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Research paper

On the inferences and applications of transmuted exponential Lomax distribution

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Abstract

This article proposed a new distribution referred to as the transmuted Exponential Lomax distribution as an extension of the popular Lomax distribution in the form of Exponential Lomax by using the Quadratic rank transmutation map proposed and studied in earlier research. Using the transmutation map, we defined the probability density function (PDF) and cumulative distribution function (CDF) of the transmuted Exponential Lomax distribution. Some properties of the new distribution were extensively studied after derivation. The estimation of the distribution's parameters was also done using the method of maximum likelihood estimation. The performance of the proposed probability distribution was checked in comparison with some other generalizations of Lomax distribution using three real-life data sets. The results obtained indicated that TELD performs better than the other distributions comprising power Lomax, Exponential-Lomax, and the Lomax distributions.

Keywords: *Lomax Distribution; Quadratic Rank Transmutation Map; Moments; Properties; Maximum Likelihood Estimation; Order Statistics; Transmuted Exponential Lomax Distribution; Parameters; Analysis.*

1. Introduction

The Pareto II or Lomax (1954) distribution proposed by [21] for modeling business failure data moreover it has been widely applied in a variety of contexts. [16] mentioned that it used for reliability modeling and life testing. The distribution has been used for modeling different data, which studied by so many authors, [15] used Lomax distribution for income and wealth data, [5] used it for modeling business failure data, while [7] used it to model firm size and queuing problems. It has also found application in the biological sciences and even for modeling the distribution of the sizes of computer files on servers, [17]. Some authors, such as [6], suggested the use of this distribution as an alternative to the exponential distribution when the data are heavy-tailed.
A random variable X is said to follow a Lomax distribution with parameters α and β if its probability density function (pdf) is given by.

$$f(x)=\frac{\alpha}{\beta}\left[1+\left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)} \tag{1}$$

And the corresponding cumulative distribution function (cdf) is given as

$$F(x)=1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha} \tag{2}$$

For $x>0,\alpha>0,\beta>0$ where α and β are the shape and scale parameters respectively.
In the literature, there are several extensions of the Lomax distribution; these among others include the Marshall–Olkin, extended-Lomax by [12] and [14], Beta–Lomax, Kumaraswamy Lomax, McDonald-Lomax by [20], Gamma-Lomax by [8] and Exponentiated Lomax by [1]. [8] presented a three-parameter Gamma–Lomax distribution based on a versatile and flexible gamma generator proposed by [27] using Stacy's generalized gamma distribution and record value theory. [26] Introduced the four parameters Weibull Lomax distribution, [3] introduced Poisson-Lomax distribution and also the Power Lomax distribution was introduced by [23]. The Extended Poisson-Lomax distribution was introduced by [2] and [4] proposed the transmuted exponentiated Lomax distribution.
Recently, a new extension of the Lomax distribution has been proposed in the literature by considering the Exponential distribution as a based model, where the random variable X is said to have follow the Exponential distribution with parameter θ. The distribution of X according to [9] is referred to as Exponential Lomax distribution.
The pdf of the Exponential Lomax distribution is defined as;

$$g(x)=\frac{\alpha\theta}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha+1}e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}},x\geq-\beta,\alpha,\beta,\theta>0 \tag{3}$$

The corresponding cumulative distribution function (CDF) of Exponential Lomax distribution is given by

$$G(x)=1-e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}},x\geq-\beta,\alpha,\beta,\theta>0 \tag{4}$$

Where, $x\geq-\beta,\alpha>0,\beta>0,\theta>0$ α and β are the shape and scale parameters respectively and θ is the scale parameter of the exponential distribution.
The cdf and pdf of the transmuted Exponential Lomax distribution are obtained using the steps proposed by [25]. A random variable

X is said to have a transmuted distribution function if its pdf and cdf are respectively given by;

$$f(x) = g(x)[1 + \lambda - 2\lambda G(x)] \quad (5)$$

And

$$F(x) = (1+\lambda)G(x) - \lambda[G(x)]^2 \quad (6)$$

Where; x > 0, and $-1 \leq \lambda \leq 1$ is the transmuted parameter, G(x) is the cdf of any continuous distribution while f(x) and g(x) are the associated pdf of F(x) and G(x), respectively.
The aim of this paper is to introduce a new continuous distribution called the transmuted Exponential Lomax distribution (TELD) from the proposed quadratic rank transmutation map by [25]. The remaining parts of this paper are presented in sections as follows: We defined the new distribution and give its plots in section 2. Section 3 derived some properties of the new distribution. The estimation of parameters using maximum likelihood estimation (MLE) is provided in section 4. In section 5, we carry out application of the proposed model with others using some real life datasets. Lastly, in section 6, we make some useful conclusions.

