Equiradial designs under changing axial distances, design sizes and varying center runs with their relationships to the central composite designs

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Abstract

In assessing the preferences of equiradial designs based on design size, axial distance and number of center points in relation to the central composite designs, D-absolute deviation (D-AD) and G-absolute deviation (G-AD) are proposed as new measures of similarity of experimental designs. Each absolute deviation is positive or zero. The G-absolute deviation is zero or approximately zero at n_c equals 1 center point. For n_c greater than 1, G-absolute deviation decreases for increasing values of n_c. On the other hand, the D-absolute deviation decreases as the design size increases. Designs having G-AD values of zero or approximately zero are identical or near identical in G-optimality properties. Also, designs having D-AD values of zero or approximately zero are identical or near identical in D-optimality properties. It is conjecturally hoped that at some increased design size, the equiradial design and the central composite design, having same axial or radial distance will coincide (be identical) in their properties, with D-AD value of zero or approximately zero.

Keywords: *Equiradial Design; Central Composite Design; Axial Distance; Design Size; Center Point; D-Absolute Deviation; G-Absolute Deviation.*

1. Introduction

Design performance may be greatly influenced by the design region, the model type, the positioning of the design points, missing design points and a number of several other factors. Varying design regions have been considered in optimal design theory and include spherical, cuboidal, simplex and even irregular regions (See e.g. Atkinson and Donev [3]). Each design has been related to a particular model type and the composition of design points is generally a factor of some axial distance and the axial distance specifies the nature of the design geometry. Three commonly encountered axial distances have been explored in Iwundu [5]. Antille and Weinberg [2] employed perturbations of design points to study the conjecture that the less central a design point is the greater its influence on D-efficiency is. Robustness of an experimental design to one or more missing runs has been studied by Akram [1], Srisuradetchai [14] and Smucker *et al.* [13] as an important characteristic of experimental design. Yakubu *et al.* [15] considered the effects of missing observations on predictive capability of central composite designs. Kinai [10] examined the loss in efficiency when D-optimal designs are used instead of Ds optimal designs for data with missing observations. Iwundu [6] considered the behavior of alternative second-order N-point equiradial designs under variations of model parameters for design radius $\rho = 1.0$ and established relationships among some alphabetic optimality criteria with regards to the designs and the models. Iwundu and Jaja [7] considered the precision of employing full polynomial response surface designs on models with missing coefficients (reduced models) using efficiency measures. The loss in D- and G-efficiency of constructed first- and second-order exact designs were established for models with missing interaction coefficients.

As defined by Khuri and Cornel [9], an equiradial design consists of two or more sets of points where the design points for each set have the same distance from the design center. For a two-dimensional design region, an equiradial design comprises of a set of five points on the circle of radius $\rho \geq 1$ from the design center. The center point (or points) forms a second set in a circle of radius zero. The rotatable central composite design due to Box and Wilson [4] is a member of the larger class of equiradial designs. For example, with k = 2 and $\alpha = \sqrt{2}$, the four factorial points and the four axial points form a set of eight points on the circle of radius $\rho = \sqrt{2}$ and the center point (or points) forms a second set on a circle of radius zero. Hence, the inscribed central composite design and the circumscribed central composite design automatically belong to the larger class of equiradial designs. Obviously, two or more equiradial sets are needed to provide a design for fitting a second-order model. The second-order equiradial designs that are rotatable include those equiradial set of points that are equally spaced on a circle, a sphere or a hypersphere and which form the vertices of a regular polygon, polyhedron or polytope. Alphabetic optimality criteria and Design efficiency have been employed in most comparative studies involving experimental designs. We shall in this work examine the equiradial designs for second-order model defined on design regions under changing axial distances, varying design sizes and varying center runs. In addition, the behavior of the designs shall be compared with central composite designs on similar design regions.

