International Journal of Physical Research, *5 (2) (2017) 109-114*

International Journal of Physical Research

Website: www.sciencepubco.com/index.php/IJPR
doi: 10.14419/ijpr.v5i2.8403
Research paper

Analytical treatment for the conformable space-time fractional Benney-Luke equation via two reliable methods

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Abstract

In this study, with help of the Mathematica software, we employ the Kudryashov method and the modified extended tanh expansion method with the Riccati differential equation to analytically treat the Benney-Luke equation. The Benney-Luke equation considered in this study features fractional derivatives in both the spatial and the temporal variables of the newly introduced conformable fractional derivative. We extensively examine the equation via the two methods, and we construct various structures such as the exponential functions, trigonometric functions and hyperbolic functions. Finally, we depict the graphs of all solutions.

Keywords: *Conformable Fractional Derivative; Space-Time Fractional Benney-Luke Equation; Singular Solution.*

1. Introduction

Nonlinear partial differential equations featuring fractional order derivatives play important roles in modeling many real-life applications. The study of such equations becomes imperative since they best describe physical situations in many instances. In particular, one can find equations such as the Boussinesq-Burgers equation [1] and Benney-Luke equation [2] to be of paramount importance in the study of shallow water, fluid dynamics and plasma physics among others. More recently, the classical Benney-Luke equation Eq. (1)

$$u_{tt} - u_{xx} + pu_{xxxx} - qu_{xxtt} + u_t u_{xx} + u_x u_{xt} = 0, \; x \in [0,1] \quad (1)$$

Where $p - q = w - 1/3$, and w is the surface tension parameter; was studied by Islam et al. in [3] by employing the G'/G expansion method. Moreover, one can find many other analytical methods in the literature to study such class of equations, see [4-20].

On the other hand, the theory of the fractional calculus is gaining weight with various definition ranging from the Riemann-Liouville fractional derivative and Caputo's fractional derivative definition [21-22]; and the recent conformable and Atangana-Baleanu fractional derivative definitions [23-24], [25], respectively.

Further, for the conformable fractional derivative; for $u: [0, \infty) \to \mathbb{R}$, the α' s order conformable derivative of u is defined by

$$D_t^{\alpha}(u(t)) = \lim_{\varepsilon \to 0} \frac{u(t + \varepsilon t^{1-\alpha}) - u(t)}{\varepsilon}, t > 0, \alpha \in (0,1]. \quad (2)$$

However, in this study, the space-time fractional Benney-Luke equation Eq. (3)

$$D_{tt}^{2\alpha} u - D_{xx}^{2\alpha} u + pD_{xxxx}^{4\alpha} u - qD_{xxtt}^{4\alpha} u + 3D_{xxt}^{3\alpha} u^2 = 0, \quad \alpha \in (0,1] \quad (3)$$

Will be examined using the newly introduced conformable fractional derivative definition [23-24] by employing two promising and reliable analytical methods. The methods involve are the Kudryashov method [4] and the modified extended tanh expansion method with the Riccati differential equation [5]. The Mathematica software will be fully utilized in the solution aspect as well as in the graphical representation. The paper is organized as follows: Section 2 gives properties of the conformable fractional derivative and methodology of solution. Section 3 gives the outline of the solution of Benney-Luke equation. In Section 4, we give the results and discussion. Section 5 gives the conclusion.

2. The properties of the conformable fractional derivative and methodology of solution

Some properties of the conformable fractional derivative is given using the following theorems:

Theorem 1: Let $\alpha \in (0,1]$ and suppose $u(t)$ and $v(t)$ are α-differentiable at $t > 0$. Then

a) $D_t^{\alpha}(t^c) = ct^{c-\alpha}$, for all $c \in \mathbb{R}$.

b) $D_t^{\alpha}(a) = 0$, for all constant function $u(t) = a$.

c) $D_t^{\alpha}(au(t)) = aD_t^{\alpha}(u(t))$, for all a constant.

