Squeezing properties of nondegenerate three-level laser with degenerate parametric amplifier

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Abstract

We study the squeezing properties of the light generated by nondegenerate three-level laser in which the three-level atoms are in cascade configuration with degenerate parametric oscillator. We obtain the c-number Langevin equations associated with the normal ordering using the pertinent master equation. Making use of the solutions of the c-number Langevin equations, we calculate the quadrature variances. The results show that the two-mode light produced by the system under consideration is in squeezed state and the squeezing occurs in the minus quadrature. The degree of squeezing increase with the linear gain coefficient and almost perfect squeezing can be obtained for large values of the linear gain coefficient and the atoms initially prepared in lower level slightly greater than in the upper level. Moreover, the presence of nonlinear crystal increases the degree of squeezing for single mode and two-mode lights.

Keywords: *Three-Level Laser; Parametric Oscillator; Quadrature Squeezing.*

1. Introduction

Three-level laser is a quantum optical system in which three-level atoms in a cascade configuration, initially prepared in coherent superposition of the top and bottom levels are injected into the cavity coupled to a vacuum reservoir via a single-port mirror. When a three-level atom makes a transition from the top to bottom level via the intermediate level, two photons are generated. If the generated light modes have the same frequencies, it is called degenerate three level laser; otherwise, it is called non-degenerate three-level laser. The two generated photons are highly correlated, and this correlation is responsible for the squeezing of the light produced by the three-level laser [1] - [3]. Squeezed states are non-classical states characterized by reduction of quantum fluctuations (noise) in one quadrature below that of the quantum standard limit (QSL) or below that achievable in coherent state at the expense increased fluctuations in another component such that the product of these fluctuations still obeys the uncertainty relation. This means that when we turn on the squeezed light, we see less noise than no light at all. This is apparently paradoxical feature which is a direct consequence of quantum nature of light and which cannot be explained with in the classical frame work [4] [5].

An optical parametric oscillator (OPO) is a quantum optical system most widely used to generate squeezed coherent state and entangled state of light. This quantum optical system consists of a non-linear crystal pumped by coherent light and the cavity modes are coupled to a vacuum reservoir via a single-port mirror. In parametric oscillator a pump photon of frequency 2ω is down-converted into a pair of correlated photons. If the produced paired of photons are with the same frequency, the system is called degenerate parametric oscillator; otherwise it is called non-degenerate parametric oscillator [6].

Berihu Tekle has studied the squeezing properties of light produced by degenerate parametric oscillator coupled to squeezed vacuum reservoir and the cavity mode is driven by coherent light. The study has shown that the driving coherent light has no effect on the squeezing properties of the cavity mode. However, both the squeezed vacuum reservoir and the driving coherent light increase the mean photon number of the cavity mode [7]. Fesseha. K has studied the squeezing properties of the light produced by a degenerate three-level laser with the atoms in a closed cavity and pumped by electron bombardment. He has shown that the maximum quadrature squeezing of the light generated by the laser, operating far below threshold, is 50% below the coherent-state level. [8] - [9].

Sintayehu Tesfa studied the squeezing and entanglement properties of light produced by degenerate parametric oscillator coupled to a two-mode vacuum reservoir. The investigation indicates that there is a quadrature entanglement between the harmonically related fundamental and residual pump modes where the super imposed radiation exhibits a higher degree of two-mode squeezing. It turned out that the two-mode squeezing can exist when there is no entanglement, since the correlations leading to these phenomena are essentially different. He has also shown that the more the external pumping radiation is down-converted by the nonlinear crystal, the stronger the entanglement and intensity of the two-mode radiation would be; this condition is not generally true for squeezing [10].

2. Objectives

2.1. General objective

- To study the squeezing properties of the radiation produced by non-degenerate three level laser whose cavity contains degenerate parametric amplifiers.

2.2. Specific objective

- To drive master equation of the system under consideration
- To obtain normally ordering c-number langevin and the solution of these equations
- To calculate the quadrature variance of the cavity modes

3. Methods

We first obtain c-number langevin equations associated with the normal ordering for the cavity mode variables using the pertinent Master equation. With the aid of the resulting equations and correlation properties of the noise forces, we calculate the quadrature variances of the cavity modes.
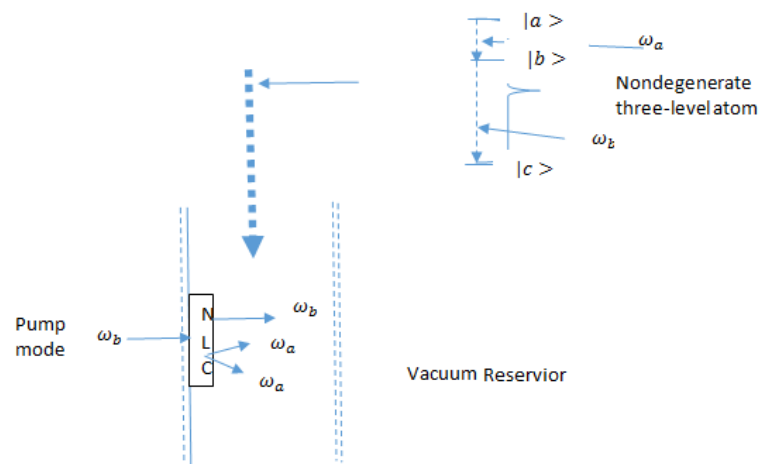

Fig. 1: Schematic Representation of Nondegenerate Three-Level Laser with Degenerate Parametric Oscillator.

3.1. Nondegenerate three-level laser

The interaction of the three-level atom with the cavity modes can be described by the Hamiltonian in the interaction picture by

$$\hat{H} = ig\left(|a\rangle\langle b|\hat{a} - \hat{a}^+|b\rangle\langle a| + |b\rangle\langle c|\hat{b} - \hat{b}^+|c\rangle\langle b|\right), \tag{1}$$

Where g is the coupling constant and $\hat{a}$ and $\hat{b}$ are annihilation operators for the cavity modes. We take initially the state of the atom to be

$$|\psi_A(0)\rangle = C_a|a\rangle + C_c|c\rangle. \tag{2}$$

The density operator corresponding to this state for a single atom is

$$\hat{\rho}_A(0) = \rho_{aa}^{(0)}|a\rangle\langle a| + \hat{\rho}_{ac}^{(0)}|a\rangle\langle c| + \hat{\rho}_{ca}^{(0)}|c\rangle\langle a| + \hat{\rho}_{cc}^{(0)}|c\rangle\langle c|, \tag{3}$$