2. Methodology

We commence our investigation with 6-point equiradial design ξ_6 whose set of five equidistant design points consists of the equally spaced points, supported on the vertices of a regular pentagon and

the second set of equidistant design points consists of $n_c \geq 1$ center points. The design ξ_6 and its design matrix, X, are as expressed in Khuri and Cornel [9], Myers *et al.* [11] and Iwundu [6]. The 6-point equiradial design with $\rho = 1.0$ and a center point is

$$\xi_6 = \begin{pmatrix} 1 & 0 \\ 0.31 & 0.95 \\ -0.81 & 0.59 \\ -0.81 & -0.59 \\ 0.31 & -0.95 \\ 0 & 0 \end{pmatrix}$$

The design measures associated with the equiradial designs are such that the design matrices are expressed as

$$\begin{array}{cc} x_1 & x_2 \\ \{\rho \, \text{Cos} \, (\theta + 2\pi u/n_1) & \rho \, \text{Sin} \, (\theta + 2\pi u/n_1)\}; \\ u = 0, 1, 2, \ldots, n_1 - 1 \end{array}$$

where x_1 and x_2 represent the two controllable variables, ρ is the radius of the design, n_1 represents the number of points on the sphere and n_c represents the number of center points. This standard expression shall be employed in obtaining the design measures for the required N-point equiradial designs. Following Myers *et al.* [11], the value of θ is assumed equal to zero since θ has no effect on the information matrix, X^TX, of the design. For the design matrix, X, the associated information matrix, $X'X$, is assumed nonsingular and normalized as $\frac{X'X}{N} = M$, where N is the design size. For the purpose of this research, the non-center points n_1, called the radial points, shall lie between 5 and 8, inclusive. In addition to the n_1 radial points of the design, $1 \leq n_c \leq 5$ center points shall be included in the design. Thus, the N-point design shall comprise of a set of n_1 radial points and n_c center points. There exists the inverse matrix M^{-1} which is symmetric and represents a matrix of estimates of variances of parameters and covariances among parameters. The inverse matrix is cardinal when comparing designs as most optimality criteria, as in Rady *et al.* [12], are defined as functional of M^{-1}. In establishing the preferences of the equiradial designs over the central composite designs under changing axial distances, design sizes and varying center runs, we shall concern ourselves with the D- and G-efficiency. These criteria have been well explained and illustrated in a good number of literatures on optimal design of experiments with recent works offering modifications to suit experimental conditions. An example of such documentations is due to Srisuradetchai [14]. The determinant of information matrix as well as the scaled predictive variances associated with each design shall be computed. Two axial distances $\alpha = 1.0$ and $\alpha = 1.414$ shall define the geometry of the design region. Specifically, the equidistant points (different from the center points) of the equiradial designs shall make radius of $\rho = 1.0$ or $\rho = 1.414$ thus resulting in axial distance of $\alpha = 1.0$ or $\alpha = 1.414$, respectively. The rotatable central composite design shall be defined for $\alpha = 1.414$. Such rotatable central composite design is called circumscribed central composite design and comprises of the design points $(-1, -1)$, $(1, -1)$, $(-1, 1)$, $(1, 1)$, $(\alpha, 0)$, $(-\alpha, 0)$, $(0, \alpha)$, $(0, -\alpha)$ and n_c centre points $(0, 0)$, where $\alpha \cong 1.414$. The inscribed central composite design also uses α value of 1.414 to describe a circular geometric region. However, the inscribed central composite design restricts the actual design region to the defined variable ranges ± 1 by locating the axial points at the lower and upper bounds of the variable ranges. Thus, the factorial points are brought into (inscribed into) the interior of the design space. This way, α value becomes 1.0. Unlike the rotatable circumscribed central composite design which uses design points ranging between -1.414 and $+1.414$, the rotatable inscribed central composite design uses design points ranging between -1 and $+1$ and comprises of the design points $(-0.7, -0.7)$, $(0.7, -0.7)$, $(-0.7, 0.7)$, $(0.7, 0.7)$, $(1, 0)$, $(-1, 0)$, $(0, 1)$, $(0, -1)$ and n_c centre points $(0, 0)$. These design points shall make up the N-point exact design compositions. Iwundu and Otaru [8] presented the composition of N-

