced by Gompertz [8].
It may be noticed that from (1.2) and (1.3) the relation between pdf and cdf is given by,

$$f(x) = \alpha e^{\beta x}[1 - F(x)]. \quad (1.4)$$

For any continuous distribution, we shall denote the i^{th}single moment of the GPTIICOS in view of Eq. (1.1) as

$$\mu_{q:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)^{(i)}} = E\left[X_{q:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)}\right]^i =$$

$$K_{(n,m-1)} \iint \dots \int_{0<x_{r+1}<\dots<x_m<\infty} x_q^i [F(x_{r+1})]^r f(x_{r+1}) \times$$

$$[1 - F(x_{r+1})]^{R_{r+1}} f(x_{r+2})[1 - F(x_{r+2})]^{R_{r+2}} \dots f(x_m)$$

$$[1 - F(x_m)]^{R_m} dx_{r+1} \dots dx_m, \quad (1.5)$$

And the i^{th} and j^{th} product moments as

$$\mu_{q,s:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)^{(i,j)}} = E\left[X_{q:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)^i} X_{s:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)^j}\right] =$$

$$K_{(n,m-1)} \iint \dots \int_{0<x_{r+1}<\dots<x_m<\infty} x_q^i x_s^j [F(x_{r+1})]^r f(x_{r+1}) \times$$

$$[1 - F(x_{r+1})]^{R_{r+1}} f(x_{r+2})[1 - F(x_{r+2})]^{R_{r+2}} \dots f(x_m)$$

$$[1 - F(x_m)]^{R_m} dx_{r+1} \dots dx_m. \quad (1.6)$$

2. Recurrence relations of single and product moments

In this section, we introduce the recurrence relations for single and product moments based on GPTIICOS.
In the next theorem we introduce the recurrence relations for single moments based on GPTIICOS.

Theorem 2.1: *If $X_{r+1:n} \leq X_{r+2:n} \leq \dots \leq X_{n:n}$ be the order stasistics of a random sample of size $(n-r)$ following GD, for $r+2 \leq q \leq m-1, m \leq n$ and $i \geq 0$, then*

$$\mu_{q:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)^{(i)}} = \sum_{k=0}^{\infty} \frac{\alpha\beta^k}{k!\,(i+k+1)} \times$$

$$\{(R_q+1)\mu_{q:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)^{(i+k+1)}}$$

$$+ (n - R_{r+1} - \dots - R_q - q)\mu_{q:m-1:n}^{(R_{r+1},\dots,(R_q+R_{q+1}+1),R_{q+2},\dots,R_m)^{(i+k+1)}}$$

$$-(n - R_{r+1} - \dots - R_{q-1} - q + 1) \times$$

$$\mu_{q-1:m-1:n}^{(R_{r+1},\dots,(R_{q-1}+R_q+1),R_{q+1},\dots,R_m)^{(i+k+1)}}\}. \quad (1.2)$$

Proof
From Eq. (1.4) and Eq. (1.5), we get

$$\mu_{q:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)^{(i)}} = \sum_{k=0}^{\infty} \frac{\alpha\beta^k}{k!} K_{(n,m-1)} \times$$

$$\iint \int_{0<x_{r+1}<\dots<x_{q-1}<x_{q+1}<\dots<x_m<\infty} \gamma_1(x_{q-1},x_{q+1})[F(x_{r+1})]^r \times$$

$$f(x_{r+1})[1 - F(x_{r+1})]^{R_{r+1}} \dots f(x_{q-1})[1 - F(x_{q-1})]^{R_{q-1}} f(x_{q+1})$$

$$[1 - F(x_{q+1})]^{R_{q+1}} \dots f(x_m)[1 - F(x_m)]^{R_m}$$

$$dx_{r+1} \dots dx_{q-1} dx_{q+1} \dots dx_m, \quad (2.2)$$

where

$$\gamma_1(x_{q-1},x_{q+1}) = \int_{x_{q-1}}^{x_{q+1}} x_q^{i+k} [1 - F(x_q)]^{R_q+1} dx_q. \quad (2.3)$$

Now, integrating by parts gives

$$\gamma_1(x_{q-1},x_{q+1}) =$$

$$\frac{x_{q+1}^{i+k+1}[1 - F(x_{q+1})]^{R_q+1} - x_{q-1}^{i+k+1}[1 - F(x_{q-1})]^{R_q+1}}{i+k+1}$$

$$+ \left(\frac{R_q+1}{i+k+1}\right) \int_{x_{q-1}}^{x_{q+1}} x_q^{i+k+1} f(x_q)[1 - F(x_q)]^{R_q} dx_q. \quad (2.4)$$

Substituting by Eq. (2.4) in Eq. (2.2) and simplifying, yields Eq. (2. 1).
This completes the proof.

