

Periodic wave analysis to the time-fractional phi-four and (2+1) dimensional CBS equations

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Abstract

In this paper, we investigate some innovative and exact travelling wave solutions to the time-fractional phi-four equation and the (2+1) dimensional Calogero-Bogoyavlanskiil schilf (CBS) equation using the (G'/G^2) -expansion method. Periodic solutions are displayed in hyperbolic, trigonometric, and rational function solutions. Subsequently, we construct some new solutions connecting the free parameters of the phi-four equation and the CBS equation, which are characterized into three complete forms: rational function, trigonometric function, and hyperbolic functions. Graphical representations of some attained solutions are also presented in this article. Hence, this study shows the efficiency and the easiness of the (G'/G^2) -expansion technique with the assistance of emblematically computational software MATLAB and Mathematica.

Keywords: The (G'/G^2) -Expansion Technique; Travelling Wave Solution; Phi-Four Equation; Periodic Wave; Calogero-Bogoyavlanskiil Schilf Equation.

1. Introduction

In engineering, fluid dynamics, plasma physics, applied mathematics, and mathematical physics, nonlinear fractional partial differential equations (FPDEs) model significant phenomena and applications. To obtain some new analytical solutions, many authors absorbed enhanced new approaches or adjustments to standing systems. To find traveling wave solutions, efforts have increased in the literature of practical mathematics. It is broadly used to perceive the complex physical amplification of mathematical physics, nuclear physics, plasma physics, solid-state physics, statistical physics, astrophysics, mechanical engineering, fractional dynamics, biomechanics, strong state material science, neural material science, fluid mechanics, geo-optical filaments, stochastic dynamics, nonlinear optics, etcetera, etc. [1]. Thus, many influential and effective approaches have been established to find analytical solutions for traveling waves, such as the modified extended tanh-function technique, [2]-[4], the advanced $\exp(-\phi(\psi))$ -expansion technique, [5],[6], the $(G'/G, 1/G)$ -expansion technique, [7],[8], method of characteristics, [9], the variational iteration process, [10]-[12], the improved simple equation system, [13], the novel exponential rational function technique, [14], the multiple Exp-function system, [15],[16], the improved $\tan(\phi(\xi)/2)$ and $\tanh(\phi(\xi)/2)$ -expansion approaches, [17],[18], the Darboux transform process, [19], the exponential rational function technique, [20], extended simple equation method, [21], the updated simple equation method, [22], the Hirota bilinear method, [23], the first integral method, [24], the expanded trial equation method, [25], symbolic computations, [26],[27], a transformed rational function technique, [28], the ansatz structure, [29], the sine-cosine system, [30], the new extended direct algebraic scheme, [31], the (G'/G^2) -expansion technique, [32], the (G'/G) -expansion technique, [33], etc.

The goal of this study is to develop precise travelling waves analysis using the (G'/G^2) -expansion technique to better understand the physical meaning of a diversity of the phi-four and CBS equations. The unique solutions eliminated by the (G'/G^2) -expansion approach is articulated by the arrangement of the sinh, cosh, sin, and cos functions. Specify periodic waves solutions when creating joint solutions. The remainder of the paper is prepared in the following manner. The (G'/G^2) -expansion approach is briefly designated in Section 2. In Section 3, the mathematical formulation of the phi-four and CBS equations is provided, as well as its application using the (G'/G^2) -expansion technique. Section 4 is where this paper's findings are drawn.

2. Explanation of the (G'/G^2) -expansion technique

In this part, the (G'/G^2) -expansion technique is discoursed and assessed using the recommended methodology.

Suppose a nonlinear FDE presumed by

$$F(u, D_t^\gamma u, D_x^\gamma u, D_y^\gamma u, D_z^\gamma u, D_t^\gamma D_x^\gamma u, D_t^\gamma D_y^\gamma u, D_t^\gamma D_z^\gamma u) = 0, 0 < \gamma < 1. \quad (2.1)$$

In the previous FDE, $u(x, y, t)$ is an enigmatic purpose, and F is a polynomial of u and partial fractional derivatives. By disbursing the complex fractional transformation, we acquire.

$$\begin{aligned} u(x, t) &= U(\psi), \\ \psi &= p \frac{x^\gamma}{\gamma} + q \frac{y^\gamma}{\gamma} + r \frac{z^\gamma}{\gamma} - s \frac{t^\gamma}{\gamma}, \end{aligned} \quad (2.2)$$

Where $p, q, r,$ and s are nonzero constants. Equation (2.2) can be distorted into an ODE of the form:

$$Q(U, U', U'', U''', \dots) = 0, \quad (2.3)$$

The formula explanation of ODE can be written as:

$$U(\psi) = A_0 + \sum_{k=1}^N \left[A_k \left(\frac{G'}{G^2} \right)^k + B_k \left(\frac{G'}{G^2} \right)^{-k} \right], \quad (2.4)$$

$$\left(\frac{G'}{G^2} \right)' = \mu + \lambda \left(\frac{G'}{G^2} \right)^2, \quad (2.5)$$

Where, $\lambda \neq 0$, and $A_0, A_k, B_k (k = 1, 2, 3, \dots, N)$ are constants to be determined.