point exact designs for each of the inscribed and circumscribed central composite designs. The 6-point circumscribed central composite design contains the design points $(-1, -1)$, $(1, -1)$, $(-1, 1)$, $(1, 1)$, $(1, 0)$, $(0, 0)$. The 7-point circumscribed central composite design contains the design points $(-1, -1)$, $(1, -1)$, $(-1, 1)$, $(1, 1)$, $(1, 0)$, $(0, -1)$, $(0, 0)$. The 8-point circumscribed central composite design contains the design points $(-1, -1)$, $(1, -1)$, $(-1, 1)$, $(1, 1)$, $(1, 0)$, $(-1, 0)$, $(0, 1)$, $(0, 0)$. The 9-point circumscribed central composite design contains the design points $(-1, -1)$, $(1, -1)$, $(-1, 1)$, $(1, 1)$, $(1, 0)$, $(-1, 0)$, $(0, 1)$, $(0, -1)$, $(0, 0)$. The 6-point inscribed central composite design contains the design points $(-0.7, 0.7)$, $(0.7, 0.7)$, $(0, -1)$, $(-1, 0)$, $(1, 0)$, $(0, 0)$. The 7-point inscribed central composite design contains the design points $(-0.7, 0.7)$, $(0.7, 0.7)$, $(-0.7, -0.7)$, $(0, -1)$, $(-1, 0)$, $(1, 0)$, $(0, 0)$. The 8-point inscribed central composite design contains the design points $(-0.7, 0.7)$, $(0.7, 0.7)$, $(-0.7, -0.7)$, $(0.7, -0.7)$, $(0, -1)$, $(-1, 0)$, $(1, 0)$, $(0, 0)$. The 9-point inscribed central composite design contains the design points $(-0.7, 0.7)$, $(0.7, 0.7)$, $(-0.7, -0.7)$, $(0.7, -0.7)$, $(0, -1)$, $(0, 1)$, $(-1, 0)$, $(1, 0)$, $(0, 0)$.

With these arrangements, we shall (i) Examine the behaviour of the spherical equiradial designs for changing axial distances. (ii) Examine the behaviour of the spherical equiradial designs for changing design sizes. (iii) Examine the behaviour of the spherical equiradial designs for increased center points. (iv) Examine the behaviour of the spherical central composite designs for changing axial distances. (v) Examine the behaviour of the spherical central composite designs for changing design sizes. (vi) Examine the behaviour of the spherical central composite designs for increased center points. (vii) Compare the spherical equiradial designs and the spherical central composite designs on the basis of D- and G- efficiencies. (viii) Propose new measures of similarity of experimental designs. A simple layout comprising the radial points (n_1), the center points (n_c), the design size (N), the determinant of information matrix (det (M)), the maximum scaled predictive variance (SPV_{max}), the D-efficiency (D_{eff}) and the G-efficiency (G_{eff}) shall be presented as summary of computations. Each layout component has been well explained in Myers *et al.* [11], which serves as a major reference in Response Surface Methodology.

3. Results

3.1. Spherical equiradial designs ($\rho = 1.0$, $n_c \geq 1$)

The computations involving equiradial designs constructed using $\rho = 1.0$ and $n_c \geq 1$ are as in Table 1.

3.2. Spherical equiradial designs ($\rho = 1.414$, $n_c \geq 1$)

The computations involving equiradial designs constructed using $\rho = 1.414$ and $n_c \geq 1$ are as in Table 2.

3.3. Circumscribed central composite designs ($\alpha = 1.414$, $n_c \geq 1$)

The computations involving Circumscribed Central Composite designs constructed using $\alpha = 1.414$ and $n_c \geq 1$ are as in Table 3.

3.4. Inscribed central composite designs ($\alpha = 1.0$, $n_c \geq 1$)

The computations involving Inscribed Central Composite designs constructed using $\alpha = 1.0$ and $n_c \geq 1$ are as in Table 4.