2. The transmuted Exponential Lomax distribution (TELD)

Using equation (3) and (4) in (5) and (6) and simplifying, we obtain the cdf and pdf of the transmuted Exponential Lomax distribution as follows:

$$F(x) = 1 - e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\left[1 - \lambda + \lambda e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right] \quad (7)$$

And

$$f(x) = \frac{\alpha\theta}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha+1} e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\left[1 - \lambda + 2\lambda e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right] \quad (8)$$

Respectively. Where, $x > 0, \alpha > 0, \theta > 0, \beta > 0, -1 \leq \lambda \leq 1$, α and β are the shape and scale parameters respectively and θ is the exponential parameter while λ is called the transmuted parameter.
The pdf and cdf of the TELD using some parameter values are displayed in figures 1 and 2 as follows:

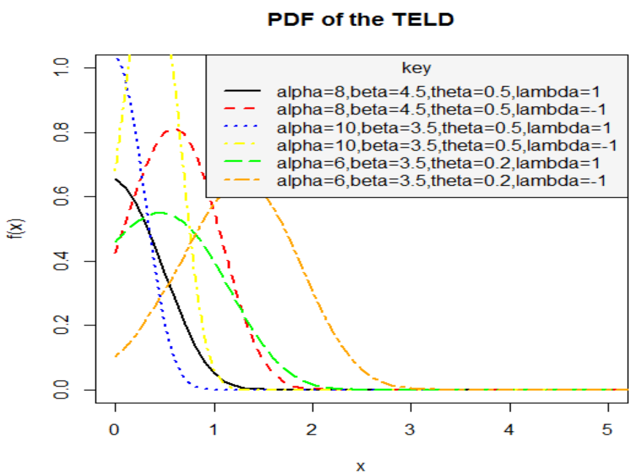

Fig. 1: The Graph of PDF of the TELD at Different Parameter Values As Displayed on the Key in the Plot Above.

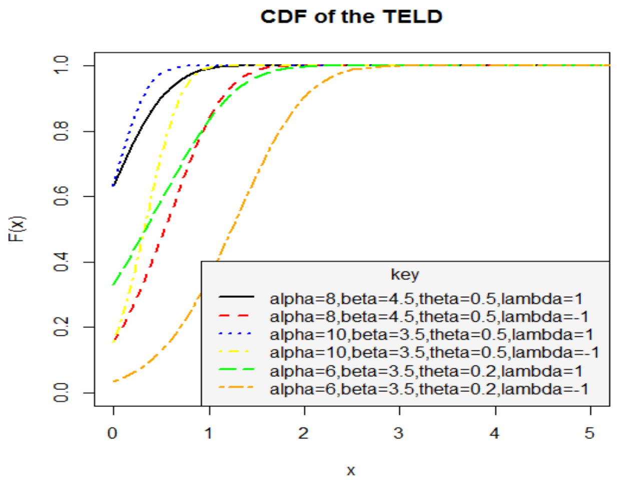

Fig. 2: The Graph of CDF of the TELD at Some Parameter Values Shown in the Key on the Figure Above.

The plot for the PDF reveals that the TELD is positively skewed and therefore will be a good model for positively skewed data sets.

3. Properties

In this section, we defined and discuss some properties of the TELD distribution.

