International Journal of Engineering & Technology, 7 (3.7) (2018) 340-344

International Journal of Engineering & Technology

Website: www.sciencepubco.com/index.php/IJET

Research paper

Numerical Solution of the Equation Modified Equal Width Equation by Using Cubic Trigonometric B-Spline Method

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Abstract

A numerical solution for the modified equal width was achieved via cubic trigonometric-B-spline (C. T.-B-spline) method approach which is based on finite difference scheme with the help of θ weighted scheme. In other words, the finite difference scheme is used to discretize the time derivative, while a cubic trigonometric B-spline is used as an interpolation map in the space dimension. The performance of the scheme was validated through two examination problems. The performance of the process was validated via using L_2 and L_∞ error norms and conserved laws. Thus, stability analysis was applied by the von-Neumann method. Finally, the efficiency and accuracy of the suggested scheme was determined through comparison with the exact solution for different time and some other published numerical methods.

Keywords: Cubic trigonometric B-spline; Finite difference; Modified equal width equation; Von-Neumann.

1. Introduction

The modified equal width waves (MEW) equation based upon the equal width wave (EW) equation. (Karakoç and Geyikli,2012).

These waves propagate in non-linear media by keeping wave form and velocity even after interaction occurs, which was introduced as a model equation for describe the non-linear dispersive waves has the normalized form:

$$v_t + \varepsilon v^2 v_x - \mu v_{xxt} = 0 \qquad (1)$$

with boundary conditions (BCs)

$$v(a,t) = f_0 \quad , v(b,t) = f_1 \ t \in [0,T] \qquad (2)$$

and an initial condition (IC)

$$v(x,t) = g(x) \quad , a \le x \le b \qquad (3)$$

where ε is an arbitrary constant and μ is a positive parameter. $g(x)$ is a localized disturbance inside the interval [a, b].Few analytical solution of the MEW equation known. Thus numerical solutions of the MEW equation can be important and comparison between analytic solutions can be made. Numerous researchers suggested some numerical solutions to solve the MEW

Equation. (Karakoç and Geyikli2012) used the lumped galerkin method based on the cubic b-spline finite element method to solve MEW equation. The same (Geyikli and Karakoç, 2011) author extend their work to solve the same equation by extending cubic B-Spline to Septic B-Spline. (Islam, S. U., Haq, F. I., &Tirmizi, I. A 2010) used the quartic B-spline method for numerical solution of the MEW equation. (Saka, 2007) solved the MEW equation via the quintic B-spline collection method. (Esen and Kutluay, 2008) solved the MEW equation by alinearized numerical scheme based on finite difference method.

The outline of this article is as follows: In section 2, the cubic trigonometric B-spline method is explained and numerical solution of proposed is discussed. In section 3, the stability of proposed method is investigated. In section 4, the results of numerical experiments are presented and compared with some previous methods. Finally, in section 5, the conclusion of this study is given.

The objective of this work is to solve one dimensional MEW equation accurately, based on the motion of a single solitary wave through two problems. As an example to the application of numerical method, this incorporates the finite difference approach with cubic trigonometric B-spline.

2. Materials and Methods

2.1 Cubic Trigonometric B-Spline-(C.T.B-Spline)

In this part, the cubic trigonometric basis map (C. T. B. F) is discussed as below (Ersoy and Dog, 2016; Salih et al., 2016):

$$TB_i(x)=\frac{1}{z}\begin{cases} a^3(x_i), & x\in[x_i,x_{i+1}) \\ \mathrm{a}(x_i)\left(\mathrm{a}(x_i)\,\mathrm{b}(x_{i+2})+b(x_{i+3})\,\mathrm{a}(x_{i+1})\right)+b(x_{i+4})\,\mathrm{a}^2(x_{i+1}), & x\in[x_{i+1},x_{i+2}) \\ \mathrm{b}(x_{i+4})\left(\mathrm{a}(x_{i+1})\,\mathrm{b}(x_{i+3})+b(x_{i+4})\,\mathrm{a}(x_{i+2})\right)+a(x_i)\,\mathrm{b}^2(x_{i+3}), & x\in[x_{i+2},x_{i+3}) \\ b^3(x_{i+4}), & x\in[x_{i+3},x_{i+4}] \end{cases} \quad (4)$$

Where,

$\mathrm{a}(x_i)=\sin\left(\frac{x-x_i}{2}\right),\ b(x_i)=\sin\left(\frac{x_i-x}{2}\right)$ and

$z=\sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right)$

where $h=(b-a)/n$ and $TB_i(x)$ is a piece-wise cubic trigonometric map with various geometric properties like C^2 continuity, non-negativity and partition of unity .The values of $TB_i(x)$ and its derivatives at nodal points are required and these derivatives are tabulated in Table 1.