Based on the general solutions to equation (2.5), the ratio $\left(\frac{G'}{G^2} \right)$ can be separated into three cases as follows:

Case-I. Hyperbolic function solution, when $(\mu\lambda < 0)$

$$\frac{G'}{G^2} = -\frac{\sqrt{|\mu\lambda|}}{\lambda} \left[\frac{a \sinh(2\sqrt{|\mu\lambda}\psi) + a \cosh(2\sqrt{|\mu\lambda}\psi) + b}{a \sinh(2\sqrt{|\mu\lambda}\psi) + a \cosh(2\sqrt{|\mu\lambda}\psi) - b} \right], \quad (2.6)$$

Case-II. Rational function solution, when $(\mu\lambda = 0)$

$$\frac{G'}{G^2} = -\frac{a}{\lambda(a\psi + b)}, \mu = 0, \lambda \neq 0, \quad (2.7)$$

Case-III. Trigonometric function solution, when $(\mu\lambda > 0)$

$$\frac{G'}{G^2} = \frac{\mu}{\lambda} \left[\frac{a \cos(\sqrt{\mu\lambda}\psi) + b \sin(\sqrt{\mu\lambda}\psi)}{b \cos(\sqrt{\mu\lambda}\psi) - a \sin(\sqrt{\mu\lambda}\psi)} \right], \quad (2.8)$$

Where $a, b \neq 0$ for all three cases.

3. Applications of the (G'/G^2) -expansion technique

3.1. The nonlinear time-fractional Phi-four equation

The Phi-four equation is a specific form of the Klein-Gordon equation.

$$D_t^{2\theta} - u_{xx} + \beta^2 u + \gamma u^3 = 0, \gamma > 0, 0 < \theta \leq 1 \quad (3.1)$$

Where β and γ are real numbers.

Employing the following travelling wave transformation

$$u(x, t) = U(\psi), \text{ where } \psi = qx - p \frac{t^\theta}{\theta}.$$

On Eq. (3.1), we get

$$(p^2 - q^2)U'' + \beta^2 U + \gamma U^3 = 0, \quad (3.2)$$

With the asset of homogeneous balancing of the highest order derivative term U'' and nonlinear term U^3 in Eq. 3.2, we find that $N = 1$. Thus, our recommended technique allows us to use the auxiliary equation of the form:

$$U(\psi) = A_0 + A_1 \alpha + B_1 \alpha^{-1}, \quad (3.3)$$

Now putting the value of U and U'' in Eq. (3.2), we get,

$$\begin{aligned} \beta^2 A_0 + \gamma A_0^3 + G\beta^2 A_1 + 2G^3 p^2 \lambda^2 A_1 - 2G^3 q^2 \lambda^2 A_1 + 2Gp^2 \lambda \mu A_1 - 2Gq^2 \lambda \mu A_1 + 3G\gamma A_0^2 A_1 + 3G^2 \gamma A_0 A_1^2 + G^3 \gamma A_1^3 + \frac{\beta^2 B_1}{G} + \frac{2p^2 \lambda \mu B_1}{G} - \\ \frac{2q^2 \lambda \mu B_1}{G} + \frac{2p^2 \mu^2 B_1}{G^3} - \frac{2q^2 \mu^2 B_1}{G^3} + \frac{3\gamma A_0^2 B_1}{G} + 6\gamma A_0 A_1 B_1 + 3G\gamma A_1^2 B_1 + \frac{3\gamma A_0 B_1^2}{G^2} + \frac{3\gamma A_1 B_1^2}{G} + \frac{\gamma B_1^3}{G^3} = 0 \end{aligned}$$

Compeering the coefficients of like power of α from both sides of the above equation, we get the following SAE:

$$\begin{aligned}
\beta^2 A_0 + \gamma A_0^3 + 6\gamma A_0 A_1 B_1 &= 0, \\
\beta^2 A_1 + 2p^2 \lambda \mu A_1 - 2q^2 \lambda \mu A_1 + 3\gamma A_0^2 A_1 + 3\gamma A_1^2 B_1 &= 0, \\
\beta^2 B_1 + 2p^2 \lambda \mu B_1 - 2q^2 \lambda \mu B_1 + 3\gamma A_0^2 B_1 + 3\gamma A_1 B_1^2 &= 0, \\
3\gamma A_0 A_1^2 &= 0, \\
3\gamma A_0 B_1^2 &= 0, \\
2p^2 \lambda^2 A_1 - 2q^2 \lambda^2 A_1 + \gamma A_1^3 &= 0, \\
2p^2 \mu^2 B_1 - 2q^2 \mu^2 B_1 + \gamma B_1^3 &= 0,
\end{aligned} \tag{3.4}$$