3.1. Moments

Let X denote a continuous random variable, the nth moment of X is given by;

$$\mu_n' = E(X^n) = \int_{-\infty}^{\infty} x^n f(x)dx = \int_0^{\infty} x^n f(x)dx \quad (9)$$

Taking f(x) as the PDF of the transmuted Exponential Lomax distribution as given in equation (8), the nth moment of X is obtained using integration by substitution and is given as:
Hence,

$$\mu_n' = \sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{\beta^n \Gamma\left(\frac{n-k+\alpha}{\alpha}\right)\left[1-\lambda-2^{\frac{n-k}{\alpha}}\lambda\right]}{(2\theta)^{\frac{n-k}{\alpha}}} \quad (10)$$

The Mean
The mean of the TELD can be obtained from the nth moment of the distribution when n=1 as follows:

$$\mu_1' = \frac{\beta\Gamma\left(\frac{1+\alpha}{\alpha}\right)\left[1-\lambda-2^{\frac{1}{\alpha}}\lambda\right]}{(2\theta)^{\frac{1}{\alpha}}} - \beta[1-2\lambda] \quad (11)$$

Also the second moment of the TELD is obtained from the nth moment of the distribution when n=2 as

$$\mu_2' = \frac{\beta^2\Gamma\left(\frac{2+\alpha}{\alpha}\right)\left[1-\lambda-2^{\frac{2}{\alpha}}\lambda\right]}{(2\theta)^{\frac{2}{\alpha}}} - 2\frac{\beta^2\Gamma\left(\frac{1+\alpha}{\alpha}\right)\left[1-\lambda-2^{\frac{1}{\alpha}}\lambda\right]}{(2\theta)^{\frac{1}{\alpha}}} + \beta^2[1-2\lambda] \quad (12)$$

The Variance
The nth central moment or moment about the mean of X, say μ_n, can be obtained as

$$\mu_n = E(X - \mu_1')^n = \sum_{i=0}^{n}(-1)^i \binom{n}{i} \mu_1'^i \mu_{n-i}' \quad (13)$$

The variance of X for TELD is obtained from the central moment when n=2, that is,

$$Var(X)=E(X^2)-\{E(X)\}^2 \quad (14)$$

$$Var(X)=\mu_2'-\{\mu_1'\}^2 \quad (15)$$

Where μ_1' and μ_2' are the mean and second moment of the TELD all obtainable from equation (10).
The variation, skewness and kurtosis measures can also be calculated from the non-central moments using some well-known relationships.

3.2. Moment generating function

The moment generating is an important shape characteristic of a distribution and is always in one function that represents all the moments. In other words, the mgf produces all the moments of the random variable X by differentiation.
The mgf of a random variable X can be obtained by

$$M_x(t)=E\left(e^{tx}\right)=\int_0^\infty e^{tx}f(x)dx \quad (16)$$

$$M_x(t)=E\left(e^{tx}\right)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\mu_n'$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{t^n}{n!}(-1)^k\binom{n}{k}\frac{\beta^n\Gamma\left(\frac{n-k+\alpha}{\alpha}\right)\left[1-\lambda-2^{\frac{n-k}{\alpha}}\lambda\right]}{(2\theta)^{\frac{n-k}{\alpha}}} \quad (17)$$

3.3. Characteristics function

This is useful and has some properties which give it a genuine role in mathematical statistics. It is used for generating moments, characterization of distributions and in analysis of linear combination of independent random variables.
The characteristics function of a random variable X is given by;

$$\varphi_x(t)=E\left(e^{itx}\right)=E\left[\cos(tx)+i\sin(tx)\right]=E\left[\cos(tx)\right]+E\left[i\sin(tx)\right] \quad (18)$$

Simple algebra and power series expansion proves that

$$\phi_x(t)=\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!}\mu_{2n}'+i\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}\mu_{2n+1}'$$

Where μ_{2n}' and μ_{2n+1}' are the moments of X for n=2n and n=2n+1 respectively and can be obtained from μ_n'.
Therefore;

$$\phi_x(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{2n}\frac{(-1)^{n+k}t^{2n}}{(2n)!}\binom{2n}{k}\frac{\beta^{2n}\Gamma\left(\frac{2n-k+\alpha}{\alpha}\right)\left[1-\lambda-2^{\frac{2n-k}{\alpha}}\lambda\right]}{(2\theta)^{\frac{2n-k}{\alpha}}}$$
$$+i\sum_{n=0}^{\infty}\sum_{k=0}^{2n+1}\frac{(-1)^{n+k}t^{2n+1}}{(2n+1)!}\binom{2n+1}{k}\frac{\beta^{2n+1}\Gamma\left(\frac{2n-k+\alpha+1}{\alpha}\right)\left[1-\lambda-2^{\frac{2n-k+1}{\alpha}}\lambda\right]}{(2\theta)^{\frac{2n-k+1}{\alpha}}} \quad (19)$$