Where

$$\rho_{aa}^{(0)} = C_a^* C_a \ , \ \rho_{cc}^{(0)} = C_c^* C_c \tag{4}$$

Are the probability of finding the atom in the upper level and in the lower level, respectively, and

$$\rho_{ac}^{(0)} = C_a^* C_c, \ or \ \left|\rho_{ac}^{(0)}\right|^2 = \rho_{aa}^{(0)}\rho_{cc}^{(0)}. \tag{5}$$

Suppose $\hat{\rho}_{AC}(t, t_j)$ is the density operator for a single atom injected at time t_j such that extended from $(t-\tau)$ to t and the cavity modes at time t. Then the density operator for all atoms in the cavity modes at time t can be written as

$$\hat{\rho}_{AC}(t) = r_a \sum_j \hat{\rho}_{AC}(t, t_j)\Delta t_j, \tag{6}$$

Where r_a represents the rate at which the atoms are injected into the cavity. Now converting the summation into integration in the limit $\Delta t_j \to 0,$ we have

$$\hat{\rho}_{Ac}(t) = r_a \int_{t-\tau}^{t} \hat{\rho}_{Ac}(t,t')dt',\tag{7}$$

And on differentiating with respect to t, we see that

$$\frac{d}{dt}\hat{\rho}_{Ac}(t) = r_a(\hat{\rho}_{Ac}(t,t) - \hat{\rho}_{Ac}(t,t-\tau)) + r_a\int_{t-\tau}^{t}\frac{\partial}{\partial t}\hat{\rho}_{Ac}(t,t')dt'.\tag{8}$$

We observe that $\hat{\rho}_{AC}(t,t)$ is the density operator for the cavity mode with an atom being injected into the cavity at time t and $\hat{\rho}_{AR}(t,t-\tau)$ represents the density operator for an atom and the cavity mode at time t with the atom being removed from the cavity at this time. Hence these operators can be put in the form

$$\hat{\rho}_{Ac}(t,t) = (\hat{\rho}_A(0)\hat{\rho}(t), \ \hat{\rho}_{Ac}(t,t-\tau) = \hat{\rho}_A(t-\tau)\hat{\rho}(t),\tag{9}$$

With $\hat{\rho}(t)$ being the density operator for the cavity modes alone. On account of Eqs. (9), we see that

$$\frac{d}{dt}\hat{\rho}_{Ac}(t) = r_a(\hat{\rho}_A(0) - \hat{\rho}_A(t-\tau))\hat{\rho}(t) + r_a\int_{t-\tau}^{t}\frac{\partial}{\partial t}\hat{\rho}_{Ac}(t,t')dt'.\tag{10}$$

In the absence of damping for the cavity modes by a vacuum reservoir, the density operator $\hat{\rho}_{AC}(t,t')$ involves in time is

$$\frac{\partial}{\partial t}\hat{\rho}_{Ac}(t,t') = -i\left[\hat{H},\hat{\rho}_{Ac}(t,t')\right].\tag{11}$$

Using this relation along with Eq. (7), Eq. (10) can be written as

$$\frac{d\hat{\rho}_{Ac}(t)}{dt} = r_a(\hat{\rho}_A(0) - \hat{\rho}_A(t-\tau))\hat{\rho}(t) - i\left[\hat{H},\hat{\rho}_{Ac}(t)\right].\tag{12}$$

Upon tracing over atomic variables and taking into account the fact that $Tr_A\hat{\rho}_A(t) = Tr_A\hat{\rho}_A(t-\tau) = 1$, we have

$$\frac{d\hat{\rho}(t)}{dt} = -iTr_A\left[\hat{H},\hat{\rho}_{Ac}(t)\right].\tag{13}$$

Employing Eq. (1), we can write in the form

$$\frac{d\hat{\rho}(t)}{dt} = g(\hat{a}\hat{\rho}_{ba} - \hat{a}^+\hat{\rho}_{ab} + \hat{b}\hat{\rho}_{cb} - \hat{b}^+\hat{\rho}_{bc} - \hat{\rho}_{ba}\hat{a} + \hat{\rho}_{ab}\hat{a}^+ - \hat{\rho}_{cb}\hat{b} + \hat{\rho}_{bc}\hat{b}^+,\tag{14}$$

In which the matrix element $\hat{\rho}_{\alpha\beta} = \langle\alpha|\hat{\rho}_{ac}(t)|\beta\rangle, a,b,c$ with $\alpha,\ \beta\ = a,b,c.$

We next proceed to determine the matrix elements involved in Eq. (14). Multiplying Eq. (12) on the left by $\langle\alpha|$ and on the right by $|\beta\rangle$, we see that

$$\frac{d\hat{\rho}_{Ac}(t)}{dt} = r_a(\langle\alpha|\hat{\rho}_A(0)|\beta\rangle - \langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle)\hat{\rho}(t) - i\langle\alpha|\left[\hat{H},\hat{\rho}_{Ac}(t)\right]|\beta\rangle - \gamma\hat{\rho}_{\alpha\beta}.\tag{15}$$

In which the last term is included to account for the decay of the atoms due to spontaneous emission, and, γ considered to be the same, is the atomic decay rate. We assume that the atoms are removed from the cavity after they have decayed to the level other than middle and bottom level. We then see that

$$\langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle = 0.\tag{16}$$

With the aid of Eqs. (1), (3), and (15), we easily find that

$$\frac{d\hat{\rho}_{ab}}{dt} = g\left(\hat{a}\hat{\rho}_{bb} - \hat{\rho}_{aa}\hat{a} + \hat{\rho}_{ac}\hat{b}^+\right) - \gamma\hat{\rho}_{ab.}\tag{17}$$

$$\frac{d\hat{\rho}_{ab}}{dt} = g\left(\hat{b}\hat{\rho}_{cc} - \hat{\rho}_{bb}\hat{b} - \hat{a}^+\hat{\rho}_{ac}\right) - \gamma\hat{\rho}_{bc.}\tag{18}$$

$$\frac{d\hat{\rho}_{ac}}{dt} = r_a\hat{\rho}_{ac}^{(0)}\hat{\rho} + g\left(\hat{a}\hat{\rho}_{bc} - \hat{\rho}_{ab}\hat{b}\right) - \gamma\hat{\rho}_{ac.}\tag{19}$$

$$\frac{d\hat{\rho}_{aa}}{dt} = r_a \hat{\rho}_{aa}^{(0)} \hat{\rho} + g\left(\hat{a}\hat{\rho}_{ba} + \hat{\rho}_{ab}\hat{a}^+\right) - \gamma\hat{\rho}_{aa},$$ (20)

$$\frac{d\hat{\rho}_{bb}}{dt} = g\left(\hat{b}\hat{\rho}_{cb} - \hat{a}^+\hat{\rho}_{ab} - \hat{\rho}_{ba}\hat{a} + \hat{\rho}_{bc}\hat{b}^+\right) - \gamma\hat{\rho}_{bb},$$ (21)

$$\frac{d\hat{\rho}_{cc}}{dt} = r_a \hat{\rho}_{cc}^{(0)} \hat{\rho} - g\left(\hat{b}^+\hat{\rho}_{bc} + \hat{\rho}_{cb}\hat{b}^+\right) - \gamma\hat{\rho}_{cc},$$ (22)