Special case: Theorem 2.1 *will be valid for the PTIICOS as a special case from the GPTIICOS when $r = 0$,*

$$\mu_{q:m:n}^{(R_1,R_2,\dots,R_m)^{(i)}} = \sum_{k=0}^{\infty} \frac{\alpha\beta^k}{k!(i+k+1)} \{(R_q+1)\,\mu_{q:m:n}^{(R_1,R_2,\dots,R_m)^{(i+k+1)}}$$

$$-(n - R_1 - \dots - R_{q-1} - q + 1)\mu_{q-1:m-1:n}^{(R_1,\dots,(R_{q-1}+R_q+1),R_{q+1},\dots,R_m)^{(i+k+1)}}$$

$$+ (n - R_1 - \dots - R_q - q)\mu_{q:m-1:n}^{(R_1,R_2,\dots,(R_q+R_{q+1}+1),R_{q+2},\dots,R_m)^{(i+k+1)}}\}.$$

In the next two theorems, we shall introducerecurrence relations for product moments based on GPTIICOS.

Theorem 2.2: *If $X_{r+1:n} \leq \dots \leq X_{n:n}$ be the order stasistics of a random sample of size $(n-r)$ following GD, for $r+1 \leq q < s \leq m-1, m \leq n$ and $i,j \geq 0$, then*

$$\mu_{q,s:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)^{(i,j)}} = \sum_{k=0}^{\infty} \frac{\alpha\beta^k}{k!\,(i+k+1)} \times$$

$$\{(R_q+1)\mu_{q,s:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)^{(i+k+1,j)}}$$

$$+ (n - R_{r+1} - \dots - R_q - q)\mu_{q,s-1:m-1:n}^{(R_{r+1},\dots,(R_q+R_{q+1}+1),R_{q+2},\dots,R_m)^{(i+k+1,j)}}$$

$$-(n - R_{r+1} - \dots - R_{q-1} - q + 1) \times$$

$$\mu_{q-1,s-1:m-1:n}^{(R_{r+1},R_{r+2},\dots,(R_{q-1}+R_q+1),R_{q+1},\dots,R_m)^{(i+k+1,j)}}\}. \quad (2.5)$$

Proof
Similarly as proved in theorem 2.1.

Special case: This theorem will be valid for the PTIICOS as a special case from the GPTIICOS when $r = 0$,

$$\mu_{q,s:m:n}^{(R_1,R_2,\dots,R_m)^{(i,j)}} = \sum_{k=0}^{\infty} \frac{\alpha\beta^k}{k!(i+k+1)} \{(R_q+1)\,\mu_{q,s:m:n}^{(R_1,R_2,\dots,R_m)^{(i+k+1,j)}}$$

$$+ (n - R_1 - \dots - R_q - q)\mu_{q,s-1:m-1:n}^{(R_1,R_2,\dots,(R_q+R_{q+1}+1),R_{q+2},\dots,R_m)^{(i+k+1,j)}}$$

$$-(n - R_1 - \dots - R_{q-1} - q + 1) \times$$

$\mu_{q-1,s-1:m-1:n}^{(R_1,\ldots,(R_{q-1}+R_q+1),R_{q+1},\ldots,R_m)^{(i+k+1,j)}}\}.$

Theorem 2.3: *If $X_{r+1:n} \le \ldots \le X_{n:n}$ be the order stasistics of a random sample of size $(n-r)$ following GD, for $r+1 \le q < s \le m-1$, $m \le n$ and $i,j \ge 0$, then*

$$\mu_{q,s:m:n}^{(R_{r+1},\ldots,R_m)^{(i,j)}} = \sum_{k=0}^{\infty} \frac{\alpha\beta^k}{k!(i+k+1)} \{(R_s+1)\,\mu_{q,s:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i,j+k+1)}}$$

$$+(n-R_{r+1}-\cdots-R_s-s)\mu_{q,s:m-1:n}^{(R_{r+1},\ldots,(R_s+R_{s+1}+1),R_{s+2},\ldots,R_m)^{(i,j+k+1)}}$$

$$-(n-R_{r+1}-\cdots-R_{s-1}-s+1)\times$$

$$\mu_{q,s-1:m-1:n}^{(R_{r+1},R_{r+2},\ldots,(R_{s-1}+R_s+1),R_{s+1},\ldots,R_m)^{(i,j+k+1)}}\}. \quad (2.6)$$

Proof

Similarly as proved in theorem 2.1.