d) $D_t^{\alpha}(au(t) + bv(t)) = aD_t^{\alpha}u(t) + bD_t^{\alpha}v(t)$, for all $a, b \in \mathbb{R}$

e) $D_t^{\alpha}(v(t)u(t)) = D_t^{\alpha}v(t)(u(t)) + u(t)D_t^{\alpha}(v(t))$.

f) $D_t^{\alpha}\left(\frac{u(t)}{v(t)}\right)=\frac{v(t)D_t^{\alpha}u(t)-u(t)D_t^{\alpha}v(t)}{v^2(t)},\ v(t)\neq 0.$

g) If, in addition to $u(t)$ is differentiable, then $D_t^{\alpha}u(t)=t^{1-\alpha}\frac{du}{dt}$.

Theroem 2: *Let $\alpha\in(0,1]$ such $u(t)$ is differentiable and also α-differentiable. Let $v(t)$ be a function defined in the range of $u(t)$ also differentiable, then*

$$D_t^{\alpha}\left(u(t)\circ v(t)\right)=t^{1-\alpha}v'(t)u'(v(t)).$$

See also [23-24].

Now, considering the following conformable fractional differential equation, we present the method:

$$P(u,D_t^{\alpha_1}u,D_x^{\alpha_2}u,D_t^{\alpha_1}D_x^{\alpha_2}u,D_{tt}^{2\alpha_1}u,D_{xx}^{2\alpha_2}u,...)=0,\quad 0<\alpha_1,\alpha_2<1. \tag{4}$$

By the wave transformation, we set

$$u(x,t)=U(\xi),\ \ \xi=a\frac{x^{\alpha_1}}{\alpha_1}-b\frac{t^{\alpha_2}}{\alpha_2}, \tag{5}$$

where α_1,α_2 are fractional orders, a and b are nonzero constants. Substitution of transformation (5) into (4), we get a reduced ordinary differential equation of the polynomial form

$$Q(U(\xi),\frac{d}{d\xi}U(\xi),\frac{d^2}{d\xi^2}U(\xi),...)=0. \tag{6}$$

3. Solution of conformable space-time Benney-Luke equation

Consider the conformable space-time Benney-Luke equation

$$D_{tt}^{2\alpha}u-D_{xx}^{2\alpha}u+pD_{xxxx}^{4\alpha}u-qD_{xxtt}^{4\alpha}u+3D_{xxt}^{3\alpha}u^2=0,\ \ \alpha\in(0,1]. \tag{7}$$

On using the transformation Eq. (5), we get a reduced ordinary differential equation as follows

$$b^2u''-a^2u''+a^4pu''''-qa^2b^2u''''-3a^2b(u^2)'''=0. \tag{8}$$

Balancing the highest order derivative with the highest nonlinear order in Eq. (8) after integrating trice with zero constant of integration assumption,we get

$$N=1. \tag{9}$$

3.1. Soliton solution by Kudryashov method

From Eq. (9), Kudryashov method offers a truncated series solution of the form:

$$U(\xi)=a_0+a_1\Phi(\xi), \tag{10}$$

Where, a_0 and a_1 are constants and $\Phi(\xi)$ is given by the function

$$\Phi(\xi)=\frac{1}{1+d\ \exp(\xi)}, \tag{11}$$

Which satisfies the differential equation

$$\Phi'(\xi)=\Phi^2(\xi)-\Phi(\xi). \tag{12}$$

Thus, substituting Eq. (10) with its necessary derivatives alongside Eq. (12) into Eq. (8); equating the coefficients of $\Phi(\xi)$ to zero we get the following algebraic equations with the help of Mathematica software:

$$-a^2a_1+b^2a_1+a^4pa_1-a^2b^2qa_1+6a^2ba_0a_1=0,$$

$$3a^2a_1-3b^2a_1-15a^4pa_1+15a^2b^2qa_1-42a^2ba_0a_1+24a^2ba_1^2=0,$$

$$-2a^2a_1+2b^2a_1+50a^4pa_1-50a^2b^2qa_1+72a^2ba_0a_1-114a^2ba_1^2=0,$$

$$-60a^4pa_1+60a^2b^2qa_1-36a^2ba_0a_1+162a^2ba_1^2=0,$$

$$24a^4pa_1-24a^2b^2qa_1-72a^2ba_1^2=0.$$

Solving the above system, we get the following:

Case 1:

$$a_0=a_0,\qquad a_1=-2a_0,\qquad a=\pm\frac{6a_0}{\sqrt{p^2-2pq+q^2}},\qquad b=-\frac{6a_0}{p-q}.$$

Which produces

$$u_{1,2}(x,t)=a_0-\frac{2a_0}{1+d\ \exp(\pm\frac{6a_0}{\sqrt{p^2-2pq+q^2}}\frac{x^{\alpha}}{\alpha}+\frac{6a_0}{p-q}\frac{t^{\alpha}}{\alpha})}, \tag{13}$$

(see Fig. 1).

3.2. Soliton solution by modified extended *tanh* method

From Eq. (9), modified extended tanh method offers a truncated series solution of the form:

$$U(\xi)=a_0+a_1\Phi(\xi)+\frac{b_1}{\Phi(\xi)}, \tag{14}$$

Where, A_0,A_1 and B_1 are constants and $\Phi(\xi)$ is given by the function

$$\Phi(\xi)=\tanh\xi, \tag{15}$$

Which satisfies the differential equation

$$\Phi'(\xi)=r+\Phi^2(\xi), \tag{16}$$

Where r is constant. Also, Eq. (16) has the following structure of solutions:

a) If $r<0$, then

$$\Phi(\xi)=-\sqrt{-r}\tanh(\sqrt{-r}\xi),$$
$$\Phi(\xi)=-\sqrt{-r}\coth(\sqrt{-r}\xi),$$

b) If $r=0$, then

$$\Phi(\xi)=-\frac{1}{\xi},$$

c) If $r>0$, then

$$\Phi(\xi)=\sqrt{r}\tan(\sqrt{r}\xi),$$
$$\Phi(\xi)=-\sqrt{r}\cot(\sqrt{r}\xi).$$

Therefore, substituting Eq. (14) with its necessary derivatives alongside Eq. (16) into Eq. (8); equating the coefficients of $\Phi(\xi)$ to zero we get the following algebraic equations with the help of Mathematica software:

$$24a^4pr^4b_1-24a^2b^2qr^4b_1+72a^2br^3b_1^2=0, 36a^2br^3a_0b_1=0,$$

$$-2a^2r^2b_1+2b^2r^2b_1+40a^4pr^3b_1-40a^2b^2qr^3b_1+120a^2br^2b_1^2=0,$$

$$-2a^2rb_1+2b^2rb_1+16a^4pr^2b_1-16a^2b^2qr^2b_1+48a^2brb_1^2=0,$$

$$-12a^2br^2a_0a_1+12a^2bra_0b_1=0, 48a^2br^2a_0b_1=0, -48a^2bra_0a_1=0,$$

$$-2a^2ra_1+2b^2ra_1+16a^4pr^2a_1-16a^2b^2qr^2a_1-48a^2br^2a_1^2=0,$$

$$-2a^2a_1+2b^2a_1+40a^4pra_1-40a^2b^2qra_1-120a^2bra_1^2=0,$$

$$-36a^2ba_0a_1=0, 24a^4pa_1-24a^2b^2qa_1-72a^2ba_1^2=0.$$

Solving the above system, we get the following:

Case 1:

$$a_0=0,\quad a_1=a_1,\quad b_1=0,$$
$$a=\pm\frac{3a_1}{\sqrt{p^2-2pq+q^2}},\quad b=\frac{3a_1}{p-q},\quad r=r.$$

Which produces

$$u_{1,2}(x,t)=-a_1\sqrt{-r}\tanh\left(\sqrt{-r}\left(\pm\frac{3a_1}{\sqrt{p^2-2pq+q^2}}\frac{x^\alpha}{\alpha}-\frac{3a_1}{p-q}\frac{t^\alpha}{\alpha}\right)\right), r<0, \tag{17}$$

(See Fig. 2).