Table 1: $TB_i(x)$ values and their derivatives.

x	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}
TB_i	0	p_1	p_2	p_1	0
TB_i'	0	p_3	0	p_4	0
TB_i''	0	p_5	p_6	p_5	0

where

$$p_1=\frac{\sin^2\left(\frac{h}{2}\right)}{\sin(h)\sin\left(\frac{3h}{2}\right)},\ p_2=\frac{2}{1+2\cos(h)},\ p_3=-\frac{3}{4\sin\left(\frac{3h}{2}\right)}$$

$$p_4=\frac{3}{4\sin\left(\frac{3h}{2}\right)},\ p_5=\frac{3\left(1+3\cos(h)\right)}{16\sin^2\left(\frac{h}{2}\right)\left(2\cos\left(\frac{h}{2}\right)+\cos\left(\frac{3h}{2}\right)\right)}$$

$$p_6=-\frac{3\cos^2\left(\frac{h}{2}\right)}{\sin^2\left(\frac{h}{2}\right)\left(2+4\cos(h)\right)}\ ,$$

The (C.T.B-Spline) collocation method is also discussed in order to solve Eq. (1).The solution domain $a\le x\le b$ was divided equally by knots x_i into n subintervals $[x_i,x_{i+1}]$, $i=0,1,2,...,n-1$ where $a=x_0<x_1<...<x_n=b$ .Our approach for MEW equation using (C.T-B-spline) was to seek an approximate solution as:

$$V_i(x,t)=\sum_{i=-3}^{n-1}C_i(t)TB_i(x) \quad (5)$$

where $C_i(\mathrm{t})$ is determined for the approximated solutions $V_i(x,t)$ to the exact solution $v(\mathrm{x},\mathrm{t})$ at the point (x_i,t_n) over subinterval $[x_i,x_{i+1}]$. In order to get the approximations to the solution, the values of $TB_i(x)$ and its derivatives at nodal points are required. These derivative were tabulated using approximate maps (4) and (5) whereby, the values at the knots of V_i^n and their derivatives up to the second order are:

$$\begin{cases} (V)_i^n = p_1C_{i-3}^n + p_2C_{i-2}^n + p_1C_{i-1}^n, \\ \left(\frac{\partial V}{\partial x}\right)_i^n = p_3C_{i-3}^n + p_4C_{i-1}^n \\ \left(\frac{\partial^2 V}{\partial x^2}\right)_i^n = p_5C_{i-3}^n + p_6C_{i-2}^n + p_5C_{i-1}^n \end{cases} \quad (6)$$

The approximations for the solutions of the MEW Eq. (1) at t_{n+1} th time level can be given as:

$$[\frac{(\mathrm{v}-v_{xx})^{n+1}-(\mathrm{v}-v_{xx})^n}{\Delta t}]+\varepsilon[\frac{(\mathrm{v}^2\mathrm{v}_x)^{n+1}+(\mathrm{v}^2\,v_x)^n}{2}]=0 \quad (7)$$

where $n=0,1,2,..$ and Δt is the time step. The non-linear term in Eq. (7) was approximated using the Taylor series (Islam et al.,2010):

$$(\mathrm{v}^2)^{n+1}\mathrm{v}_x^{n+1}=(v^n)^2v_x^{n+1}+2v^nv_x^nv^{n+1}-2(\mathrm{v}^n)^2\mathrm{v}_x^n \quad (8)$$

Eq.(8) with nodal values v and derivatives via (6) leads to the following difference equation with variable C_i , $\mathrm{i}=-3,...,\mathrm{n}-1$. It was also noted that the system becomes a Crank-Nicolson scheme when $\theta=\frac{1}{2}$.