Solving the SAE Eq. (3.4) for p, A_0, A_1, B_1 we get several solutions sets as follows:

$$\begin{aligned}
q &= \pm \frac{\sqrt{\beta^2 + 2p^2 \lambda \mu}}{\sqrt{2\sqrt{\lambda}\sqrt{\mu}}}, A_0 = 0, A_1 = \pm \frac{\beta\sqrt{\lambda}}{\sqrt{\gamma\sqrt{\mu}}}, B_1 = 0. \\
q &= \pm \frac{\sqrt{\beta^2 + 2p^2 \lambda \mu}}{\sqrt{2\sqrt{\lambda}\sqrt{\mu}}}, A_0 = 0, A_1 = 0, B_1 = \pm \frac{\beta\sqrt{\mu}}{\sqrt{\gamma\sqrt{\lambda}}}. \\
q &= \pm \frac{\sqrt{-\beta^2 + 4p^2 \lambda \mu}}{2\sqrt{\lambda}\sqrt{\mu}}, A_0 = 0, A_1 = -\frac{i\beta\sqrt{\lambda}}{\sqrt{2\sqrt{\gamma}\sqrt{\mu}}}, B_1 = -\frac{i\beta\sqrt{\mu}}{\sqrt{2\sqrt{\gamma}\sqrt{\lambda}}}. \\
q &= \pm \frac{\sqrt{-\beta^2 + 4p^2 \lambda \mu}}{2\sqrt{\lambda}\sqrt{\mu}}, A_0 = 0, A_1 = \frac{i\beta\sqrt{\lambda}}{\sqrt{2\sqrt{\gamma}\sqrt{\mu}}}, B_1 = \frac{i\beta\sqrt{\mu}}{\sqrt{2\sqrt{\gamma}\sqrt{\lambda}}}. \\
q &= \pm \frac{\sqrt{\beta^2 + 8p^2 \lambda \mu}}{2\sqrt{2\sqrt{\lambda}\sqrt{\mu}}}, A_0 = 0, A_1 = -\frac{\beta\sqrt{\lambda}}{2\sqrt{\gamma}\sqrt{\mu}}, B_1 = \frac{\beta\sqrt{\mu}}{2\sqrt{\gamma}\sqrt{\lambda}}. \\
q &= \pm \frac{\sqrt{\beta^2 + 8p^2 \lambda \mu}}{2\sqrt{2\sqrt{\lambda}\sqrt{\mu}}}, A_0 = 0, A_1 = \frac{\beta\sqrt{\lambda}}{2\sqrt{\gamma}\sqrt{\mu}}, B_1 = -\frac{\beta\sqrt{\mu}}{2\sqrt{\gamma}\sqrt{\lambda}}.
\end{aligned}$$

Expanding these solution sets, we build the solutions to Eq. (3.1) as follows:

When $\mu\lambda < 0$, we obtain the following hyperbolic function solutions:

$$\begin{aligned}
U_{1,2}(x, t) &= \pm \beta \frac{|\lambda\mu| \left[\frac{b + a\text{Cosh}[2\sqrt{\lambda\mu}\psi] + a\text{Sinh}[2\sqrt{\lambda\mu}\psi]}{-b + a\text{Cosh}[2\sqrt{\lambda\mu}\psi] + a\text{Sinh}[2\sqrt{\lambda\mu}\psi]} \right]}{\sqrt{\gamma\lambda\mu}}; \psi = qx - p \frac{t^\theta}{\theta}; q = \pm \frac{\sqrt{\beta^2 + 2p^2 \lambda \mu}}{\sqrt{2\sqrt{\lambda}\sqrt{\mu}}} \\
U_{3,4}(x, t) &= \pm \beta \frac{\mu\lambda \left[\frac{-b + a\text{Cosh}[2\sqrt{\lambda\mu}\psi] + a\text{Sinh}[2\sqrt{\lambda\mu}\psi]}{b + a\text{Cosh}[2\sqrt{\lambda\mu}\psi] + a\text{Sinh}[2\sqrt{\lambda\mu}\psi]} \right]}{\gamma|\lambda\mu|}; \psi = qx - p \frac{t^\theta}{\theta}; q = \pm \frac{\sqrt{\beta^2 + 2p^2 \lambda \mu}}{\sqrt{2\sqrt{\lambda}\sqrt{\mu}}} \\
U_{5,6}(x, t) &= \pm \frac{i\beta \left(\mu + \sqrt{|\lambda\mu| \left[\frac{b + a\text{Cosh}[2\sqrt{\lambda\mu}\psi] + a\text{Sinh}[2\sqrt{\lambda\mu}\psi]}{-b + a\text{Cosh}[2\sqrt{\lambda\mu}\psi] + a\text{Sinh}[2\sqrt{\lambda\mu}\psi]} \right]^2 \right)}{\sqrt{\frac{2\gamma\mu|\lambda\mu|}{\lambda} \left[\frac{b + a\text{Cosh}[2\sqrt{\lambda\mu}\psi] + a\text{Sinh}[2\sqrt{\lambda\mu}\psi]}{-b + a\text{Cosh}[2\sqrt{\lambda\mu}\psi] + a\text{Sinh}[2\sqrt{\lambda\mu}\psi]} \right]}}; \psi = qx - p \frac{t^\theta}{\theta}; q = \pm \frac{\sqrt{-\beta^2 + 4p^2 \lambda \mu}}{2\sqrt{\lambda}\sqrt{\mu}} \\
U_{7,8}(x, t) &= \pm \frac{\beta \left(\mu - \sqrt{|\lambda\mu| \left[\frac{b + a\text{Cosh}[2\sqrt{\lambda\mu}\psi] + a\text{Sinh}[2\sqrt{\lambda\mu}\psi]}{-b + a\text{Cosh}[2\sqrt{\lambda\mu}\psi] + a\text{Sinh}[2\sqrt{\lambda\mu}\psi]} \right]^2 \right)}{\sqrt{\frac{2\gamma\mu|\lambda\mu|}{\lambda} \left[\frac{b + a\text{Cosh}[2\sqrt{\lambda\mu}\psi] + a\text{Sinh}[2\sqrt{\lambda\mu}\psi]}{-b + a\text{Cosh}[2\sqrt{\lambda\mu}\psi] + a\text{Sinh}[2\sqrt{\lambda\mu}\psi]} \right]}}; \psi = qx - p \frac{t^\theta}{\theta}; q = \pm \frac{\sqrt{\beta^2 + 8p^2 \lambda \mu}}{2\sqrt{2\sqrt{\lambda}\sqrt{\mu}}}
\end{aligned}$$