We confine ourselves to linear analysis and this can be achieved by dropping the g terms in Eqs. (19) - (22). Dropping the g terms in Eqs. (19) - (22) and then applying the large time approximation scheme, we get

$$\hat{\rho}_{ac} = \frac{r_a \hat{\rho}_{ac}^{(0)} \hat{\rho}}{\gamma},$$ (23)

$$\hat{\rho}_{aa} = \frac{r_a \hat{\rho}_{aa}^{(0)} \hat{\rho}}{\gamma},$$ (24)

$$\hat{\rho}_{bb} = 0,$$ (25)

$$\hat{\rho}_{cc} = \frac{r_a \hat{\rho}_{cc}^{(0)} \hat{\rho}}{\gamma}.$$ (26)

Substituting the above results into Eqs. (17) and (18), and pplying once more the large time approximation, we get

$$\hat{\rho}_{ab} = \frac{gr_a}{\gamma^2}\left(\hat{\rho}_{ac}^{(0)}\hat{\rho}\hat{b}^+ - \hat{\rho}_{aa}^{(0)}\hat{\rho}\hat{a}\right),$$ (27)

$$\hat{\rho}_{bc} = \frac{gr_a}{\gamma^2}\left(\hat{\rho}_{cc}^{(0)}\hat{b}\hat{\rho} - \hat{\rho}_{ac}^{(0)}\hat{a}^+\hat{\rho}\right).$$ (28)

Finally, on account of these results, Eq. (16) can be put in the form

$$\frac{d\hat{\rho}(t)}{dt} = \frac{A\hat{\rho}_{aa}^{(0)}}{2}\left(2\hat{a}^+\hat{\rho}\hat{a} - \hat{a}\hat{a}^+\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^+\right) + \frac{A\hat{\rho}_{cc}^{(0)}}{2}\left(2\hat{b}\hat{\rho}\hat{b}^+ - \hat{\rho}\hat{b}^+\hat{b} - \hat{b}^+\hat{b}\hat{\rho}\right)$$
$$- \frac{A\hat{\rho}_{ac}^{(0)}}{2}\left(2\hat{a}^+\hat{\rho}\hat{b}^+ - \hat{b}^+\hat{a}^+\hat{\rho} - \hat{\rho}\hat{b}^+\hat{a}^+\right) - \frac{A\hat{\rho}_{ca}^{(0)}}{2}\left(2\hat{b}\hat{\rho}\hat{a} - \hat{a}\hat{b}\hat{\rho} - \hat{\rho}\hat{a}\hat{b}\right),$$ (29)

In which

$$A = \frac{2g^2 r_a}{\gamma^2}$$ (30)

Is referred to as the atomic linear gain coefficient. Eq. (30) represents the evolution of the density operator for nondegenerate three-level laser.

3.2. Degenerate parametric oscillator

With the pump and driving residue modes which is emerged from nonlinear crystal without down converted treated classically, the sub-harmonic degenerate parametric oscillator and the interaction of the driving light with the cavity mode can be described by the interaction Hamiltonian,

$$\hat{H}_{po} = \frac{1}{2}i\lambda\left(\hat{a}^{+2} - \hat{a}^2\right) + i\varepsilon\left(\hat{b}^+ - \hat{b}\right),$$ (31)

In which λ and ε are the amplitudes proportional to the pump and driving residue modes, respectively, Hence the equation of evolution of the density operator corresponding to this Hamiltonian is

$$\frac{d\hat{\rho}(t)}{dt} = \frac{1}{2}\lambda(\hat{a}^{+2}\hat{\rho} - \hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^{+2} + \hat{\rho}\hat{a}^2) + \varepsilon(\hat{b}^+\hat{\rho} - \hat{b}\hat{\rho} - \hat{\rho}\hat{b}^+ + \hat{\rho}\hat{b}).$$ (32)

Eq. (32) represents the equation of evolution of the density operator for the degenerate parametric oscillator.

3.3. Nondegenerate three-level laser with degenerate parametric oscillator coupled to vacuum reservoir

With the aid of Eqs. (29), and (32), the master equation for nondegenerate three-level laser with degenerate parametric oscillator coupled to vacuum reservoir can be expressed as

$$\frac{d\hat{\rho}(t)}{dt} = \frac{1}{2} A\rho_{aa}^{(0)} \left(2\hat{a}^+\hat{\rho}\hat{a} - \hat{a}\hat{a}^+\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^+\right) + \frac{1}{2}\left(\kappa + A\rho_{cc}^{(0)}\right)\left(2\hat{b}\hat{\rho}\hat{b}^+ - \hat{\rho}\hat{b}^+\hat{b} - \hat{b}^+\hat{b}\hat{\rho}\right)$$

$$-\frac{1}{2} A\rho_{ac}^{(0)} \left(2\hat{a}^+\hat{\rho}\hat{b}^+ - \hat{b}^+\hat{a}^+\hat{\rho} - \hat{\rho}\hat{b}^+\hat{a}^+\right) - \frac{1}{2}A\rho_{ac}^{(0)}\left(2\hat{b}\hat{\rho}\hat{a} - \hat{a}\hat{b}\hat{\rho} - \hat{\rho}\hat{a}\hat{b}\right) + \frac{1}{2}\kappa\left(2\hat{a}\hat{\rho}\hat{a}^+ - \hat{a}^+\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^+\hat{a}\right) \tag{33}$$

$$+\frac{1}{2}\lambda(\hat{a}^{+2}\hat{\rho} - \hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^{+2} + \hat{\rho}\hat{a}^2) + \varepsilon\left(\hat{b}^+\hat{\rho} - \hat{b}\hat{\rho} - \hat{\rho}\hat{b}^+ + \hat{\rho}\hat{b}\right).$$

3.4. C-number langevien equation

We next proceed to obtain the c-number Langevin equations for the cavity mode variables. To this end, employing Eq. (33) and the relation: $\frac{d}{dt}\langle\hat{A}\rangle = Tr\left(\frac{d}{dt}\hat{\rho}\hat{A}\right)$.