Special cases: For $r=0$, we obtain the recurrence relations of PTIICOS.

$$\mu_{q,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j)}} = \sum_{k=0}^{\infty} \frac{\alpha\beta^k}{k!(i+k+1)} \{(R_s+1)\,\mu_{q,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j+k+1)}}$$

$$-(n-R_1-\cdots-R_{s-1}-s+1)\times$$

$$\mu_{q,s-1:m-1:n}^{(R_1,R_2,\ldots,(R_{s-1}+R_s+1),R_{s+1},\ldots,R_m)^{(i,j+k+1)}}$$

$$+(n-R_1-\cdots-R_s-s)\mu_{q,s:m-1:n}^{(R_1,R_2,\ldots,(R_s+R_{s+1}+1),R_{s+2},\ldots,R_m)^{(i,j+k+1)}}\},$$

and for $i=0$

$$\mu_{s:m:n}^{(R_1,R_2,\ldots,R_m)^{(j)}} = \sum_{k=0}^{\infty} \frac{\alpha\beta^k}{k!(i+k+1)} \{(R_s+1)\,\mu_{s:m:n}^{(R_1,R_2,\ldots,R_m)^{(j+k+1)}}$$

$$-(n-R_1-\cdots-R_{s-1}-s+1)\times$$

$$\mu_{s-1:m-1:n}^{(R_1,R_2,\ldots,(R_{s-1}+R_s+1),R_{s+1},\ldots,R_m)^{(j+k+1)}}$$

$$+(n-R_1-\cdots-R_s-s)\mu_{s:m-1:n}^{(R_1,R_2,\ldots,(R_s+R_{s+1}+1),R_{s+2},\ldots,R_m)^{(j+k+1)}}\},$$

and for $s=m$

$$\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(j)}} = \sum_{k=0}^{\infty} \frac{\alpha\beta^k}{k!(i+k+1)} \{(R_m+1)\,\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(j+k+1)}}$$

$$-(n-R_1-\cdots-R_{m-1}-m+1)\mu_{m-1:m-1:n}^{(R_1,R_2,\ldots,(R_{m-1}+R_m+1))^{(j+k+1)}}\}.$$

3. The characterization

In this section, we introduce the characterization of the GD using the relation between pdf and cdfand using recurrence relations for single and product moments based on GPTIICOS.

3.1. Characterization via differential equation for the GD distribution

In the next theorem, we introduce the characterization of the GD using relation between pdf and cdf.

Theorem 3.1: *Let X be a continuous random variable with pdf $f(\cdot)$, cdf $F(\cdot)$ and survival function $[1-F(\cdot)]$. Then X has GD iff*

$$f(x) = \alpha e^{\beta x}[1-F(x)]. \quad (3.1)$$

Proof

Necessity:
From Eq. (1.2) and Eq. (1.3) we can easily obtain Eq. (3.1).
Sufficiency:
Suppose that Eq. (3.1) is true. Then we have:

$$\frac{-d[1-F(x)]}{1-F(x)} = \alpha e^{\beta x}dx.$$

By integrating, we get

$$-\ln|1-F(x)| = \frac{\alpha}{\beta}e^{\beta x} + C \quad (3.2)$$

Where C is an arbitrary constant.
Now, since$[1-F(\mu)]=1$, then putting $x=0$ in Eq. (3.2), we get $C = \frac{-\alpha}{\beta}$.
Therefore,

$$\ln|[1-F(x)]| = -\frac{\alpha}{\beta}e^{\beta x} + \frac{\alpha}{\beta},$$

Or

$$[1-F(x)] = \exp\{-\frac{\alpha}{\beta}e^{\beta x} + \frac{\alpha}{\beta}\}.$$

Hence,

$$F(x) = 1-\exp\{\frac{\alpha}{\beta} - \frac{\alpha}{\beta}e^{\beta x}\}$$

That is the distribution function of GD.
This completes the proof.

3.2. Characterization via recurrence relations for single moments

In the next theorem, we will introduce the characterization of the GD using recurrence relations for single moments based on GPTIICOS.