When $\mu\lambda > 0$, we obtain the following trigonometric function solution

$$\begin{aligned}
U_{9,10}(x, t) &= \pm \frac{\beta}{\sqrt{\gamma}} \frac{a\text{Cos}[\sqrt{\lambda\mu}\psi] + b\text{Sin}[\sqrt{\lambda\mu}\psi]}{b\text{Cos}[\sqrt{\lambda\mu}\psi] - a\text{Sin}[\sqrt{\lambda\mu}\psi]}; \psi = qx - p \frac{t^\theta}{\theta}; q = \pm \frac{\sqrt{\beta^2 + 2p^2 \lambda \mu}}{\sqrt{2\sqrt{\lambda}\sqrt{\mu}}} \\
U_{11,12}(x, t) &= \pm \frac{\beta}{\sqrt{\gamma}} \frac{b\text{Cos}[\sqrt{\lambda\mu}\psi] - a\text{Sin}[\sqrt{\lambda\mu}\psi]}{a\text{Cos}[\sqrt{\lambda\mu}\psi] + b\text{Sin}[\sqrt{\lambda\mu}\psi]}; \psi = qx - p \frac{t^\theta}{\theta}; q = \pm \frac{\sqrt{\beta^2 + 2p^2 \lambda \mu}}{\sqrt{2\sqrt{\lambda}\sqrt{\mu}}} \\
U_{13,14}(x, t) &= \pm \frac{i\beta \left(\mu + \sqrt{\lambda\mu} \left[\frac{a\text{Cos}[\sqrt{\lambda\mu}\psi] + b\text{Sin}[\sqrt{\lambda\mu}\psi]}{b\text{Cos}[\sqrt{\lambda\mu}\psi] - a\text{Sin}[\sqrt{\lambda\mu}\psi]} \right]^2 \right)}{\mu\sqrt{2\gamma} \left[\frac{a\text{Cos}[\sqrt{\lambda\mu}\psi] + b\text{Sin}[\sqrt{\lambda\mu}\psi]}{b\text{Cos}[\sqrt{\lambda\mu}\psi] - a\text{Sin}[\sqrt{\lambda\mu}\psi]} \right]}; \psi = qx - p \frac{t^\theta}{\theta}; q = \pm \frac{\sqrt{-\beta^2 + 4p^2 \lambda \mu}}{2\sqrt{\lambda}\sqrt{\mu}}
\end{aligned}$$

$$U_{15,16}(x, t) = \pm \frac{\beta \left(\mu - \sqrt{\lambda\mu} \left[\frac{a \cos[\sqrt{\lambda\mu}\psi] + b \sin[\sqrt{\lambda\mu}\psi]}{b \cos[\sqrt{\lambda\mu}\psi] - a \sin[\sqrt{\lambda\mu}\psi]} \right] \right)^2}{2\mu\sqrt{\gamma} \left[\frac{a \cos[\sqrt{\lambda\mu}\psi] + b \sin[\sqrt{\lambda\mu}\psi]}{b \cos[\sqrt{\lambda\mu}\psi] - a \sin[\sqrt{\lambda\mu}\psi]} \right]}; \psi = qx - p \frac{t^\theta}{\theta}; q = \pm \frac{\sqrt{\beta^2 + 8p^2\lambda\mu}}{2\sqrt{2}\sqrt{\lambda}\sqrt{\mu}}$$