Along with the cyclic property of the trace operation, we easily find the c-number equations corresponding to the normal ordering as the following

$$\frac{d}{dt}\langle\alpha\rangle = -\frac{1}{2}\mu_a\langle\alpha\rangle - \frac{1}{2}\nu\langle\beta^*\rangle + \lambda\langle\alpha^*\rangle, \tag{34}$$

$$\frac{d}{dt}\langle\beta\rangle = -\frac{1}{2}\mu_c\langle\beta\rangle + \frac{1}{2}\nu\langle\alpha^*\rangle + \varepsilon, \tag{35}$$

$$\frac{d}{dt}\langle\alpha^2\rangle = -\mu_a\langle\alpha^2\rangle - \nu\langle\beta^*\alpha\rangle + 2\lambda\langle\alpha^*\alpha\rangle + \lambda, \tag{36}$$

$$\frac{d}{dt}\langle\beta^2\rangle = -\mu_c\langle\beta^2\rangle + \nu\langle\alpha^*\beta\rangle + 2\varepsilon\langle\beta\rangle, \tag{37}$$

$$\frac{d}{dt}\langle\alpha^*\alpha\rangle = -\mu_a\langle\alpha^*\alpha\rangle - \frac{1}{2}\nu(\langle\alpha^*\beta^*\rangle + \langle\alpha\beta\rangle) + \lambda(\langle\alpha^{*2}\rangle + \langle\alpha^2\rangle) + A\rho_{aa}^{(0)} \tag{38}$$

$$\frac{d}{dt}\langle\beta^*\beta^*\rangle = -\mu_c\langle\beta^*\beta\rangle + \frac{1}{2}\nu(\langle\alpha^*\beta^*\rangle + \langle\alpha\beta\rangle) + \varepsilon(\langle\beta^*\rangle + \langle\beta\rangle), \tag{39}$$

$$\frac{d}{dt}\langle\alpha\beta\rangle = -\frac{1}{2}(\mu_a + \mu_c)\langle\alpha\beta\rangle + \frac{1}{2}\nu(\langle\alpha^*\alpha\rangle - \langle\beta^*\beta\rangle) + \lambda\langle\alpha^*\beta\rangle + \varepsilon\langle\alpha\rangle + \frac{1}{2}\nu, \tag{40}$$

$$\frac{d}{dt}\langle\alpha^*\beta\rangle = -\frac{1}{2}(\mu_a + \mu_c)\langle\alpha^*\beta\rangle + \frac{1}{2}\nu(\langle\alpha^{*2}\rangle - \langle\beta^2\rangle) + \lambda\langle\alpha\beta\rangle + \varepsilon\langle\alpha^*\rangle. \tag{41}$$

Where

$$\mu_a = \kappa - A\rho_{aa}^{(0)}, \ \mu_c = \kappa + A\rho_{cc}^{(0)}, \ \nu = A\rho_{ac}^{(0)}. \tag{42}$$

On the base of Eqs. (34) and (35), we can write as

$$\frac{d}{dt}\alpha = -\frac{1}{2}\mu_a\alpha - \frac{1}{2}\nu\beta^* + \lambda\alpha^* + f_\alpha(t), \tag{43}$$

$$\frac{d}{dt}\beta = -\frac{1}{2}\mu_c\beta + \frac{1}{2}\nu\alpha^* + \varepsilon + f_\beta(t), \tag{44}$$

Where $f_\alpha(t)$ and $f_\beta(t)$ are the noise forces corresponding to the normal order and the properties of them to be determine. The formal solutions of Eqs. (43) and (44) can be put in the form

$$\alpha(t) = \alpha(0)e^{-\mu_a t/2} + \int_0^t dt' e^{\frac{-1}{2}\mu_a(t-t')}\left(-\frac{1}{2}\nu\beta^*(t') + \lambda\alpha^*(t') + f_\alpha(t')\right), \tag{45}$$

$$\beta(t) = \beta(0)e^{-\mu_c t/2} + \int_0^t dt' e^{\frac{-1}{2}\mu_c(t-t')}\left(-\frac{1}{2}\nu\alpha^*(t') + \varepsilon + f_\beta(t')\right). \tag{46}$$

We next proceed to determine the properties of the noise forces. We note that Eq. (34) and the expectation value of Eq. (45) as well as Eq. (35) and the expectation value of (46) will have the same form provided that

$$\langle f_\alpha(t)\rangle = \langle f_\beta(t)\rangle = 0. \tag{47}$$

Employing the relation $\dfrac{d}{dt}\langle \alpha^2 \rangle = 2\left\langle \alpha \dfrac{d}{dt}\alpha \right\rangle$, along with Eq. (45), we see that

$$\frac{d}{dt}\langle \alpha^2 \rangle = -\mu_a \langle \alpha^2 \rangle - v\langle \beta^* \alpha \rangle + 2\lambda \langle \alpha^* \alpha \rangle + 2\langle \alpha(t)f(t)\rangle. \tag{48}$$

Comparison of Eq. (48) with Eq. (36) shows that

$$\langle \alpha(t)f(t)\rangle = \lambda/2. \tag{49}$$

In view of Eq. (45), we see

$$\langle \alpha(0)f_\alpha(t)\rangle e^{-\mu_a t/2} - \int_0^t dt' e^{\frac{-1}{2}\mu_a(t-t')}\left(\frac{1}{2}v\langle \beta^*(t')f_\alpha(t)\rangle - \lambda\langle \alpha^*(t')f_\alpha(t)\rangle - \langle f_\alpha(t')f_\alpha(t)\rangle \right) = \lambda/2. \tag{50}$$

Taking into account the fact that the noise forces at a certain instant do not affect the cavity mode variables at earlier time, Eq. (50) reduces to

$$\int_0^t dt' e^{\frac{-1}{2}\mu_a(t-t')}\langle f_\alpha(t')f_\alpha(t)\rangle = \lambda/2. \tag{51}$$

Or this equation can be rewritten in the form

$$\int_0^t dt' e^{\frac{-1}{2}\mu_a(t-t')}\langle f_\alpha(t')f_\alpha(t)\rangle = \int_0^t dt' e^{\frac{-1}{2}\mu_a(t-t')}\lambda\delta(t-t'). \tag{52}$$

It then follows that

$$\langle f_\alpha(t')f_\alpha(t)\rangle = \lambda\delta(t-t'). \tag{53}$$

Similarly, we can find

$$\langle f_\alpha^*(t)f_\alpha(t')\rangle = A\rho_{aa}^{(0)}\delta(t-t'). \tag{54}$$

$$\langle f_\alpha(t)f_\beta(t')\rangle = \frac{1}{2}v\delta(t-t'). \tag{55}$$

$$\langle f_\beta(t')f_\beta(t)\rangle = \langle f_\beta^*(t')f_\beta(t)\rangle = \langle f_\alpha^*(t')f_\beta(t)\rangle = 0, \tag{56}$$

Eqs. (47) , (53) - (56) are the correlation properties of the noise forces, $f_\alpha(t)$ and $f_\beta(t)$, associated with the normal ordering.