3.5. D-absolute deviation and G-absolute deviation

An absolute deviation is a measure of variability or variation. In particular, each calculated distance between any two points is called an absolute deviation. In the context of design efficiency, we shall refer to the absolute deviation of two D-efficiency values

Table 1: Summary of Computations for Spherical Equiradial designs ($\rho = 1.0$, $N_C \geq 1$)

Design size N	Radial point n_1	Center point n_c	Determinant of information matrix det(M)	Maximum scaled predictive variance SPV_{max}	D-efficiency D_{eff} %	G-efficiency G_{eff} %
6	5	1	2.6100×10^{-4}	6	25.28	100.00
7		2	2.0700×10^{-4}	7	24.32	85.71
8		3	1.3954×10^{-4}	8	22.77	75.00
9		4	9.1756×10^{-5}	9	21.24	66.67
10		5	6.1000×10^{-5}	10	19.84	60.00
7	6	1	2.5845×10^{-4}	7	25.24	85.71
8		2	2.3198×10^{-4}	6.671	24.79	89.94
9		3	1.717×10^{-4}	7.505	23.58	79.95
10		4	1.2163×10^{-4}	8.339	22.26	71.95
11		5	8.5819×10^{-5}	9.173	21.00	65.41
8	7	1	2.5016×10^{-4}	8	25.10	75
9		2	2.4679×10^{-4}	6.492	25.05	92.33
10		3	1.9673×10^{-4}	7.146	24.12	83.96
11		4	1.4807×10^{-4}	7.861	23.00	76.33
12		5	1.0981×10^{-4}	8.576	21.88	69.96
9	8	1	2.4189×10^{-4}	9	24.96	66.67
10		2	2.5710×10^{-4}	6.259	25.22	95.86
11		3	2.1769×10^{-4}	6.886	24.53	87.13
12		4	1.7220×10^{-4}	7.510	23.59	79.89
13		5	1.3316×10^{-4}	8.136	22.60	73.75

Table 2: Summary of Computations for Spherical Equiradial designs ($\rho = 1.414$, $n_c \geq 1$)

Design size N	Radial point n_1	Center point n_c	Determinant of information matrix det(M)	Maximum scaled predictive variance SPV_{max}	D-efficiency D_{eff} %	G-efficiency G_{eff} %
6	5	1	6.6822×10^{-2}	6	63.70	100.00
7		2	5.2999×10^{-2}	7	61.29	85.71
8		3	3.5679×10^{-2}	8	57.38	75.00
9		4	2.3466×10^{-2}	9	53.51	66.67
10		5	1.5588×10^{-2}	10	49.98	60.00
7	6	1	6.5951×10^{-2}	7	63.56	85.71
8		2	5.9197×10^{-2}	6.670	62.43	89.96
9		3	4.3800×10^{-2}	7.504	59.37	79.97
10		4	3.1036×10^{-2}	8.338	56.06	71.96
11		5	2.1899×10^{-2}	9.171	52.89	65.42
8	7	1	6.4659×10^{-2}	7.995	63.35	75.05
9		2	6.3769×10^{-2}	6.498	63.21	92.34
10		3	5.0829×10^{-2}	7.174	60.86	83.64
11		4	3.8254×10^{-2}	7.941	58.05	75.56
12		5	2.8369×10^{-2}	8.608	55.23	69.70
9	8	1	6.1549×10^{-2}	9	62.84	66.67
10		2	6.5419×10^{-2}	6.257	63.48	95.89
11		3	5.5391×10^{-2}	6.883	61.74	87.17
12		4	4.3817×10^{-2}	7.508	59.38	79.91
13		5	3.3883×10^{-2}	8.134	56.88	73.76

Table 3: Summary of Computations for Circumscribed Central Composite Designs ($\alpha = 1.414$, $n_c \geq 1$)