$$u_{3,4}(x,t)=-a_1\sqrt{-r}\coth\left(\sqrt{-r}\left(\pm\frac{3a_1}{\sqrt{p^2-2pq+q^2}}\frac{x^\alpha}{\alpha}-\frac{3a_1}{p-q}\frac{t^\alpha}{\alpha}\right)\right), r<0, \tag{18}$$

(See Fig. 3).

$$u_{5,6}(x,t)=a_1\sqrt{r}\tan\left(\sqrt{r}\left(\pm\frac{3a_1}{\sqrt{p^2-2pq+q^2}}\frac{x^\alpha}{\alpha}-\frac{3a_1}{p-q}\frac{t^\alpha}{\alpha}\right)\right), r>0, \tag{19}$$

(See Fig. 4).

$$u_{7,8}(x,t)=-a_1\sqrt{r}\cot\left(\sqrt{r}\left(\pm\frac{3a_1}{\sqrt{p^2-2pq+q^2}}\frac{x^\alpha}{\alpha}-\frac{3a_1}{p-q}\frac{t^\alpha}{\alpha}\right)\right), r>0, \tag{20}$$

(See Fig. 5).
Case 2:

$$a_0=0,\quad a_1=a_1,\quad b_1=0,\quad a=\pm\frac{3a_1}{\sqrt{p^2-2pq+q^2}},$$
$$b=\frac{3a_1}{p-q},\quad r=0.$$

Which produces

$$u_{9,10}(x,t)=\frac{a_1}{\mp\frac{3a_1}{\sqrt{p^2-2pq+q^2}}\frac{x^\alpha}{\alpha}+\frac{3a_1}{p-q}\frac{t^\alpha}{\alpha}}. \tag{21}$$

(See Fig. 6).
Case 3:

$$a_0=0,\ a_1=a_1,\ b_1=b_1,\quad a=\pm\frac{3a_1}{\sqrt{p^2-2pq+q^2}},$$
$$b=\frac{3a_1}{p-q},\quad r=-\frac{9a_1b_1}{a^2(p-q)^2},<0.$$

Which produces

$$\begin{aligned}u_{11,12}(x,t)&=-a_1\sqrt{\frac{9a_1b_1}{a^2(p-q)^2}}\tanh\left(\sqrt{\frac{9a_1b_1}{a^2(p-q)^2}}\xi\right)\\&-\frac{b_1}{\sqrt{\frac{9a_1b_1}{a^2(p-q)^2}}}\coth\left(\sqrt{\frac{9a_1b_1}{a^2(p-q)^2}}\xi\right),\\ \xi&=\pm\frac{3a_1}{\sqrt{p^2-2pq+q^2}}\frac{x^\alpha}{\alpha}-\frac{3a_1}{p-q}\frac{t^\alpha}{\alpha}.\end{aligned} \tag{22}$$

(See Fig. 7).

4. Graphical representations

In the present section, we give the graphical representations of the conformable space-time fractional Benney-Luke equation'ssolutions obtained and presented in Eq. (13) and Eqs. (17-22). The graphs are ploted for $0\leq x\leq 2$ and $0\leq t\leq 1$and the used papameters are prescribed below each figure given in Fig1-7 as follows:

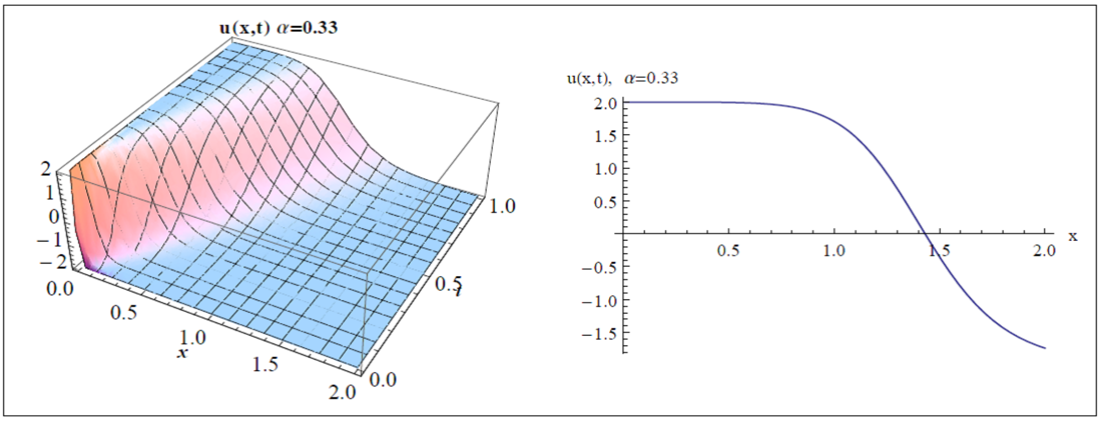

Fig. 1: Profiles of Eq. (13), Subsituting the Values of the Parameters $p = 2; q = 0.2, a_0 = 2; d = 0.7; \alpha = 0.33.$ and $t = 1$ for the 2D Graphs.

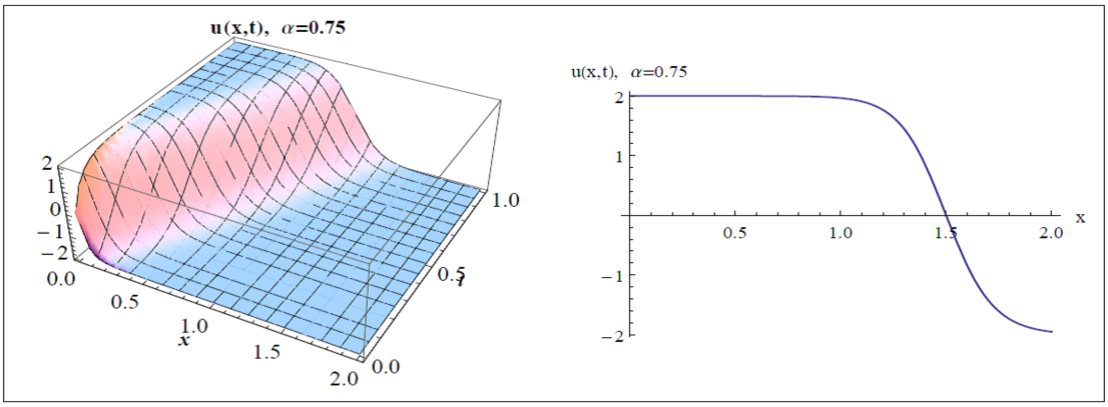

Fig. 2: Profiles of Eq. (17), Subsituting the Values of the Parameters $a_1 = 2, p = 1.8, q = 0.6, r = -1, \alpha = 0.75$ and $t = 1.5$ for the 2D Graph.

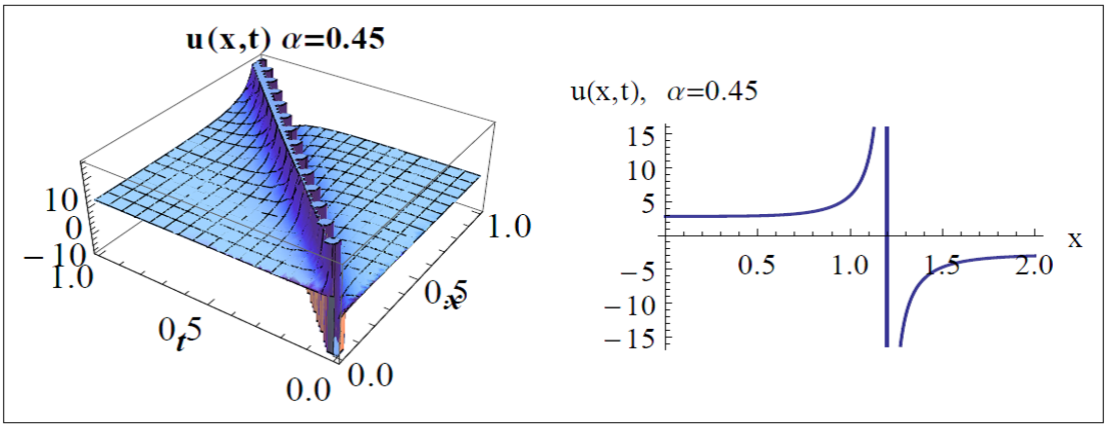

Fig. 3: Profiles of Eq. (18), Subsituting the Values of the Parameters $a_1 = 2, p = 2, q = 1, r = -4, \alpha = 0.45$ and $t = 1$ for 2D Graph.