$$a_1C_{i-3}^{n+1}+a_2C_{i-2}^{n+1}+a_3C_{i-1}^{n+1}=b_1C_{i-3}^n+b_2C_{i-2}^n+b_3C_{i-1}^n \quad (9)$$

where

$a_1=(2+2\Delta\,\mathrm{t}\,\varepsilon\,\mathrm{v}^n\mathrm{v}_x^n)\,p_1+\Delta t\varepsilon(\mathrm{v}^n)^2\,\mathrm{p}_3-2\mu\,\mathrm{p}_5$,

$a_2=(2+2\Delta\,\mathrm{t}\,\varepsilon\,\mathrm{v}^n\mathrm{v}_x^n)\,p_2+0-2\mu\,\mathrm{p}_6$,

$a_3=(2+2\Delta\,\mathrm{t}\,\varepsilon\,\mathrm{v}^n\mathrm{v}_x^n)\,p_1+\Delta t\varepsilon(\mathrm{v}^n)^2\,\mathrm{p}_4-2\mu\,\mathrm{p}_5$,

$b_1=2\,p_1+\Delta\,\mathrm{t}\,\varepsilon(v^n)^2\,p_3-2\mu\,\mathrm{p}_5$,

$b_2=2\,p_2+0-2\mu\,\mathrm{p}_6$,

$b_3=2\,p_2+\Delta\,\mathrm{t}\,\varepsilon(v^n)^2\,p_4-2\mu\,\mathrm{p}_5$,

when (9) is simplified the system consists of $(N+1)$ linear equation in $(N+3)$ unknown $C^n=[\mathrm{C}_{j-3}^n,...,\mathrm{C}_{N-1}^n]$ at the time level $t=t_{i+1}$, via applying Eq.(5) to obtain the unique solution on the boundary conditions (3) as follows:

$$p_1C_{i-3}^n+p_2C_{i-2}^n+p_1C_{i-1}^n=0, \qquad j=0,$$

$$p_1C_{i-3}^n+p_2C_{i-2}^n+p_1C_{i-1}^n=0, \qquad j=N, (10)$$

From Eqs.(9-10) the system consists $N+3\times N+3$ in the following form:

$$M_{N+3xN+3}\,C_{1xN+3}^{n+1}=N_{N+3xN+3}\,C_{1xN+3}^{n}$$

Initial state

The initial vector C^0 was computed from the initial conditions. The approximate solution V_i^{n+1} at a particular time can be calculated repeatedly through the recurrence relation. C^0 can be obtained from the initial condition and boundary values of the derivatives of the initial condition as follows :

$$\begin{cases}(V_i^0)_x = g'(\mathrm{x}_i) & i=0\\ V_i^0 = g(\mathrm{x}_i) & i=0,1,\dots,N\ (11)\\ (V_i^0)_x = g'(\mathrm{x}_i) & i=N\end{cases}$$

Thus, the system of equations in (11) can be represented as a matrix of order $N+3\times N+3$, of the form: $A_{(N+3)\times(N+3)}F^0_{1\times N+3} = d_{1\times N+3}$

where $F^0 = [\mathrm{C}^0_{-3}, \mathrm{C}^0_{-2}, \dots, \mathrm{C}^0_{n-1}]^T$,

$$d = [g'(\mathrm{x}_0), g(\mathrm{x}_0), g(\mathrm{x}_1), \dots, g(\mathrm{x}_{n-1}), g(\mathrm{x}_n), g'(\mathrm{x}_n)]^T$$

3. Stability Analysis

In this section the stability analysis of the proposed scheme was investigated using the Von Neumann method and assume that the quantity v^2 in the non-linear term $\varepsilon v^2 v_x$ in Eq. (1) is constant β (Sajjadian, 2012). As a result, the linearized form of the proposed method after simplifications as follows:

$$2\mathrm{v}^{n+1} - 2\mu v_{xx}^{\ n+1} + \Delta t\varepsilon\beta v_x^{\ n+1} = 2\mathrm{v}^{n} - 2\mu v_{xx}^{\ n} - \Delta t\varepsilon\beta v_x^{\ n} \quad (12)$$