We next proceed to obtain the solutions of Eqs. (43) and (44) which are coupled differential equations. To this end, introducing new variables defined by

$$\alpha_\pm = \alpha^* \pm \alpha, \tag{57}$$

$$\beta_\pm = \beta^* \pm \beta \tag{58}$$

And using Eqs. (43) and (44), we can write in matrix as

$$\frac{d}{dt}\Upsilon_\pm(t) = -\frac{1}{2}M_\pm \Upsilon_\pm(t) + f_\pm(t), \tag{59}$$

In which

$$\Upsilon_\pm(t) = \begin{pmatrix} \alpha_\pm(t) \\ \beta_\pm(t) \end{pmatrix}, \tag{60}$$

$$M_\pm(t) = \begin{pmatrix} \mu_a \mp 2\lambda & \pm v \\ \mp v & \mu_c \end{pmatrix}, \tag{61}$$

$$f_\pm(t) = \begin{pmatrix} f_{a\pm}(t) \\ f_{\beta\pm}(t) + \varepsilon_\pm \end{pmatrix}. \tag{62}$$

With

$$f_{a\pm}(t) = f_\alpha^*(t) \pm f_\alpha(t), \quad f_{\beta\pm}(t) = f_\beta^*(t) \pm f_\beta(t), \text{ and } \varepsilon_\pm = \varepsilon \pm \varepsilon. \tag{63}$$

Introducing a unitary matrix defined by

$$U_{\pm} = \begin{pmatrix} u_{11\pm} & u_{12\pm} \\ u_{21\pm} & u_{22\pm} \end{pmatrix}, \tag{64}$$

With

$$u_{1\pm} = \begin{pmatrix} u_{11\pm} \\ u_{21\pm} \end{pmatrix} u_{2\pm} = \begin{pmatrix} u_{12\pm} \\ u_{22\pm} \end{pmatrix}, \text{ Being the eigenvectors of the matrix M± in Eq. (59), we see that}$$

$$\frac{d}{dt}\Upsilon_{\pm}(t) = -\frac{1}{2} U_{\pm} U_{\pm}^{-1} M_{\pm} U_{\pm} U_{\pm}^{-1} \Upsilon_{\pm}(t) + f_{\pm}(t) \tag{65}$$

Multiplying both sides on the left by $U_{\pm}^{-1}$, we have

$$\frac{d}{dt}(U_{\pm}^{-1}\Upsilon_{\pm}(t)) = -\frac{1}{2} R(U_{\pm}^{-1}\Upsilon_{\pm}(t) + U_{\pm}^{-1} f_{\pm}(t), \tag{66}$$

Where

$$R_{\pm} = U_{\pm}^{-1} M_{\pm} U_{\pm} = \begin{pmatrix} \lambda_{1\pm} & 0 \\ 0 & \lambda_{2\pm} \end{pmatrix}. \tag{67}$$

Here $\lambda_{1,2\pm}$ are the eigenvalues of the matrix $M_{\pm}$. We notel that Eq. (66) will have well defined solution if and only if both $\lambda_{1,2\pm}$ have a value greater than zero. Hence the solution of Eq. (66) can be put in the form

$$\Upsilon_{\pm}(t) = U_{\pm} e^{\frac{-1}{2} R_{\pm} t} U_{\pm} \Upsilon_{\pm}(0) + \int_0^t dt' U_{\pm} e^{\frac{-1}{2} R_{\pm}(t-t')} U_{\pm}^{-1} f_{\pm}(t'). \tag{68}$$

We next proceed to find out the eigenvalues and eigenvectors of the matrix $M_{\pm}$. With the aid of Eq. (61) and the eigenvalue equation, we easily find the characteristics equation

$$\lambda_{\pm}^2 - (\mu_a + \mu_c \mp 2\lambda)\lambda + \mu_c(\mu_a \mp 2\lambda) + v^2 = 0. \tag{69}$$

Taking into account Eq. (42) and $\rho_{aa}^{(0)} + \rho_{cc}^{(0)} = 1$, the roots of this quadratic equation can be

$$\lambda_{1\pm} = \frac{1}{2}\{2\kappa + A\eta \mp 2\lambda + \varepsilon_{\pm}\}, \tag{70}$$

$$\lambda_{2\pm} = \frac{1}{2}\{2\kappa + A\eta \mp 2\lambda - \varepsilon_{\pm}\}, \tag{71}$$

Where

$$\varepsilon_{\pm} = \sqrt{A^2\eta^2 + 4\lambda^2 \pm 4A\lambda}, \qquad \eta = \rho_{cc}^{(0)} - \rho_{aa}^{(0)}. \tag{72}$$

We next proceed to find the elements of eigenvectors for the matrix $M_{\pm}$. With the help of the eigenvalue equation along with Eqs. (61) and (70), we get

$$A_{1\pm} u_{11\pm} \mp 2v u_{21\pm} = 0, \tag{73}$$

Where

$$A_{1\pm} = A \pm 2\lambda + \varepsilon_{\pm}. \tag{74}$$

Taking into account the normalization condition: $u_{11\pm}^2 + u_{21\pm}^2 = 1$, we get

$$u_{11\pm} = \frac{\pm 2v}{\sqrt{A_{1\pm}^2 - 4v^2}}, \quad u_{21\pm} = \frac{A_{1\pm}}{\sqrt{A_{1\pm}^2 - 4v^2}}. \tag{75}$$

Following a similar procedure, we also easily find the elements of the eigenvector corresponding to the eigenvalue $\lambda_{2\pm}$ as

$$u_{12\pm} = \frac{\pm 2v}{\sqrt{A_{2\pm}^2 - 4v^2}}, \quad u_{22\pm} = \frac{A_{2\pm}}{\sqrt{A_{2\pm}^2 - 4v^2}}, \tag{76}$$