3.4. Reliability analysis of the TELD

3.4.1. The Survival function describes the likelihood that a system or an individual will not fail after a given time. Mathematically, the survival function is given by:

$$S(x)=1-F(x) \quad (20)$$

Applying the cdf of the TELD in (7), the survival function for the TELD is obtained as:

$$S(x)=e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\left[1-\lambda+\lambda e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right] \quad (21)$$

The following is a plot for the survival function of the TELD using different parameter values as shown in Figure 3 below;

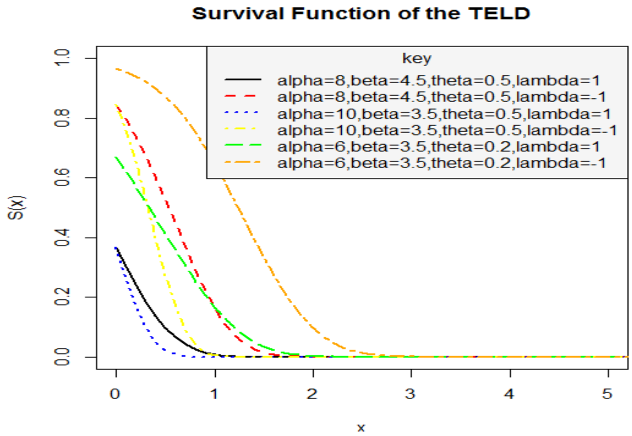

Fig. 3: Survival Function of the TELD at Different Parameter Values.

The graph in figure 3 shows that the value of the survival function equals one (1) at initial time or early age and it decreases as X increases and equals zero (0) as X becomes larger.

3.4.2. Hazard function is the probability that a component will fail or die for an interval of time. The hazard function is defined as;

$$h(x)=\frac{f(x)}{S(x)}=\frac{f(x)}{1-F(x)} \quad (22)$$

Meanwhile, the expression for the hazard rate of the TELD is given by

$$h(x)=\frac{\alpha\theta\beta^{-\alpha}\left[1-\lambda+2\lambda e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right]}{(x+\beta)^{-\alpha+1}\left[1-\lambda+\lambda e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right]} \quad (23)$$

Where $\alpha,\beta,\theta>0$ and $-1\le\lambda\le1$.

The following are some possible curves for the hazard rate at various values of the model parameters

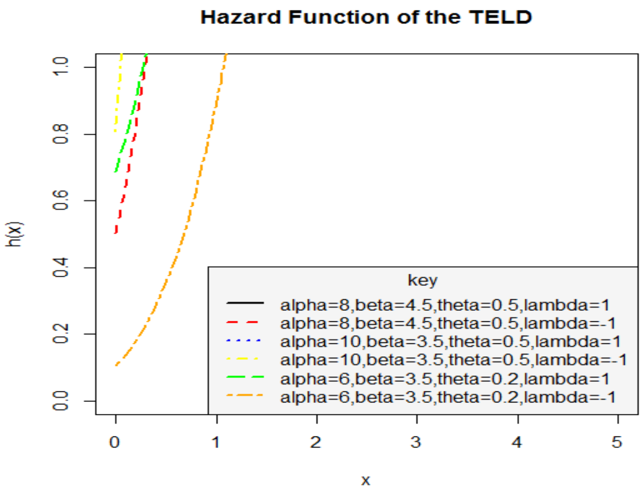

Fig. 4: Hazard Function of the TELD at Different Parameter Values.

We can see from figure 4 that the value of the hazard function increases at the beginning and slowly drop as X increases. This means that the TELD may be appropriate for modeling time dependent events, where risk or hazard increases at early stage and then drops with time.

3.5. Quantile function

Taking F(x) to be the cdf of the TELD and inverting it as above will give us the quantile function as follows:

$$F(x)=1-e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\left[1-\lambda+\lambda e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right]=u \tag{24}$$

Simplifying equation (24) above, we obtain:

$$Q(u)=X_q=\beta\left\{\left(-\frac{\ln(1-u)}{2\theta}\right)^{\frac{1}{\alpha}}-1\right\} \tag{25}$$

This function is derived above is used for obtaining some moments like skewness and kurtosis as well as the median and for generation of random variables from the distribution in question.