Simplifying Eq.(12) gives us:

$$a_1C_{j-3}^{n+1} + a_2C_{j-2}^{n+1} + a_3C_{j-1}^{n+1} = b_1C_{j-3}^{n} + b_2C_{j-2}^{n} + b_3C_{j-1}^{n} \quad (13)$$

where

$$a_1 = 2\,\mathrm{p}_1 + \Delta t\varepsilon\beta\,\mathrm{p}_3 - 2\mu\,\mathrm{p}_5 \quad ,$$
$$a_2 = 2\,\mathrm{p}_2 + 0 - 2\mu\,\mathrm{p}_6 \quad ,$$
$$a_3 = 2\,\mathrm{p}_1 + \Delta t\varepsilon\beta\,\mathrm{p}_4 - 2\mu\,\mathrm{p}_5 \quad ,$$
$$b_1 = 2\,\mathrm{p}_1 - \Delta t\varepsilon\beta\,\mathrm{p}_3 - 2\mu\,\mathrm{p}_5 \quad ,$$
$$b_2 = 2\,\mathrm{p}_2 - 0 - 2\mu\,\mathrm{p}_6 \quad ,$$
$$b_3 = 2\,\mathrm{p}_1 - \Delta t\varepsilon\beta\,\mathrm{p}_4 - 2\mu\,\mathrm{p}_5 \quad ,$$

by substituting $C_j^n = \xi^n e^{(\mathrm{im}\eta h)}$ with $i = \sqrt{-1}$ in (13) , Eq.(13) can be re-written as follows

$$\xi = \frac{X - iY}{X + iY}$$

where

$\mathrm{X} = (4\,\mathrm{p}_1 - 4\mu\,\mathrm{p}_5)\cos(\eta h) + (2\,\mathrm{p}_2 - 2\mu\,\mathrm{p}_6)$ and

$Y = (2\varepsilon\beta\,\mathrm{p}_4)\sin(\eta\,\mathrm{h})$.

The modulus, $|\xi| \le 1$, which means that the linearized scheme is unconditionally stable.

4. Results and Discussion

In this part, two examples are given and L_∞ and L_2 error norms were calculated by

$$L_\infty = \max_i |v_i - V_i| \text{ and } L_2 = \sqrt{h\left(\sum_i^n |v_i - V_i|^2\right)}$$

The conservation laws were applied to Eq.(1) as follows [Evans and Raslan,2005]:

$$I_1 = \int_a^b \mathrm{v(x,t)dx}, I_2 = \int_a^b \mathrm{v(x,t)}^2\mathrm{dx}, I_3 = \int_a^b [\mathrm{v(x,t)}^2 + \tfrac{1}{3}\mathrm{v(x,t)}^3]\,\mathrm{dx},$$

Where I_1 , I_2 , I_3 correspond to the mass, momentum and energy, respectively.

4.1 Problem.1

The modified equal width equation problem (Karakoç and Geyikli, 2012) were considered with $\varepsilon = 3$ and $\mu = 1$,

$$v_t + 3v^2v_x - v_{xxt} = 0 \qquad 0 \le x \le 80$$

The exact solitary wave solution of MEW equation is $v(\mathrm{x,t}) = \mathrm{A\,sech(k(x-ct-x_0))}$ where c is the wave velocity, $c = \dfrac{\varepsilon A^2}{6}$ and $k = \sqrt{\dfrac{1}{\mu}}$. with initial condition $v(\mathrm{x},0) = \mathrm{A\,sech(k(x-x_0))}$ and boundary conditions $v(0,\mathrm{t}) = 0, \mathrm{v}(80,\mathrm{t}) = 0$.

The (C.T.B-Spline) method was used to compute the numerical solutions of this problem. For the purpose of comparison, the numerical results obtained in this paper were found to be more accurate in comparison to (Esen and Kutluay ,2008) (see Table 2). L_∞ and L_2 errors at different time levels and I_1 , I_2 and I_3 with A=0.25, $x_0 = 30$, $\Delta t = 0.2$ and $\Delta x = 0.1$ are also shown .It is important to point out that to make our method more accurate, we used smaller time steps. It can clearly be seen that the suggested method achieves remarkable reduction in errors for the smaller time step (see Table 3) with $\Delta t = 0.05$..

Table 2: Comparison error norms and invariants for single wave at $\Delta t = 0.2$.