Where

$$A_{2\pm} = A \pm 2\lambda - \varepsilon_{\pm}. \tag{77}$$

Substituting Eqs. (75) and (76) into (64), we see that

$$U_{\pm} = \begin{pmatrix} \dfrac{\pm 2v}{\sqrt{A_{1\pm}^2 - 4v^2}} & \dfrac{\pm 2v}{\sqrt{A_{2\pm}^2 - 4v^2}} \\ \dfrac{A_{1\pm}}{\sqrt{A_{1\pm}^2 - 4v^2}} & \dfrac{A_{2\pm}}{\sqrt{A_{2\pm}^2 - 4v^2}} \end{pmatrix}. \tag{78}$$

With that aid of Eq. (67) which is a diagonal matrix Eq. (78) and its inverse, we find

$$U_{\pm} e^{\frac{-1}{2} R_{\pm} t} U_{\pm}^{-1} = \begin{pmatrix} p_{1\pm}(t) & q_{1\pm}(t) \\ q_{2\pm}(t) & p_{2\pm}(t) \end{pmatrix}, \tag{79}$$

$$U_{\pm} e^{\frac{-1}{2} R_{\pm}(t-t')} U_{\pm}^{-1} = \begin{pmatrix} p_{1\pm}(t-t') & q_{1\pm}(t-t') \\ q_{2\pm}(t-t') & p_{2\pm}(t-t') \end{pmatrix}, \tag{80}$$

Where

$$p_{1\pm}(t-t') = \frac{A_{1\pm}}{2\varepsilon_{\pm}} e^{\frac{-1}{2}\lambda_{2\pm}(t-t')} - \frac{A_{2\pm}}{2\varepsilon_{\pm}} e^{\frac{-1}{2}\lambda_{1\pm}(t-t')}, \tag{81}$$

$$p_{2\pm}(t-t') = \frac{A_{1\pm}}{2\varepsilon_{\pm}} e^{\frac{-1}{2}\lambda_{1\pm}(t-t')} - \frac{A_{2\pm}}{2\varepsilon_{\pm}} e^{\frac{-1}{2}\lambda_{2\pm}(t-t')}, \tag{82}$$

$$q_{1\pm}(t-t') = \pm \frac{2v}{2\varepsilon_{\pm}} e^{\frac{-1}{2}\lambda_{1\pm}(t-t')} \mp \frac{2v}{2\varepsilon_{\pm}} e^{\frac{-1}{2}\lambda_{2\pm}(t-t')}, \tag{83}$$

$$q_{2\pm}(t-t') = \mp \frac{2v}{2\varepsilon_{\pm}} e^{\frac{-1}{2}\lambda_{1\pm}(t-t')} \pm \frac{2v}{2\varepsilon_{\pm}} e^{\frac{-1}{2}\lambda_{2\pm}(t-t')}. \tag{84}$$

with t'= 0 and t'. With the help of Eqs. (68), (79), and (80), we finally obtain

$$\alpha_{\pm}(t) = p_{1\pm}(t)\alpha_{\pm}(0) + q_{1\pm}(t)\beta_{\pm}(0) + W_{1\pm}(t) + v_{1\pm}(t), \tag{85}$$

$$\beta_{\pm}(t) = p_{2\pm}(t)\beta_{\pm}(0) + q_{2\pm}(t)\alpha_{\pm}(0) + W_{2\pm}(t) + v_{2\pm}(t), \tag{86}$$

Where

$$W_{1\pm}(t) = \int_0^t \left\{ p_{1\pm}(t-t') f_{\alpha\pm}(t') + q_{1\pm}(t-t') f_{\beta\pm}(t') \right\} dt', \tag{87}$$

$$W_{2\pm}(t) = \int_0^t \left\{ p_{2\pm}(t-t') f_{\beta\pm}(t') + q_{2\pm}(t-t') f_{\alpha\pm}(t') \right\} dt', \tag{88}$$

$$v_{1\pm}(t) = \pm \frac{2v}{\lambda_{1\pm}}(1 - e^{-\lambda_{1\pm}t/2}) \mp \frac{2v}{\lambda_{2\pm}}(1 - e^{-\lambda_{2\pm}t/2}), \tag{89}$$

$$v_{2\pm}(t) = \pm \frac{A_{1\pm}}{\lambda_{1\pm}}(1 - e^{-\lambda_{1\pm}t/2}) - \frac{A_{2\pm}}{\lambda_{2\pm}}(1 - e^{-\lambda_{2\pm}t/2}). \tag{90}$$

3.5. Quadrature variances

In this section we seek to study the quadrature variances of the cavity modes produced by nondegenerate three-level laser with degenerate subharmonic parametric oscillate employing the solutions of the c-number Langevein equations and the correlation properties of the noise forces.

3.6. Quadrature variances of a single-mode light

The variances of the quadrature for a single-mode light defined by operators:

$$\hat{a}_{+} = \hat{a}^{+} + \hat{a}, \tag{91}$$

$$\hat{a}_{-} = i(\hat{a}^{+} - \hat{a}), \tag{92}$$

Can be written as

$$\Delta a_\pm^2 = \langle \hat{a}_\pm(t), \hat{a}_\pm(t) \rangle, \tag{93}$$

It can be easily verified that

$$[\hat{a}_+, \hat{a}_-] = 2i, \tag{94}$$

$$\Delta a_+ \Delta a_- \geq 1. \tag{95}$$

Eq. (59) can be expressed in term of c-number variables associated to the normal order as

$$\Delta a_\pm^2 = 1 \pm \langle \alpha_\pm(t), \alpha_\pm(t) \rangle. \tag{96}$$

Taking into account initially the cavity in vacuum state along with Eqs. (47) (85), and (87), we easily show that

$$\langle \alpha_\pm(t) \rangle = \nu_{1\pm}. \tag{97}$$

With the aid of Eqs. (85), and (87) along with the assumption that initially the cavity is in vacuum state and the fact that the noise forces don't affect the cavity mode variables at earlier time, we have

$$\langle \alpha_\pm, \alpha_\pm \rangle = \int_0^t \int_0^t \left\{ \begin{array}{l} p_{1\pm}(t-t')p_{1\pm}(t-t'')\langle f_{a\pm}(t')f_{a\pm}(t'') \rangle + p_{1\pm}(t-t')q_{1\pm}(t-t'')\langle f_{a\pm}(t')f_{\beta\pm}(t'') \rangle \\ + q_{1\pm}(t-t')p_{1\pm}(t-t'')\langle f_{\beta\pm}(t')f_{a\pm}(t'') \rangle \\ + q_{1\pm}(t-t')q_{1\pm}(t-t'')\langle f_{\beta\pm}(t')f_{\beta\pm}(t'') \rangle \end{array} \right\} dt'dt'' + \nu_{1\pm}^2(t). \tag{98}$$