Design size N	Radial point n_1	Center point n_c	Determinant of information matrix det(M)	Maximum scaled predictive variance SPV_{max}	D-efficiency D_{eff} %	G-efficiency G_{eff} %
6	5	1	3.1930×10^{-2}	6	56.32	100.00
7		2	2.5325×10^{-2}	7	54.19	85.71
8		3	1.7048×10^{-2}	8	50.73	75.00
9		4	1.1213×10^{-2}	9	47.31	66.67
10		5	7.4485×10^{-3}	10	44.19	60.00
7	6	1	3.8352×10^{-2}	7	58.07	85.71
8		2	3.4424×10^{-2}	7.547	57.04	79.50
9		3	2.5470×10^{-2}	8.490	54.24	70.67
10		4	1.8048×10^{-2}	9.434	51.22	63.60
11		5	1.2735×10^{-2}	10.377	48.32	57.82
8	7	1	4.6803×10^{-2}	8	60.03	75.00
9		2	4.6173×10^{-2}	7.811	59.90	76.81
10		3	3.6807×10^{-2}	8.679	57.68	69.13
11		4	2.7702×10^{-2}	9.547	55.01	62.85
12		5	2.0545×10^{-2}	10.415	52.33	57.61
9	8	1	6.1548×10^{-2}	9	62.84	66.67
10		2	6.5418×10^{-2}	6.251	63.48	95.99
11		3	5.5390×10^{-2}	6.876	61.74	87.26
12		4	4.3817×10^{-2}	7.501	59.38	79.99
13		5	3.3883×10^{-2}	8.126	56.88	73.84

Table 4: Summary of Computations For Inscribed Central Composite Designs ($\alpha = 1.0$, $n_c \geq 1$)

Design size N	Radial point n_1	Center point n_c	Determinant of information matrix $\det(M)$	Maximum scaled predictive variance SPV_{max}	D-efficiency D_{eff} %	G-efficiency G_{eff} %
6	5	1	1.1660×10^{-4}	6	22.10	100
7		2	9.2480×10^{-5}	7	21.27	85.71
8		3	6.2269×10^{-5}	8	19.91	75.00
9		4	4.0946×10^{-5}	9	18.57	66.67
10		5	2.7200×10^{-5}	10	17.34	60.00
7	6	1	1.3847×10^{-4}	7	22.75	85.71
8		2	1.2428×10^{-4}	7.547	22.34	79.50
9		3	9.1951×10^{-5}	8.490	21.25	70.67
10		4	6.5155×10^{-5}	9.434	20.06	63.60
11		5	4.5972×10^{-5}	10.377	18.93	57.82
8	7	1	1.6676×10^{-4}	7.996	23.46	75.04
9		2	1.6447×10^{-4}	7.795	23.41	76.97
10		3	1.3110×10^{-4}	8.661	22.54	69.28
11		4	9.8663×10^{-5}	9.527	21.50	62.98
12		5	7.3168×10^{-5}	10.393	20.45	57.73
9	8	1	2.2241×10^{-4}	8.993	24.61	66.72
10		2	2.3630×10^{-4}	6.301	24.86	95.22
11		3	2.0005×10^{-4}	6.931	24.18	86.57
12		4	1.5824×10^{-4}	7.561	23.26	79.35
13		5	1.2236×10^{-4}	8.191	22.28	73.25

as D-absolute deviation (D-AD). Similarly, we shall refer to the absolute deviation of two G-efficiency values as G-absolute deviation (G-AD). Mathematically, we shall write

D-AD = | $D_{eff\,(1)}$ - $D_{eff\,(2)}$|; $0 \leq D_{eff\,(1)}, D_{eff\,(2)} \leq 1$

where
$D_{eff\,(1)}$ is the D-efficiency associated with design ξ_1 and $D_{eff\,(2)}$ is the D-efficiency associated with design ξ_2.
In a similar form, we shall write

G-AD = | $G_{eff\,(1)}$ - $G_{eff\,(2)}$|; $0 \leq G_{eff\,(1)}, G_{eff\,(2)} \leq 1$

where
$G_{eff\,(1)}$ is the G-efficiency associated with design ξ_1 and $G_{eff\,(2)}$ is the G-efficiency associated with design ξ_2. Thus, D-absolute deviation is related to D-efficiency and G-absolute deviation is related to G-efficiency.
In comparing pairs of N-point designs, $\{\xi_{11}, \xi_{12}\}$ and $\{\xi_{21}, \xi_{22}\}$, the pair of designs ξ_{11} and ξ_{12} are assumed more closely related in D-efficiency property than the pair of designs ξ_{21} and ξ_{22} if and only if
| $D_{eff\,(\xi_{11})} - D_{eff\,(\xi_{12})}$| $\leq$ | $D_{eff\,(\xi_{21})} - D_{eff\,(\xi_{22})}$|.
If | $D_{eff\,(\xi_{11})} - D_{eff\,(\xi_{12})}$| = 0, then design ξ_{11} and design ξ_{12} are

identical in optimality property.
Also, if | $D_{eff\,(\xi_{21})} - D_{eff\,(\xi_{22})}$| = 0, then design ξ_{21} and design ξ_{22} are identical in optimality property.