Theorem 3.2: *Let$X_{r+1:n} \le X_{r+2:n} \le \ldots \le X_{n:n}$ be the order stasistics of a random sample of size $(n-r)$. Then X has GD iff, for$r+2 \le q \le m-1$, $m \le n$ and $i \ge 0$,*

$$\mu_{q:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i)}} = \sum_{k=0}^{\infty} \frac{\alpha\beta^k}{k!\,(i+k+1)} \times$$

$$\{(R_q+1)\mu_{q:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i+k+1)}}$$

$$+(n-R_{r+1}-\cdots-R_q-q)\mu_{q:m-1:n}^{(R_{r+1},\ldots,(R_q+R_{q+1}+1),R_{q+2},\ldots,R_m)^{(i+k+1)}}$$

$$-(n-R_{r+1}-\cdots-R_{q-1}-q+1)\times$$

$$\mu_{q-1:m-1:n}^{(R_{r+1},R_{r+2},\ldots,(R_{q-1}+R_q+1),R_{q+1},\ldots,R_m)^{(i+k+1)}}\}. \quad (3.3)$$

Proof

Necessity:
Theorem 2.1: proved the necessary part of this theorem.
Sufficiency:
Assuming that Eq.(3.3)holds, then we have:

$$\mu_{q:m:n}^{(R_{r+1},\ldots,R_m)^{(i)}} = \sum_{k=0}^{\infty} \frac{\alpha\beta^k}{k!(i+k+1)} \{(R_q+1)\,\mu_{q:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i+k+1)}}$$

$$+(n-R_{r+1}-\cdots-R_q-q)\mu_{q:m-1:n}^{(R_{r+1},\ldots,(R_q+R_{q+1}+1),R_{q+2},\ldots,R_m)^{(i+k+1)}}$$

$-(n-R_{r+1}-\cdots-R_{q-1}-q+1)\times$

$$\mu_{q-1:m-1:n}^{(R_{r+1},R_{r+2},\dots,(R_{q-1}+R_q+1),R_{q+1},\dots,R_m)^{(i+k+1)}} \}, \quad (3.4)$$

where,

$$\mu_{q:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)^{(i+k+1)}} =$$

$$K_{(n,m-1)} \iint \dots \int_{0<x_{r+1}\dots<x_{q-1}<x_{q+1}<\dots<x_m<\infty} \gamma_3(x_{q-1},x_{q+1})\times$$

$$[F(x_{r+1})]^r f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots f(x_{q-1})\times$$

$$[1-F(x_{q-1})]^{R_{q-1}} f(x_{q+1})[1-F(x_{q+1})]^{R_{q+1}} \dots f(x_m)\times$$

$$[1-F(x_m)]^{R_m} dx_{r+1} \dots dx_{q-1} dx_{q+1} \dots dx_m, \quad (3.5)$$

where

$$\gamma_3(x_{q-1},x_{q+1}) = \int_{x_{q-1}}^{x_{q+1}} x_q^{i+k+1} f(x_q)[1-F(x_q)]^{R_q} dx_q. \quad (3.6)$$

Integrating by parts, we obtain

$$\gamma_3(x_{q-1},x_{q+1}) = \frac{-1}{R_q+1} x_{q+1}^{i+k+1}[1-F(x_{q+1})]^{R_q+1}$$

$$+\frac{1}{R_q+1} x_{q-1}^{i+k+1}[1-F(x_{q-1})]^{R_q+1}$$

$$+\frac{i+k+1}{R_q+1}\int_{x_{q-1}}^{x_{q+1}} x_q^{i+k}[1-F(x_q)]^{R_q+1} dx_q. \quad (3.7)$$

Substituting in Eq. (3.5), we get

$$\mu_{q:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)^{(i+k+1)}} = \frac{i+k+1}{R_q+1} K_{(n,m-1)}\times$$

$$\iint \dots \int_{0<x_{r+1}\dots<x_{q-1}<x_{q+1}<\dots<x_m<\infty} [F(x_{r+1})]^r \dots\times$$

$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} f(x_{q-1})[1-F(x_{q-1})]^{R_{q-1}}$$

$$\int_{x_{q-1}}^{x_{q+1}} x_q^{i+k}[1-F(x_q)]^{R_q+1} dx_q\, f(x_{q+1})[1-F(x_{q+1})]^{R_{q+1}} \dots\times$$

$$f(x_m)[1-F(x_m)]^{R_m} dx_{r+1} \dots dx_{q-1} dx_{q+1} \dots dx_m$$

$$+\frac{K_{(n,m-1)}}{R_q+1}\iint \dots \int_{0<x_{r+1}\dots<x_{q-1}<x_{q+1}<\dots<x_m<\infty} x_{q-1}^{i+k+1}[F(x_{r+1})]^r\times$$