Applying Eqs. (53) - (57) and carrying out the integration, we find at steady state

$$\langle \alpha_\pm, \alpha_\pm \rangle_{ss} = \frac{(\lambda \pm A\rho_{aa}^{(0)})A_{1\pm}^2 \mp 2\nu^2 A_{1\pm}}{2\varepsilon_\pm^2 \lambda_{2\pm}} + \frac{(\lambda \pm A\rho_{aa}^{(0)})A_{2\pm}^2 \mp 2\nu^2 A_{2\pm}}{2\varepsilon_\pm^2 \lambda_{1\pm}}$$
$$- \frac{4(\lambda \pm A\rho_{aa}^{(0)})A_{1\pm}A_{2\pm} \mp 4(A_{1\pm} + A_{2\pm})\nu^2}{2\varepsilon_\pm^2(\lambda_{1\pm} + \lambda_{2\pm})}. \tag{99}$$

Applying Eqs. (96) and (99) in Eq. (93), we finally at steady state find

$$\Delta a_\pm^2 = 1 \pm \frac{(\lambda \pm A\rho_{aa}^{(0)})A_{1\pm}^2 \mp 2\nu^2 A_{1\pm}}{2\varepsilon_\pm^2 \lambda_{2\pm}} \pm \frac{(\lambda \pm A\rho_{aa}^{(0)})A_{2\pm}^2 \mp 2\nu^2 A_{2\pm}}{2\varepsilon_\pm^2 \lambda_{1\pm}}$$
$$\mp \frac{4(\lambda \pm A\rho_{aa}^{(0)})A_{1\pm}A_{2\pm} \mp 4(A_{1\pm} + A_{2\pm})\nu^2}{2\varepsilon_\pm^2(\lambda_{1\pm} + \lambda_{2\pm})}. \tag{100}$$

Eq. (100) represents the quadrature variances for mode a. Following the same procedure, the quadrature variances for mode b is found to be

$$\Delta b_\pm^2 = 1 \pm \frac{4\nu^2(\lambda \pm A\rho_{aa}^{(0)}) \mp 2\nu^2 A_{1\pm}}{2\varepsilon_\pm^2 \lambda_{1\pm}} \pm \frac{4\nu^2(\lambda \pm A\rho_{aa}^{(0)}) \mp 2\nu^2 A_{2\pm}}{2\varepsilon_\pm^2 \lambda_{2\pm}}$$
$$\mp \frac{16(\lambda \pm A\rho_{aa}^{(0)})\nu^2 \mp 4(A_{1\pm} + A_{2\pm})\nu^2}{2\varepsilon_\pm^2(\lambda_{1\pm} + \lambda_{2\pm})}. \tag{101}$$

3.7. Quadrature variances of two-mode light

Here we wish to calculate the quadrature variances for a two-mode light produced by the system under consideration. The quadrature operators for a two-mode light are defined by

$$\hat{c}_\pm = \sqrt{\pm 1} \; (\hat{c}^+ \pm \hat{c}) \tag{102}$$

Where

$$\hat{c} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{b}). \tag{103}$$

The variances of a two-mode light defined by

$$\Delta c_\pm^2 = \langle \hat{c}_\pm, \; \hat{c}_\pm \rangle \tag{104}$$

Are expressible in terms of c-number variables associated with the normal ordering as

$$\Delta c_\pm^2 = \frac{1}{2}\Delta a_\pm^2 + \frac{1}{2}\Delta b_\pm^2 \pm \langle \alpha_\pm,\ \beta_\pm \rangle, \tag{105}$$

In which $\Delta a_\pm^2$ and $\Delta b_\pm^2$ are quadrature variances of mode a and mode b, respectively. Employing Eqs. (53) - (57), (85), (86) , (87) and (88) and taking into account the assumption that the cavity modes are initially in vacuum state and the fact that the noise forces at a certain instant do not affect the cavity mode variables at earlier time, we have

$$\langle \alpha_\pm(t),\ \beta_\pm(t) \rangle = \int_0^t \left\{ v(p_{1\pm}(t-t')\, p_{2\pm}(t-t') + q_{1\pm}(t-t')q_{2\pm}(t-t')) + 2(\lambda \pm A\rho_{aa}^{(0)})p_{1\pm}(t-t')q_{2\pm}(t-t') \right\} dt' \tag{106}$$

Performing the integration applying Eqs. (81) - (84), this equation becomes a steady state

$$\langle \alpha_\pm,\ \beta_\pm \rangle_{ss} = \pm \frac{4v(\lambda \pm A\rho_{aa}^{(0)})A_{1\pm} \mp v(A_{1\pm}A_{2\pm} + 4v^2)}{4\varepsilon_\pm^2 \lambda_{2\pm}}$$
$$\pm \frac{4v(\lambda \pm A\rho_{aa}^{(0)})A_{2\pm} \mp v(A_{1\pm}A_{2\pm} + 4v^2)}{4\varepsilon_\pm^2 \lambda_{1\pm}} \tag{107}$$
$$\mp \frac{8v(\lambda \pm A\rho_{aa}^{(0)})(A_{1\pm} + A_{2\pm}) \mp 2v(A_{1\pm}^2 + A_{2\pm}^2 + 8v^2)}{4\varepsilon_\pm^2(\lambda_{1\pm} + \lambda_{2\pm})}.$$

Using Eqs. (100), (101), and (107) in Eq. (104), we have

$$\Delta c_\pm^2 = 1 \pm \frac{(\lambda \pm A\rho_{aa}^{(0)})(A_{1\pm} + 2v)^2 \mp v(A_{1\pm} + 2v)(A_{2\pm} + 2v)}{4\varepsilon_\pm^2 \lambda_{2\pm}}$$
$$\pm \frac{(\lambda \pm A\rho_{aa}^{(0)})(A_{1\pm} + 2v)^2 \mp v(A_{1\pm} + 2v)(A_{2\pm} + 2v)}{4\varepsilon_\pm^2 \lambda_{1\pm}} \tag{108}$$
$$\mp \frac{4(\lambda \pm A\rho_{aa}^{(0)})(A_{1\pm} + 2v)(A_{2\pm} + 2v) \mp 2v(A_{1\pm} + 2v)^2 \mp 2v(A_{2\pm} + 2v)^2}{4\varepsilon_\pm^2(\lambda_{1\pm} + \lambda_{2\pm})}.$$

Eq. (108) represents the quadrature variances for two-mode light produced by nondegenerate three-level laser with degenerate parametric oscillator.
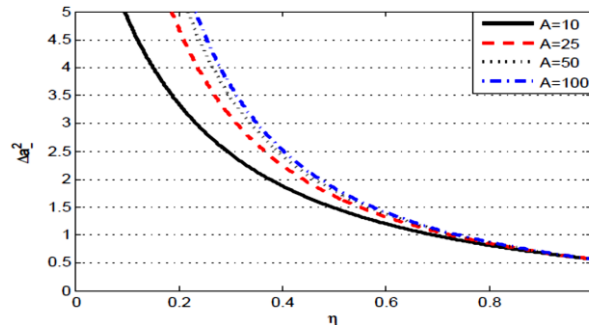
4. Result and discussion

Fig. 4.1: Plots of Represents the Minus Quadrature Variances [Eq. (100)] Versus η for $\kappa = 0.8$, $\lambda = 0.3$ for Different Values of Linear Gain Coefficient A.