3.2. The nonlinear time-fractional CBS equation

Consider the subsequent generalized (2+1)-dimensional CBS circumstances:

$$u_t + \phi(u)u_y = 0, \phi(u) = \partial_x^2 + au + bu_x\partial_x^{-1}, \quad (3.5)$$

Or homogeneously,

$$u_t + u_{xxy} + auu_y + bv_x\partial_x^{-1}v_y = 0, \quad (3.6)$$

Where $\partial_x^{-1} = \int f dx$ and a, b are constraints. Eq. (3.6) can be characterized in the probable time-fractional form of CBS equation.^{2,31}

$$u_x D_t^\theta \theta_u + 4u_x u_{xy} + 2u_{xx} u_y + u_{xxy} = 0, t > 0, x, y \in \mathbb{R}, \quad (3.7)$$

Where $0 < \theta \leq 1$.

Employing the following travelling wave transformation

$$u(x, y, t) = U(\psi) \text{ and } \psi = x + y - p \frac{t^\theta}{\theta} \quad (3.8)$$

On Eq. (3.7), we get,

$$pU' + \left(\frac{\beta+\gamma}{2}\right)(U')^2 + U''' = 0 \quad (3.9)$$

With the asset of homogeneous balancing of the highest order derivative term U''' and the nonlinear term $(U')^2$ in Eq. (3.9), we find that $N = 1$. Thus, our recommended technique allows us to use the auxiliary equation of the form:

$$U(\psi) = A_0 + A_1\alpha + B_1\alpha^{-1}, \quad (3.10)$$

From Eq. (3.9) and (3.10) we get,

$$\begin{aligned} & -G^2 p \lambda A_1 + 6G^4 \lambda^3 A_1 - p \mu A_1 + 8G^2 \lambda^2 \mu A_1 + 2\lambda \mu^2 A_1 + \frac{1}{2} G^4 \beta \lambda^2 A_1^2 + \frac{1}{2} G^4 \gamma \lambda^2 A_1^2 + G^2 \beta \lambda \mu A_1^2 + G^2 \gamma \lambda \mu A_1^2 + \frac{1}{2} \beta \mu^2 A_1^2 + \frac{1}{2} \gamma \mu^2 A_1^2 + \\ & p \lambda B_1 + \frac{p \mu B_1}{G^2} - 2\lambda^2 \mu B_1 - \frac{8\lambda \mu^2 B_1}{G^2} - \frac{6\mu^3 B_1}{G^4} - G^2 \beta \lambda^2 A_1 B_1 - G^2 \gamma \lambda^2 A_1 B_1 - 2\beta \lambda \mu A_1 B_1 - 2\gamma \lambda \mu A_1 B_1 - \frac{\beta \mu^2 A_1 B_1}{G^2} - \frac{\gamma \mu^2 A_1 B_1}{G^2} + \frac{1}{2} \beta \lambda^2 B_1^2 + \\ & \frac{1}{2} \gamma \lambda^2 B_1^2 + \frac{\beta \lambda \mu B_1^2}{G^2} + \frac{\gamma \lambda \mu B_1^2}{G^2} + \frac{\beta \mu^2 B_1^2}{2G^4} + \frac{\gamma \mu^2 B_1^2}{2G^4} = 0 \end{aligned}$$

Compeering the coefficients of like power of α from both sides of the above equation, we get the following SAE:

$$\begin{aligned} & -p \mu A_1 + 2\lambda \mu^2 A_1 + \frac{1}{2} \beta \mu^2 A_1^2 + \frac{1}{2} \gamma \mu^2 A_1^2 + p \lambda B_1 - 2\lambda^2 \mu B_1 - 2\beta \lambda \mu A_1 B_1 - 2\gamma \lambda \mu A_1 B_1 + \frac{1}{2} \beta \lambda^2 B_1^2 + \frac{1}{2} \gamma \lambda^2 B_1^2 = 0, \\ & -p \lambda A_1 + 8\lambda^2 \mu A_1 + \beta \lambda \mu A_1^2 + \gamma \lambda \mu A_1^2 - \beta \lambda^2 A_1 B_1 - \gamma \lambda^2 A_1 B_1 = 0, \\ & p \mu B_1 - 8\lambda \mu^2 B_1 - \beta \mu^2 A_1 B_1 - \gamma \mu^2 A_1 B_1 + \beta \lambda \mu B_1^2 + \gamma \lambda \mu B_1^2 = 0, \\ & 6\lambda^3 A_1 + \frac{1}{2} \beta \lambda^2 A_1^2 + \frac{1}{2} \gamma \lambda^2 A_1^2 = 0, \\ & -6\mu^3 B_1 + \frac{1}{2} \beta \mu^2 B_1^2 + \frac{1}{2} \gamma \mu^2 B_1^2 = 0, \end{aligned} \quad (3.11)$$