$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots f(x_{q-1})[1-F(x_{q-1})]^{R_{q-1}+R_q+1}\times$$

$$f(x_{q+1})[1-F(x_{q+1})]^{R_{q+1}} \dots\times$$

$$f(x_m)[1-F(x_m)]^{R_m} dx_{r+1} \dots dx_{q-1} dx_{q+1} \dots dx_m$$

$$-\frac{K_{(n,m-1)}}{R_q+1}\iint \dots \int_{0<x_{r+1}\dots<x_{q-1}<x_{q+1}<\dots<x_m<\infty} x_{q+1}^{i+k+1}[F(x_{r+1})]^r\times$$

$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots f(x_{q+1})[1-F(x_{q+1})]^{R_q+R_{q+1}+1} \dots\times$$

$$f(x_m)[1-F(x_m)]^{R_m} dx_{r+1} \dots dx_{q-1} dx_{q+1} \dots dx_m =$$

$$K_{(n,m-1)}\frac{i+k+1}{R_q+1}\iint \dots \int_{0<x_{r+1}\dots<x_{q-1}<x_{q+1}<\dots<x_m<\infty} [F(x_{r+1})]^r \dots\times$$

$$\int_{x_{q-1}}^{x_{q+1}} x_q^{i+k}[1-F(x_q)]^{R_q+1} dx_q\, f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots\times$$

$$f(x_{q-1})[1-F(x_{q-1})]^{R_{q-1}} f(x_{q+1})[1-F(x_{q+1})]^{R_{q+1}} \dots\times$$

$$f(x_m)[1-F(x_m)]^{R_m} dx_{r+1} \dots dx_{q-1} dx_{q+1} \dots dx_m$$

$$-\frac{(n-R_{r+1}-\cdots-R_{q-1}-q+1)}{R_q+1}\mu_{q-1:m-1:n}^{(R_{r+1},R_{r+2},\dots,(R_{q-1}+R_q+1),R_{q+1},\dots,R_m)^{(i+k+1)}}$$

$$+\frac{(n-R_{r+1}-\cdots-R_q-q)}{R_q+1}\mu_{q:m-1:n}^{(R_{r+1},R_{r+2},\dots,(R_q+R_{q+1}+1),R_{q+2},\dots,R_m)^{(i+k+1)}}. \quad (3.8)$$

Substituting for $\mu_{q:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)^{(i+k+1)}}$ from Eq. (3.8) in Eq. (3.4), we get

$$\mu_{q:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)^{(i)}} = \sum_{k=0}^{\infty}\frac{\alpha\beta^k}{k!}K_{(n,m-1)}$$

$$\iint \dots \int_{0<x_{r+1}<\dots<x_m<\infty} x_q^{i+k}[F(x_{r+1})]^r[1-F(x_q)]^{R_q+1} f(x_{r+1})\times$$

$$[1-F(x_{r+1})]^{R_{r+1}} \dots f(x_{q-1})[1-F(x_{q-1})]^{R_{q-1}}\times$$

$$f(x_{q+1})[1-F(x_{q+1})]^{R_{q+1}} \dots f(x_m)[1-F(x_m)]^{R_m} dx_{r+1} \dots dx_m. \quad (3.9)$$

We get

$$K_{(n,m-1)}\iint \dots \int_{0<x_{r+1}<\dots<x_m<\infty} x_q^i [F(x_{r+1})]^r f(x_q)[1-F(x_q)]^{R_q}$$