Fig (4.1) represents the minus quadrature variances of mode a [Eq. (108)] versus η for $\kappa = 0.8$, $\lambda = 0$. and for A = 10 (solid), A = 25 (dashed), A = 50 (dotted) and A = 100 (dash-dotted). The figure shows that the quadrature variance increases as the linear gain coefficient increases. Moreover, the minimum quadrature variance for A = 10, $\kappa = 0.8$, $\lambda = 0.3$ is 0.5 at $\eta = 1$. It implies that 50% of maximum squeezing can be obtained for the given values.
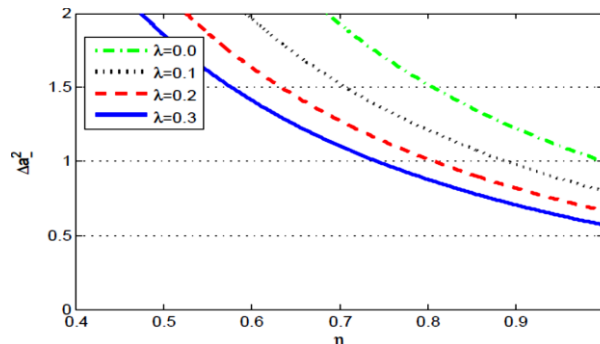

Fig. 4.2: Plots of the Minus Quadrature Variances [Eq. (100)] Versus η for $\kappa = 0.8$, A = 100 and for Different Values of λ.

In Fig. (4.2) we plot the minus quadrature variances [Eq.(100)] versus and for $\lambda = 0.0$ (dash-dot), $\lambda = 0.1$ (dotted), $\lambda = 0.2$ (dashed) and $\lambda = 0.3$ (solid). The plots indicate that the degree of squeezing increase with the amplitude proportional to the pump mode.
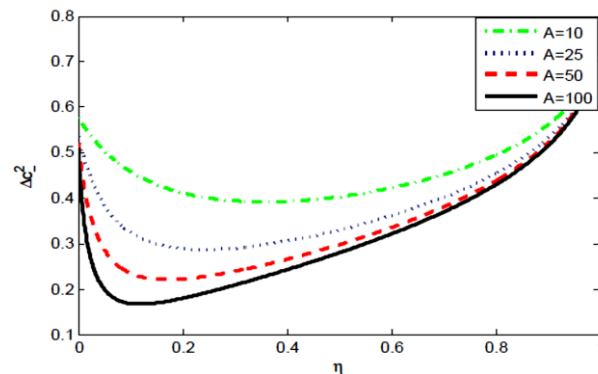

Fig. 4 3: Plots of the Minus Quadrature Variances [Eq. (108)] Versus η for $\kappa = 0.8$, $\lambda = 0$ and Different Values of Linear Gain Coefficients A.

Fig (4.3) is the plots of the minus quadrature variances for two-mode light [Eq.(108)] versus η for $\kappa = 0.8$, $\lambda = 0$ and for A = 10 (dash-dotted), A = 25 (dotted), A = 50 (dashed), and = 100 (solid). The figure shows that the degree of squeezing increases with the linear gain coefficient. Moreover, the minimum value of the quadrature variance described by Eq. (108) for A = 100, $\kappa = 0.8$ and $\lambda = 0$ is found to be $\Delta c_-^2 = 0.16$ and occurs at $\eta = 0.1$.This result implies that the maximum squeezing for the above values is 84% below the coherent state level.
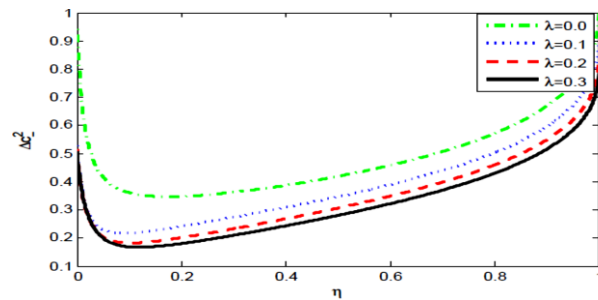

Fig. 4.4: Plots of the Minus Quadrature Variances [Eq. (108)] Versus η for A = 100, $\kappa = 0.8$, and Different Values of λ .

We Plot in Fig (4.4) the variances of the minus quadrature [Eq.(108)] versus η for A = 100, $\kappa = 0.8$, and for $\lambda = 0.0$ (dash-dot), $\lambda = 0.1$ (dot), $\lambda = 0.2$ (dash), and $\lambda = 0.3$ (solid). The figure indicates that the degree of squeezing increases with the amplitude proportional to pump mode. Moreover, the minimum value of the quadrature variance described by Eq. (108) for A = 100, $\kappa = 0.8$, and $\lambda = 0.3$ is found to be $\Delta c_-^2 = 0.14$ and occurs at $\eta = 0.1$. This result implies that the maximum squeezing for the above values is 86% below the coherent state level.

5. Conclusion

In this work we have studied the squeezing, properties of the cavity modes produced by nondegenerate three-level lasers with degenerate parametric oscillator. We have obtained, the c-number Langevin equations associated with the normal ordering using the master equation. Applying the solutions of the resulting Langevin equations, and the correlation properties noise forces, we have calculated the quadrature variances and the sum of the variances of EPR-like variables. The result shows that the two-mode light produced by the system under consideration is in squeezed state and the squeezing occurs in the minus quadrature. It is also found that the degree of squeezing increase with the linear gain coefficient. Moreover, the result shows that the minimum value of the quadrature variance for A = 100, $\kappa = 0.8$ and $\lambda = 0$ is found to be $\Delta c_-^2 = 0.16$ and occurs at $\eta = 0.1$. This result implies that the maximum squeezing for the above values is 84% below the coherent state level.

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