If | $D_{eff\,(\xi_{11})} - D_{eff\,(\xi_{12})}$| = | $D_{eff\,(\xi_{21})} - D_{eff\,(\xi_{22})}$|, then all four designs $\xi_{11}, \xi_{12}, \xi_{21}$ and ξ_{22} are identical in optimality property.
Similarly, in comparing pairs of N-point designs, $\{\xi_{11}, \xi_{12}\}$ and $\{\xi_{21}, \xi_{22}\}$, the pair of designs ξ_{11} and ξ_{12} are assumed more closely related in G-efficiency property than the pair of designs ξ_{21} and ξ_{22} if and only if
| $G_{eff\,(\xi_{11})} - G_{eff\,(\xi_{12})}$| $\leq$ | $G_{eff\,(\xi_{21})} - G_{eff\,(\xi_{22})}$|.
If | $G_{eff\,(\xi_{11})} - G_{eff\,(\xi_{12})}$| = 0, then design ξ_{11} and design ξ_{12} are identical in optimality property.
Also, if | $G_{eff\,(\xi_{21})} - G_{eff\,(\xi_{22})}$| = 0, then design ξ_{21} and design ξ_{22} are identical in optimality property.

If | $G_{eff\,(\xi_{11})} - G_{eff\,(\xi_{12})}$| = | $G_{eff\,(\xi_{21})} - G_{eff\,(\xi_{22})}$|, then all four designs $\xi_{11}, \xi_{12}, \xi_{21}$ and ξ_{22} are identical in optimality property.
For the designs studied in this research work, we present the D- and G-absolute deviations in Table 5. Each absolute deviation is positive or zero. The D- and G- absolute deviations are compared for the equiradial designs and central composite designs having same axial or radial distance.

Table 5: D- and G-Absolute Deviations for Equiradial and Central Composite Designs

Axial or Radial distance ($\alpha = 1.0$)					Axial or Radial distance ($\alpha = 1.414$)				
Design size N	Radial point n_1	Center point n_c	D-absolute deviation D-AD	G-absolute deviation G-AD	Design size N	Radial point n_1	Center point n_c	D-absolute deviation D-AD	G-absolute deviation G-AD
6	5	1	3.18	0	6	5	1	7.38	0
7		2	3.05	0	7		2	7.10	0
8		3	2.86	0	8		3	6.65	0
9		4	2.67	0	9		4	6.20	0
10		5	2.50	0	10		5	5.79	0
7	6	1	2.49	0	7	6	1	5.49	0
8		2	2.45	10.44	8		2	5.39	10.46
9		3	2.33	9.28	9		3	5.13	9.29
10		4	2.20	8.35	10		4	4.84	8.36
11		5	2.07	7.59	11		5	4.57	7.60
8	7	1	1.64	0	8	7	1	3.25	0.05
9		2	1.64	15.45	9		2	3.31	15.53
10		3	1.67	14.68	10		3	3.18	14.51
11		4	1.50	13.35	11		4	3.04	12.71
12		5	1.43	12.23	12		5	2.90	12.09
9	8	1	0.35	0.05	9	8	1	0	0
10		2	0.36	0.64	10		2	0	0.09
11		3	0.35	0.56	11		3	0	0.09
12		4	0.33	0.54	12		4	0	0.08
13		5	0.32	0.50	13		5	0	0.08