Solving the SAE Eq. (3.11) for p, A_0, A_1, B_1 we get several solutions sets as follows:

Set-I.

$$p = -4\lambda\mu, A_0 = 0, A_1 = -\frac{12\lambda}{\beta+\gamma}, B_1 = 0.$$

Set-II.

$$p = -4\lambda\mu, A_0 = 0, A_1 = 0, B_1 = \frac{12\mu}{\beta+\gamma}.$$

Set-III.

$$p = -16\lambda\mu, A_0 = 0, A_1 = -\frac{12\lambda}{\beta+\gamma}, B_1 \rightarrow \frac{12\mu}{\beta+\gamma}.$$

Expending these solution sets, we build the solutions to Eq. (3.7) as follows:

When $\mu\lambda < 0$, we obtain the following hyperbolic function solutions:

$$U_{17}(x, y, t) = \frac{12\sqrt{|\lambda\mu|}}{\beta+\gamma} \left[\frac{b+a\text{Cosh}[2\sqrt{\lambda\mu}\psi]+a\text{Sinh}[2\sqrt{\lambda\mu}\psi]}{-b+a\text{Cosh}[2\sqrt{\lambda\mu}\psi]+a\text{Sinh}[2\sqrt{\lambda\mu}\psi]} \right]; \psi = x + y - p \frac{t^\theta}{\theta}$$

$$U_{18}(x, y, t) = -\frac{12\mu\lambda}{\beta+\gamma\sqrt{|\lambda\mu|}} \left[\frac{-b+a\text{Cosh}[2\sqrt{\lambda\mu}\psi]+a\text{Sinh}[2\sqrt{\lambda\mu}\psi]}{b+a\text{Cosh}[2\sqrt{\lambda\mu}\psi]+a\text{Sinh}[2\sqrt{\lambda\mu}\psi]} \right]; \psi = x + y - p \frac{t^\theta}{\theta}$$

$$U_{19}(x, y, t) = \frac{12\left(-\mu+\sqrt{|\lambda\mu|} \left[\frac{b+a\text{Cosh}[2\sqrt{\lambda\mu}\psi]+a\text{Sinh}[2\sqrt{\lambda\mu}\psi]}{-b+a\text{Cosh}[2\sqrt{\lambda\mu}\psi]+a\text{Sinh}[2\sqrt{\lambda\mu}\psi]} \right]^2\right)}{(\beta+\gamma)\frac{\sqrt{|\lambda\mu|}}{\lambda} \left[\frac{b+a\text{Cosh}[2\sqrt{\lambda\mu}\psi]+a\text{Sinh}[2\sqrt{\lambda\mu}\psi]}{-b+a\text{Cosh}[2\sqrt{\lambda\mu}\psi]+a\text{Sinh}[2\sqrt{\lambda\mu}\psi]} \right]}; \psi = x + y - p \frac{t^\theta}{\theta}$$

When $\mu\lambda = 0$, we obtain the following rational function solutions:

$$U_{20}(x, y, t) = \frac{12a\lambda}{(\beta+\gamma)\lambda[b+a\psi]}; \psi = x + y - p \frac{t^\theta}{\theta}$$

$$U_{21}(x, y, t) = \frac{12(a^2\lambda-\mu\lambda[b+a\psi]^2)}{a(\beta+\gamma)\lambda[b+a\psi]}; \psi = x + y - p \frac{t^\theta}{\theta}$$

When $\mu\lambda > 0$, we obtain the following trigonometric function solutions:

$$U_{22}(x, y, t) = -\frac{12\lambda}{\beta+\gamma} \sqrt{\frac{\mu}{\lambda}} \left[\frac{a\text{Cos}[\sqrt{\lambda\mu}\psi]+b\text{Sin}[\sqrt{\lambda\mu}\psi]}{b\text{Cos}[\sqrt{\lambda\mu}\psi]-a\text{Sin}[\sqrt{\lambda\mu}\psi]} \right]; \psi = x + y - p \frac{t^\theta}{\theta}$$

$$U_{23}(x, y, t) = \frac{12\mu}{\beta+\gamma} \sqrt{\frac{\lambda}{\mu}} \left[\frac{b\text{Cos}[\sqrt{\lambda\mu}\psi]-a\text{Sin}[\sqrt{\lambda\mu}\psi]}{a\text{Cos}[\sqrt{\lambda\mu}\psi]+b\text{Sin}[\sqrt{\lambda\mu}\psi]} \right]; \psi = x + y - p \frac{t^\theta}{\theta}$$

$$U_{24}(x, y, t) = \frac{12\left(\mu-\lambda \sqrt{\frac{\mu}{\lambda}} \left[\frac{a\text{Cos}[\sqrt{\lambda\mu}\psi]+b\text{Sin}[\sqrt{\lambda\mu}\psi]}{b\text{Cos}[\sqrt{\lambda\mu}\psi]-a\text{Sin}[\sqrt{\lambda\mu}\psi]} \right]^2\right)}{(\beta+\gamma)\sqrt{\frac{\mu}{\lambda}} \left[\frac{a\text{Cos}[\sqrt{\lambda\mu}\psi]+b\text{Sin}[\sqrt{\lambda\mu}\psi]}{b\text{Cos}[\sqrt{\lambda\mu}\psi]-a\text{Sin}[\sqrt{\lambda\mu}\psi]} \right]}; \psi = x + y - p \frac{t^\theta}{\theta}$$