$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots f(x_{q-1})[1-F(x_{q-1})]^{R_{q-1}}$$

$$f(x_{q+1})[1-F(x_{q+1})]^{R_{q+1}} \dots f(x_m)[1-F(x_m)]^{R_m} dx_{r+1} \dots dx_m$$

$$= K_{(n,m-1)}\iint \dots \int_{0<x_{r+1}<\dots<x_m<\infty} \sum_{k=0}^{\infty}\frac{\alpha\beta^k}{k!} x_q^{i+k} \dots\times$$

$$[F(x_{r+1})]^r[1-F(x_q)]^{R_q+1} f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}\times$$

$$f(x_{q-1})[1-F(x_{q-1})]^{R_{q-1}} f(x_{q+1})[1-F(x_{q+1})]^{R_{q+1}} \dots\times$$

$$f(x_m)[1-F(x_m)]^{R_m} dx_{r+1} \dots dx_m. \quad (3.10)$$

We get

$$K_{(n,m-1)}\iint \dots \int_{0<x_{r+1}<\dots<x_m<\infty} x_q^i\, [1-F(x_q)]^{R_q} f(x_{r+1}) \dots\times$$

$$[1-F(x_{r+1})]^{R_{r+1}} f(x_{q-1})[1-F(x_{q-1})]^{R_{q-1}} f(x_{q+1})$$

$$[1-F(x_{q+1})]^{R_{q+1}} \dots f(x_m)[1-F(x_m)]^{R_m}$$

$$[f(x_q)-\alpha e^{\beta x_q}[1-F(x_q)]][F(x_{r+1})]^r dx_{r+1} \dots dx_m = 0. \quad (3.11)$$

Using Muntz-Szasz theorem, [See, Hwang and Lin [10]], we get

$$f(x_q) = \alpha e^{\beta x_q}[1-F(x_q)].$$

Using Theorem 3.1, we get

$$F(x) = 1-e^{-\frac{\alpha}{\beta}(e^{\beta x}-1)}.$$

That is the distribution function of GD.
This completes the proof.

3.3. Characterization via recurrence relations for product moments

In the next two theorems, we will introduce the characterization of GD using recurrence relations for product moments based on GPTIICOS.

Theorem 3.3: *Let $X_{r+1:n} \le X_{r+2:n} \le \ldots \le X_{n:n}$ be the order stasistics of a random sample of size $(n-r)$. Then X has GD iff, for $r+1 \le q < s \le m-1$, $m \le n$ and $i, j \ge 0$,*

$$\mu_{q,s:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)(i,j)} = \sum_{k=0}^{\infty} \frac{\alpha\beta^k}{k!\,(i+k+1)}$$

$$\{(R_q+1)\mu_{q,s:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)(i+k+1,j)}$$

$$+(n-R_{r+1}-\cdots-R_q-q)\mu_{q,s-1:m-1:n}^{(R_{r+1},\ldots,(R_q+R_{q+1}+1),R_{q+2},\ldots,R_m)(i+k+1,j)}$$

$$-(n-R_{r+1}-\cdots-R_{q-1}-q+1)\times$$

$$\mu_{q-1,s-1:m-1:n}^{(R_{r+1},R_{r+2},\ldots,(R_{q-1}+R_q+1),R_{q+1},\ldots,R_m)(i+k+1,j)}\}. \quad (3.12)$$

Proof
Necessity:
Theorem 2.2 proved the necessary part of this theorem.
Sufficiency:
Similarly as proved in theorem 3.2 we obtain the distribution function of GD given by

$$F(x) = 1 - e^{-\frac{\alpha}{\beta}(e^{\beta x}-1)}.$$

This completes the proof.

Theorem 3.4: *Let $X_{r+1:n} \le \ldots \le X_{n:n}$ be the order stasistics of a random sample of size $(n-r)$. Then X has GD iff, for $r+1 \le q < s \le m-1$, $m \le n$ and $i, j \ge 0$,*

$$\mu_{q,s:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)(i,j)} = \sum_{k=0}^{\infty} \frac{\alpha\beta^k}{k!(i+k+1)}\{(R_s+1)\,\mu_{q,s:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)(i,j+k+1)}$$

$$+(n-R_{r+1}-\cdots-R_s-s)\mu_{q,s:m-1:n}^{(R_{r+1},\ldots,(R_s+R_{s+1}+1),R_{s+2},\ldots,R_m)(i,j+k+1)}$$

$$-(n-R_{r+1}-\cdots-R_{s-1}-s+1)\times$$

$$\mu_{q,s-1:m-1:n}^{(R_{r+1},R_{r+2},\ldots,(R_{s-1}+R_s+1),R_{s+1},\ldots,R_m)(i,j+k+1)}\}. \quad (3.13)$$

Proof
Necessity:
Theorem 2.3 proved the necessary part of this theorem.
Sufficiency:
Similarly as proved in theorem 3.2 we obtain the distribution function of GD given by

$$F(x) = 1 - e^{-\frac{\alpha}{\beta}(e^{\beta x}-1)}.$$

This completes the proof.

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