3.6. Skewness and kurtosis

This paper presents the quantile based measures of skewness and kurtosis due to non-existence of the classical measures in some cases.
The Bowley's measure of skewness based on quartiles is given by;

$$SK=\frac{Q\left(3/4\right)-2Q\left(1/2\right)+Q\left(1/4\right)}{Q\left(3/4\right)-Q\left(1/4\right)} \tag{26}$$

And the Moores' (1998) kurtosis is on octiles and is given by;

$$KT=\frac{Q\left(7/8\right)-Q\left(5/8\right)-Q\left(3/8\right)+\left(1/8\right)}{Q\left(6/8\right)-Q\left(1/8\right)} \tag{27}$$

Where Q (.) is obtainable with the help of equation (25).

3.7. Order statistics

Let $X_{(1)}$ denote the smallest of $X_1,X_2,....,X_n$, $X_{(2)}$ denote the second smallest of $X_1,X_2,....,X_n$, and similarly $X_{(i)}$ denote the i^{th} smallest of $X_1,X_2,....,X_n$. Then the random variables $X_{(1)},X_{(2)},....,X_{(n)}$, called the order statistics of the sample $X_1,X_2,....,X_n$, has probability density function of the i^{th} order statistic, $X_{(i)}$, as:

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}f(x)F(x)^{i-1}\left[1-F(x)\right]^{n-i} \tag{28}$$

Where $f(x)$ and $F(x)$ are the pdf and cdf of the TELD respectively.

Using (7) and (8), the pdf of the i^{th} order statistics $X_{i:n}$, can be expressed from (28) as;

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}\left[\frac{\alpha\theta}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha+1}e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\left[1-\lambda+2\lambda e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right]\right]\left[1-e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\left[1-\lambda+\lambda e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right]\right]^{i+k-1} \tag{29}$$

Hence, the pdf of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the TELD are given by;

$$f_{1:n}(x)=n\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\left[\frac{\alpha\theta}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha+1}e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\left[1-\lambda+2\lambda e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right]\right]\left[1-e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\left[1-\lambda+\lambda e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right]\right]^{k} \tag{30}$$

And

$$f_{n:n}(x)=n\left[\frac{\alpha\theta}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha+1}e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\left[1-\lambda+2\lambda e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right]\right]\left[1-e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\left[1-\lambda+\lambda e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right]\right]^{n-1} \tag{31}$$

Respectively

4. Estimation of parameter

Let $X_1, ---, X_n$ be a sample of size 'n' independently and identically distributed random variables from the TELD with unknown parameters α, β, Θ, and λ defined previously. The pdf of the TELD is given as

$$f(x)=\frac{\alpha\theta}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha+1}e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\left[1-\lambda+2\lambda e^{-\theta\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right]$$

The likelihood function is given by;

$$L\left(X_1,X_2,.....,X_n/\alpha,\beta,\theta,\lambda\right)=\left(\frac{\alpha\theta}{\beta}\right)^n\sum_{i=1}^{n}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha+1}e^{-\theta\sum_{i=1}^{n}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}}\sum_{i=1}^{n}\left[1-\lambda+2\lambda e^{-\theta\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}}\right] \tag{32}$$

Let the log-likelihood function, $l=\log L\left(X_1,X_2,.....,X_n/\alpha,\beta,\theta,\lambda\right)$, therefore

$$l=n\log\alpha+n\log\theta-n\log\beta-(\alpha+1)\sum_{i=1}^{n}\log\left(\frac{\beta}{x_i+\beta}\right)-\theta\sum_{i=1}^{n}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}+\sum_{i=1}^{n}\log\left[1-\lambda+2\lambda e^{-\theta\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}}\right] \tag{33}$$

Differentiating l partially with respect to α, β, Θ and λ respectively gives;