4. Graphical representation

This section presents the graphical illustration of phi-four and CBS equations. Using the computational software MATLAB, we represent a combined 3D surface and contour plot view of some solutions. All the attained solutions except $U_{20}(x, y, t)$ and $U_{21}(x, y, t)$ represent periodic wave solutions. $U_{20}(x, y, t)$ and $U_{21}(x, y, t)$ are rational function solutions. Each periodic wave is presented for different values of θ . Figure 1-4 represents the periodic waves for phi-four, and CBS equations and Figure 5 represent the rational function solution for the CBS equation. Figures 1-4 are represented for $\theta = 0.3, 0.6$, and 1 , respectively.

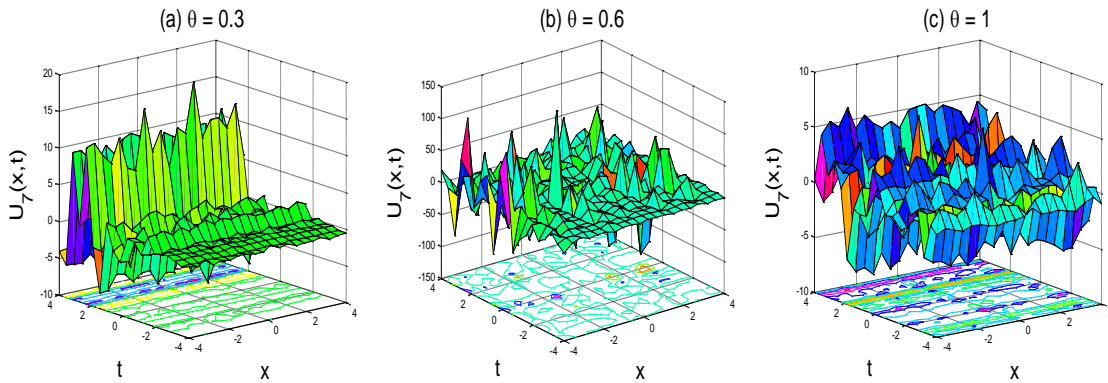


Fig. 1: Periodic Wave of $U_7(x, t)$ for the Parameters $a = 1, b = 1, p = -0.5, \lambda = 1, \mu = -1, \beta = 0.5, \gamma = 0.5$ and $\theta = 0.3, 0.6$ and 1 Respectively.

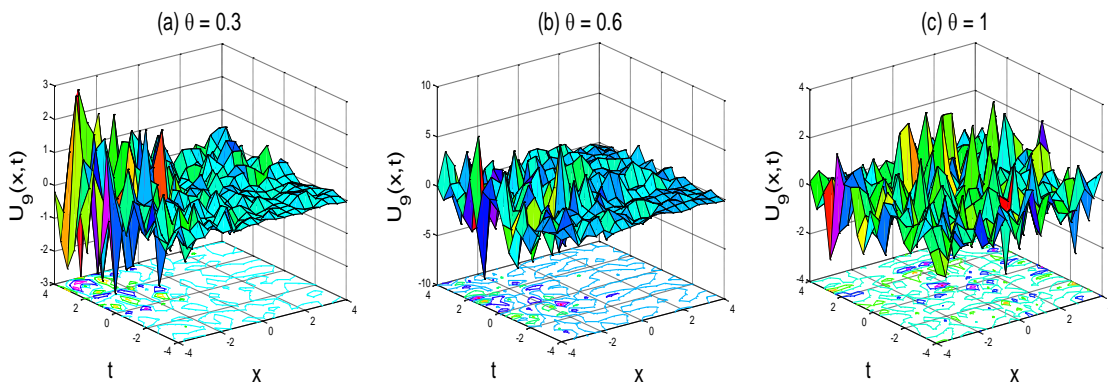


Fig. 2: Periodic Wave of $U_9(x, t)$ for the Parameters $a = 1, b = 1, p = -0.5, \lambda = 1, \mu = 1, \beta = 0.5, \gamma = 0.5$ and $\theta = 0.3, 0.6$ and 1 Respectively.