	Proposed Method					Esen&Kutluay				
t	L_2	L_∞	I_1	I_2	I_3	$L_2\times10^4$	$L_\infty\times10^4$	I_1	I_2	I_3
0	3.03E-17	2.77E-17	0.78539	0.12500	0.11111	0.00012	0.000106	0.785397	0.166473	0.00520
5	0.008378	0.005696	0.78541	0.125028	0.11113	0.68298	0.610149	0.785397	0.1664731	0.00520
10	0.016735	0.0114185	0.78462	0.12484	0.11097	1.36286	1.25559	0.78539	0.166473	0.00520
15	0.02507	0.01713	0.78305	0.12447	0.11063	2.03675	1.91682	0.78539	0.166473	0.00520
20	0.03340	0.02284	0.78078	0.123918	0.11013	2.70164	2.57637	0.78539	0.166473	0.00520

Table 3: Comparison error norms and invariants for single wave at $\Delta t = 0.05$.

	Proposed Method					Esen&Kutluay				
t	L_2	L_∞	I_1	I_2	I_3	$L_2 \times 10^4$	$L_\infty \times 10^4$	I_1	I_2	I_3
0	3.03E-17	2.77E-17	0.7853981	0.125000	0.111111	0.000121	0.000106	0.78539	0.16661	0.00520
5	0.008386	0.005619	0.7850957	0.124927	0.111045	0.682986	0.610149	0.78537	0.16660	0.00520
10	0.016769	0.011277	0.783990	0.124645	0.110794	1.362867	1.255591	0.78534	0.16659	0.00520
15	0.025154	0.016939	0.782119	0.124170	0.110367	2.036756	1.916829	0.78537	0.16659	0.00520
20	0.033548	0.022591	0.779544	0.123521	0.109786	2.701647	2.576377	0.78528	0.16658	0.00520

4. 2. Problem 2

To verify from our method is superior when compare with earlier work (Zaki, 200) we choose the a=0, b=70 with the parameter values for A are 0.25, 0.5 and 1 with $h = 0.1$, $\Delta t = 0.05$ with different times and $x_0 = 30$ were chosen. These are shown in Table 4 with I_1, I_2 and I_3. The results were compared with the previous researcher's results, and these showed that our method was clearly more accurate and efficient. Fig.2shows the space-time graph approximate solution. There is a high correlation with their exact solution at A=0.5 and t=5.

Table 4:. Error norms and invariants for single wave at different A value and different times

t	A	L_∞	L_2	I_1		I_2	I_3
0	0.25	0	0	0.785398		0.125000	0.111111
5		0.005619	0.008386	0.785097		0.124927	0.111045
10		0.011277	0.016769	0.783990		0.124645	0.110794
15		0.016939	0.025154	0.782119		0.124170	0.110367
20		0.022591	0.033548	0.775440		0.123521	0.109786
20[Saka] $\times 10^3$		0.24989	0.29051	0.784954		0.166476	0.005199
0	0.5	0	0	1.570796		0.499999	0.444444
5		0.045691	0.066810	1.561564		0.495675	.4405548
10		0.089453	0.133714	1.532299		0.482077	0.428279
15		0.130358	0.202115	1.493339		0.466062	0.413736
20		0.175771	0.272452	1.452608		0.451844	0.400865
20[Zaki,] $\times 10^3$		0.00852	0.01172	1.57078		0.666666	0.083333
0	1.0	0	0	3.141592		1.999999	1.777778
5		0.328219	0.517598	2.972905		1.896341	1.682524
10		0.616162	1.028549	2.768103		1.844196	1.642989
15		0.810218	1.423740	2.692293		1.878757	1.685219
20		0.910155	1.678853	2.693549		1.946831	1.753986
20[Zaki,] $\times 10^3$		0.08360	0.14465	3.14165		2.66676	1.33343