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\begin{matrix}\log\left(\frac{\beta}{x_i+\beta}\right)+\\ \theta\sum_{i=1}^{n}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}\log\left(\frac{\beta}{x_i+\beta}\right)+\end{matrix}\quad 2\theta\lambda\sum_{i=1}^{n}\left\{\frac{e^{-\theta\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}\log\left(\frac{\beta}{x_i+\beta}\right)}{1-\lambda+2\lambda e^{-\theta\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}}}\right\} \tag{34}$$

$$\frac{\partial l}{\partial \beta} = -\frac{n}{\beta} - \frac{(\alpha-1)}{\beta}\sum_{i=1}^{n}\left\{\frac{x_i}{x_i+\beta}\right\} + \frac{\alpha\theta}{\beta^2}\sum_{i=1}^{n} x_i\left\{\frac{\beta}{x_i+\beta}\right\}^{-\alpha+1} - \frac{\alpha\theta}{\beta^2}\sum_{i=1}^{n}\left\{\frac{x_i\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha+1} e^{-\theta\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}}}{1-\lambda+2\lambda e^{-\theta\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}}}\right\} \quad (35)$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha} + \sum_{i=1}^{n}\left\{\frac{\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha} e^{-\theta\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}}}{1-\lambda+2\lambda e^{-\theta\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}}}\right\} \quad (36)$$

$$\frac{\partial l}{\partial \lambda} = \sum_{i=1}^{n}\left\{\frac{2e^{-\theta\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}} - 1}{1-\lambda+2\lambda e^{-\theta\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}}}\right\} \quad (37)$$

Equating equations (34), (35), (36) and (37) to zero and solving for the solution of the non-linear system of equations will give us the maximum likelihood estimates of parameters $\alpha, \beta, \theta - and - \lambda$ respectively. However, the solution cannot be obtained analytically except numerically with the aid of suitable statistical software like Python, R, SAS, e.t.c when data sets are given.

5. Applications

In this section, we have compared the performance of the TELD to those of three generalizations of the Lomax model including the power Lomax distribution (PLD), Exponential Lomax distribution (ELD), Weibull-Lomax distribution (WLD), and Lomax distribution (LD).

Data Set I: This data set is the strength data of glass of the aircraft window reported by [10]. Its' summary is given as follows:

Table 1: Summary Statistics for Data Set I

parameters	N	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Values	31	18.83	25.51	29.90	35.83	30.81	45.38	52.61	0.43	2.38

Data set II: This data set represents the relief times (in minutes) of 20 patients receiving an analgesic reported by [13] and has been used by [24]. The summary of the data set is provided as follows;

Table 2: Summary Statistics for the Data Set II

parameters	N	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Values	20	1.10	1.475	1.70	2.05	1.90	4.10	0.4958	1.8625	7.1854

Data set III: This data set represents the waiting times (in minutes) before service of 100 Bank customers and examined and analyzed by [11for fitting the Lindley distribution. The summary statistics of this dataset is given below;

Table 3: Summary Statistics for Data Set III

parameters	N	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Values	100	0.80	4.675	8.10	13.020	9.877	38.500	52.3741	1.4953	5.7345

From the descriptive statistics in tables 1, 2 and 3 for the three data sets respectively, we observed that the three data sets are positively skewed, however, the second data set is highly peaked with a higher skewness coefficient followed by the third and then the first with a very low peak.

To compare this distribution, we have considered some criteria: the value of the log-likelihood function evaluated at the MLEs (ll), AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion), BIC (Bayesian Information Criterion), and HQIC (Hannan Quin Information Criterion). These statistics are given as:

$AIC = -2ll + 2k$, $BIC = -2ll + k\log(n)$, $CAIC = -2ll + \frac{2kn}{(n-k-1)}$ and $HQIC = -2ll + 2k\log[\log(n)]$

Where ll denotes the log-likelihood function evaluated at the MLEs, k is the number of model parameters and n is the sample size.

Note: In decision making, model with the lowest values for these statistics would be chosen as the best model to fit the data set in question.