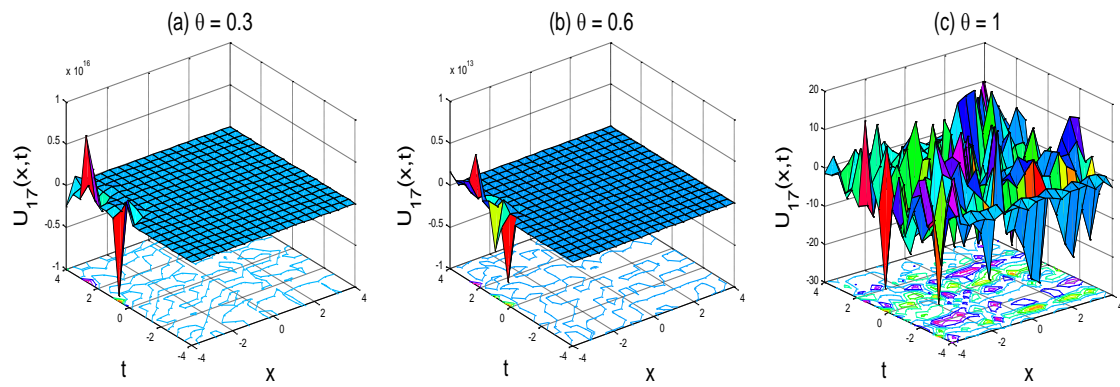


Fig. 3: Periodic Wave of $U_{17}(x,t)$ for the Parameters $a = 1, b = 1, p = -0.5, \lambda = 1, \mu = -1, \gamma = 0, \beta = 0.5, \gamma = 0.5$ and $\theta = 0.3, 0.6$ and 1 Respectively.

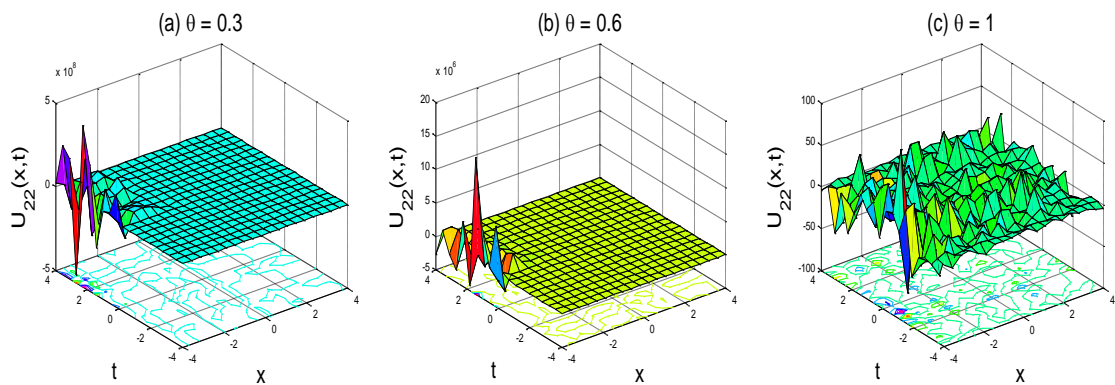


Fig. 4: Periodic Wave of $U_{22}(x,t)$ for the Parameters $a = 1, b = 1, p = -0.5, \lambda = 1, \mu = 1, \gamma = 0, \beta = 0.5, \gamma = 0.5$ and $\theta = 0.3, 0.6$ and 1 Respectively.

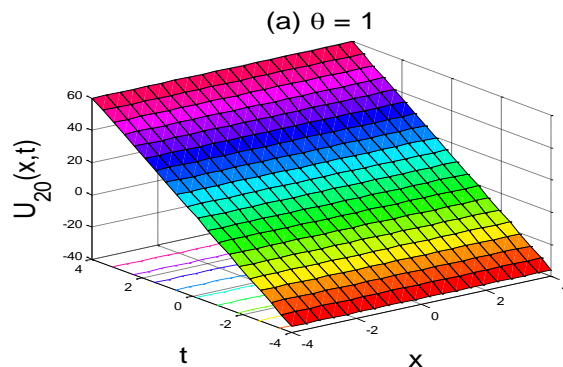


Fig. 5: Rational Function Solution of $U_{20}(x,y,t)$ for the Parameters $a = 1, b = 1, \lambda = 1, \mu = 0, \gamma = 0, \beta = 0.5, \gamma = 0.5$ and $\theta = 1$.

5. Conclusion

In this article, we investigated the periodic wave analysis of phi-four and CBS equations using the (G'/G^2) -expansion method. Using the mentioned method, we found the exact travelling wave solutions of the phi-four and CBS equations and see that most of the solutions are periodic wave solutions by the hyperbolic function \sinh , \cosh , and trigonometric function \sin and \cos . Also, we have some rational function solutions only for the CBS equation. Moreover, we analyzed the periodic wave solutions and the rational function solution for different fractional values of θ , and we found no change of wave characteristics for the change of the fractional value of θ . Eventually, it is noticed that the employed method and the relevant traveling transformation are more realistic, effective, and efficient than the other forms and that they may be used in future productive investigations in mathematical physics and engineering to understand long-wave phenomena.

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