4. Discussion of results

4.1. Findings regarding the equiradial designs

For each N-point equiradial design, the determinant value of information matrix increases for increasing axial distance. The maximum and minimum scaled variances of prediction remain constant, except for approximation error, for changing axial distances. As n_1 increases, the determinant value of information matrix decreases when the equiradial design contains only one center point. For equiradial designs having $n_c \geq 1$ center point, the determinant value of the associated information matrix decreases for increasing n_c. This holds true irrespective of the axial distance. At $n_c = 1$, the maximum scaled variance of prediction associated with equiradial designs exactly equals the number of design points, N. However, for $n_c > 1$, the maximum scaled variance of prediction increases as n_c increases with a ratio $SPV_{max} : SPV_{min}$ being 1 at N = 6, $n_c = 1$. This shows that the design is rotatable at N = 6 and $n_c = 1$. As n_1 increases, the determinant value of information matrix of the equiradial design increases for $n_c > 1$. However as n_1 increases, the maximum scaled variance of prediction increases at $n_c = 1$ and decreases at $n_c > 1$.

4.2. Findings regarding the central composite designs

For N-point central composite design, the determinant values of information matrices are maximized for increasing axial distances. On the other hand, the maximum and minimum scaled variances of prediction remain constant for changing axial distances, except for approximation error. As n_1 increases, the determinant value of information matrix increases for $n_c \geq 1$. Also, as n_1 and n_c increase, the maximum scaled variance of prediction generally increases (except at $n_1 = 8$). As regards the addition of center points, the determinant value of information matrix decreases for increasing center points and rotatability is attained at N=6, $n_c = 1$.

4.3. Findings regarding the design efficiency

The D-efficiency value associated with N-point equiradial design at $\rho = 1.414$ is higher than the D-efficiency value associated with the corresponding N-point equiradial design at $\rho = 1.0$. However, G-efficiency value remains constant for the two categories of design, except for approximation error. The D-efficiency values associated with N-point circumscribed central composite designs at $\alpha = 1.414$ are higher than the D-efficiency values associated with corresponding N-point inscribed central composite designs at $\alpha = 1.0$. Also, G-efficiency values remain constant for the two categories of design, except for approximation error. For a fixed axial distance or radius, the D-efficiency value associated with N-point equiradial design is higher than the D-efficiency value associated with the corresponding N-point central composite design. Similarly for a fixed axial radius distance, the G-efficiency associated with N-point equiradial design is generally higher than the D-efficiency associated with the corresponding N-point central composite design. At $n_c = 1$, D-efficiency associated with the equiradial design decreases as N increases but increases as N increases at fixed $n_c > 1$. At $n_c = 1$, G-efficiency associated with the equiradial design decreases as N increases but increases as N increases at fixed $n_c > 1$.

For changing n_c center points, D-efficiency of equiradial design deceases as n_c increases. For fixed n_c center points, D-efficiency associated with central composite design increases as N increases. For changing n_c center points, D-efficiency associated with central composite design decreases as n_c increases. Although, variations seem to exist with $n_1 = 8$ axial points, the G-efficiency associated with central composite design generally decreases with increasing design size, N, for fixed center point, n_c. The D- and G-absolute deviations present themselves as good measures of similarity of experimental designs. Their values are zero if the pairs of designs are identical. For the two cases of design types studied, the G-absolute deviation (G-AD) is zero at $n_c = 1$ center point and decreases for increasing values of n_c. It can further be seen from the results that as the design size N increases, D-absolute deviation (D-AD) decreases. It is conjecturally hoped that at some large design size, the equiradial design and the central composite design having the same axial or radial distance will coincide (be identical) in their properties.

5. Conclusion

The behaviours of the spherical equiradial designs and central composite designs have been examined for changing axial distances, changing design sizes and increased center points. Two new measures of similarity of experimental designs have been proposed for comparing experimental designs defined by some related properties. When two or more designs are identical in terms of D-efficiency property, the new measure called D-absolute deviation (D-AD) will be zero. Similarly, when two or more designs are identical in terms of G-efficiency property, the new measure called G-absolute deviation (G-AD) will be zero. The similarity of equiradial design to the central composite design, as measured by D-absolute deviation is stronger as the design size, N, increases. Although numerical illustrations presented in Table 5 reveal interesting properties associated with the new measures of similarity of experimental designs, there is need for analytical verification and justification.

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