Table 4: Performance Evaluation of the TELD with Some Generalizations of the Lomax Distribution Using the AIC, CAIC, BIC and HQIC Values of the Models Evaluated at the MLES Based on Data Set I

Distributions	Parameter estimates	-ll(minus log-likelihood value)	AIC	CAIC	BIC	HQIC	Ranks
TELD	$\hat{\alpha}$=2.8243 $\hat{\beta}$=5.4825 $\hat{\theta}$=0.0052 $\hat{\lambda}$=-0.9025	109.0717	226.1433	227.6818	231.8793	228.0131	3
PLD	$\hat{\alpha}$=0.5103 $\hat{\beta}$=1.1628 $\hat{\theta}$=9.1841	156.9083	319.8166	320.7055	324.1186	321.219	5
ELD	$\hat{\alpha}$=0.2029 $\hat{\beta}$=0.1873 $\hat{\theta}$=1.0511	-102.0569	-198.1139	-197.225	-193.8119	-196.7115	1
WLD	$\hat{\alpha}$=0.4109 $\hat{\theta}$=7.8208 $\hat{a}$=1.0901 $\hat{b}$=6.7338	105.2365	218.4731	220.0115	224.209	220.3428	2
LD	$\hat{\alpha}$=0.6941 $\hat{\theta}$=9.8149	155.9237	315.8474	316.2759	318.7153	316.7822	4

In table 4, we have the MLEs to each of the five fitted distributions including the TELD for the first dataset. The table also provides the corresponding values of the AIC, CAIC, BIC and HQIC for each of the models. The values of the test statistics in the table above are lower for the ELD followed by the WLD and TELD which is an indication that the TELD performed better than the other two models (PLD and LD) considered in the analysis and could be chosen as the best model compared to the two distributions. Hence, our proposed model (TELD) is better than the PLD and the conventional Lomax distribution (LD). This also provides additional evidence to the fact that generalizing probability distributions provides compound distributions that are more flexible compared to the parent distributions.

We have also considered a goodness-of-fit test in order to know which distribution fits the data better we apply the Kolmogorov-Smirnov (K-S) statistics. This statistic can be computed as:

$$K-S=D=\sup\left|F_n(x)-F_0(x)\right|$$

Where $F_n(x)$ is the empirical distribution function and n is the sample size.

Note: In decision making, any distribution or model with the lowest values for this statistic (K-S) or highest p-value would be chosen as the best model to fit the data set in question.

Table 5: Performance Evaluation of the TELD with Some Generalizations of the Lomax Distribution Using the K-S Values of the Models with Their Corresponding P-Values in Parenthesis Based on Datasets I, II and III

Distributions	Dataset I	Dataset II	Dataset III	Ranks
TELD	D = 0.1898 (0.1882)	D =0.32938 (0.02608)	D =0.22638 (7.1e-05)	2
ELD	D =0.93178 (2.2e-16)	D =0.40391 (0.00293)	D =0.52598 (2.2e-16)	5
PLD	D =0.52531 (1.7e-08)	D =0.14239 (0.8122)	D =0.2602 (2.6e-06)	3
WLD	D =0.15406 (0.412)	D =0.16597 (0.6402)	D =0.046866 (0.9806)	1
LD	D =0.52453 (1.8e-08)	D =0.4471 (0.000674)	D =0.23758 (2.5e-05)	4

From the Table above, we can observe the K-S test values of the distributions with their corresponding p-values in parenthesis based on datasets I, II and III. From the table, it is clear and we confirmed that the TELD has smaller or lower values of the K-S statistic with higher P-values for all the datasets compared to the ELD, PLD and LD which is an indication that it has a better performance compared to the three models. Hence, we can confidently conclude that the TELD is better than the ELD, PLD and LD. However, we also confirmed that the WLD performs better than all the distributions considered in the analysis though it is not far different from the TELD in performance.

6. Conclusion

A new distribution, TELD has been proposed. Some mathematical and statistical properties of the proposed distribution have been studied appropriately. The derivations of some expressions for its moments, moment generating function, characteristics function, survival function, hazard function, and ordered statistics have been done appropriately. Some plots of the distribution revealed that its shape was skewed to the right or left depending on values of the parameters. The model parameters have been estimated using the method of maximum likelihood estimation. The implications of the plots for the survival function indicate that the transmuted Exponential Lomax distribution could be used to model age-dependent events or variables whose survival decreases as time grows or where survival rates decrease with time. The results from the application showed that the proposed distribution (transmuted Exponential Lomax distribution) performs better than the power Lomax, Exponential Lomax and the Lomax distributions based on the three data sets considered. This implies that the proposed distribution (TELD) is best and can be used in placed of these three models (Exponential Lomax, power Lomax and the Lomax distributions) when we have a positively skewed data set.

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