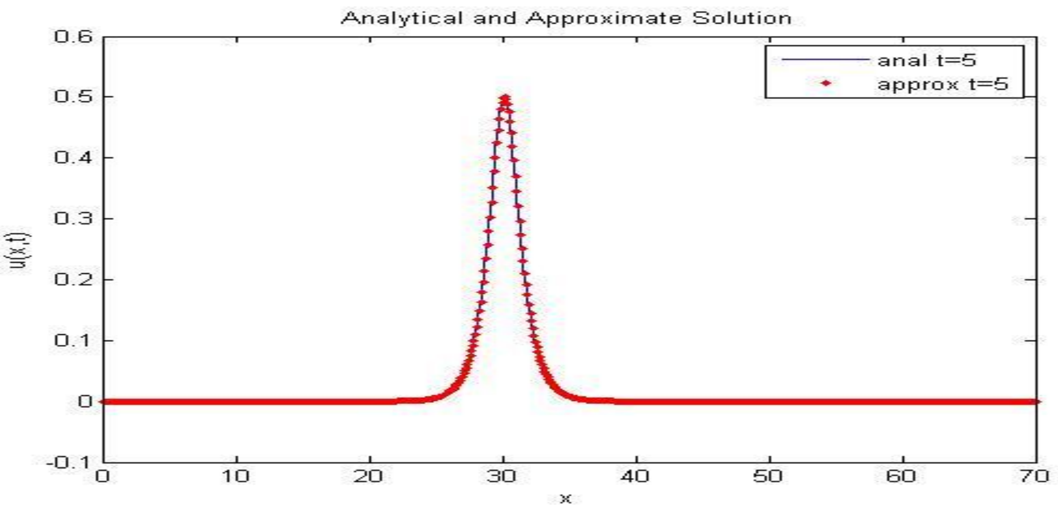

Fig.2L: . Approximate and exact solution for single solitary wave at A=0.5.

5. Conclusion

In this study, the (C. T.-B-spline) was used to solve the modified equal width equation. The performance and accuracy of the scheme was achieved by calculating L_∞ and L_2 errors at different time levels and I_1 , I_2 and I_3 on the motion of a single solitary wave through two problems has been compared with existing methods by calculating L_∞ and L_2 . The comparison indicated improved accuracy compared to (C. T.-B-spline) .The von Neumann method was used to check the stability analysis of the proposed method and it is shown that the solution is unconditionally stable.

Acknowledgements

The authors are indebted to the anonymous reviewers for their helpful valuable esteemed comments and suggestion in the improvement of this manuscript

References

[1] Arora, G., Mittal, R. C., & Singh, B. K. (2014). Numerical solution of BBM-Burger equation with quartic B-spline collocation method. *J. Engg. Sci. Tech,9* (3), 104-116.

[2] Ersoy, O., & Dag, I. (2016). A Trigonometric Cubic B-spline Finite Element Method for Solving the Nonlinear Coupled Burger Equation. *arXiv preprint arXiv:1604.04419.*

[3] Esen, A., &Kutluay, S. (2008). Solitary wave solutions of the modified equal width wave equation. *Communications in Nonlinear Science and Numerical Simulation*, *13*(8), 1538-1546.

[4] Evans, D. J., &Raslan, K. R. (2005). Solitary waves for the generalized equal width (GEW) equation. *International Journal of Computer Mathematics*, *82*(4), 445-455.

[5] Geyikli, T., &Karakoc, S. B. G. (2011). Septic B-spline collocation method for the numerical solution of the modified equal width wave equation. *Applied mathematics*, *2*(06), 739.

[6] Islam, S. U., Haq, F. I., &Tirmizi, I. A. (2010). Collocation Method Using Quartic B-spline for Numerical Solution of the Modified Equal Width Wave Equation. *Journal of applied mathematics & informatics*, *28*(3_4), 611-624.

[7] Karakoç, S. B. G., &Geyikli, T. (2012). Numerical solution of the modified equal width wave equation. *International journal of differential equations*, *2012*.

[8] Sajjadian, M. (2012). Numerical solutions of Korteweg de Vries and Korteweg de Vries-Burger's equations using computer programming. *arXiv preprint arXiv:1209.1782*.

[9] Saka, B. (2007). Algorithms for numerical solution of the modified equal width wave equation using collocation method. *Mathematical and Computer Modelling*, *45*(9), 1096-1117.

[10] Salih, H. M., Tawfiq, L. N. M., &Yahya, Z. R. (2016). Using Cubic Trigonometric B-Spline Method to Solve BBM-Burger Equation. IWNEST Conference Proceedings, 2, 1-9.

[11] Zaki, S. I. (2000). Solitary wave interactions for the modified equal width equation. *Computer Physics Communications*, *126*(3), 219-231 Saka, B. (2007). Algorithms for numerical solution of the modified equal width wave equation using collocation method. *Mathematical and Computer Modelling*, *45*(9), 1096-1117.