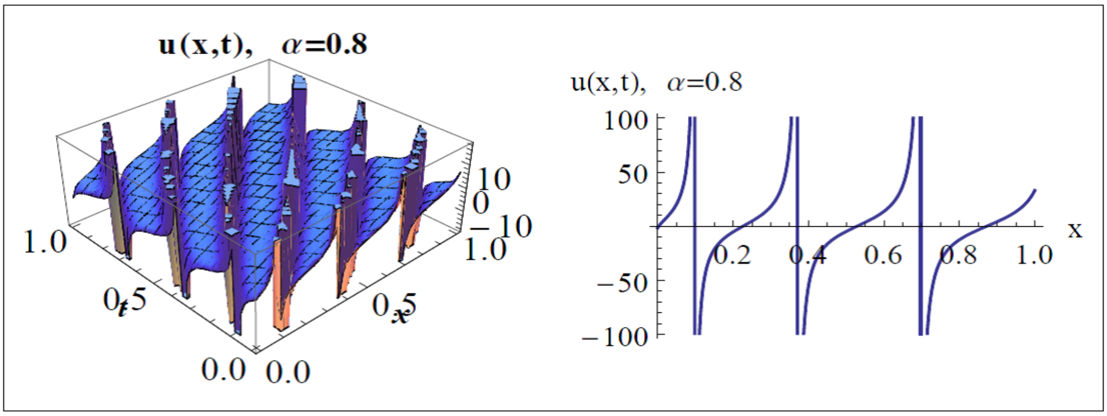

Fig. 4: Profiles of Eq. (19), Subsituting the Values of the Parameters $a_1 = 2, p = 2, q = 1, r = 2, \alpha = 0.8$ and $t = 2.5$ for the 2D Graph.

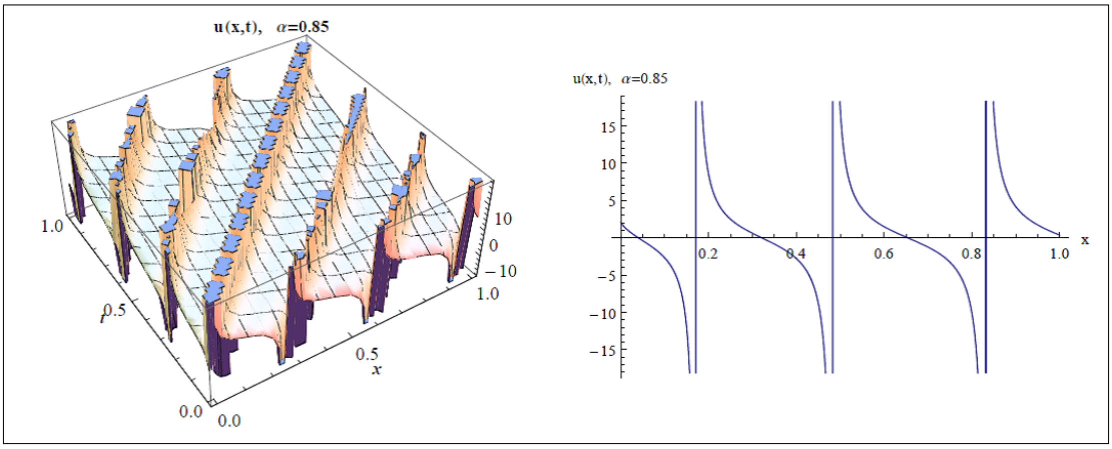

Fig. 5: Profiles of Eq. (20), Subsituting the Values of the Parameters $a_1 = 2.3, p = 2, q = 1, r = 1.5, \alpha = 0.85$ and $t = 2$ for the 2D Graph.

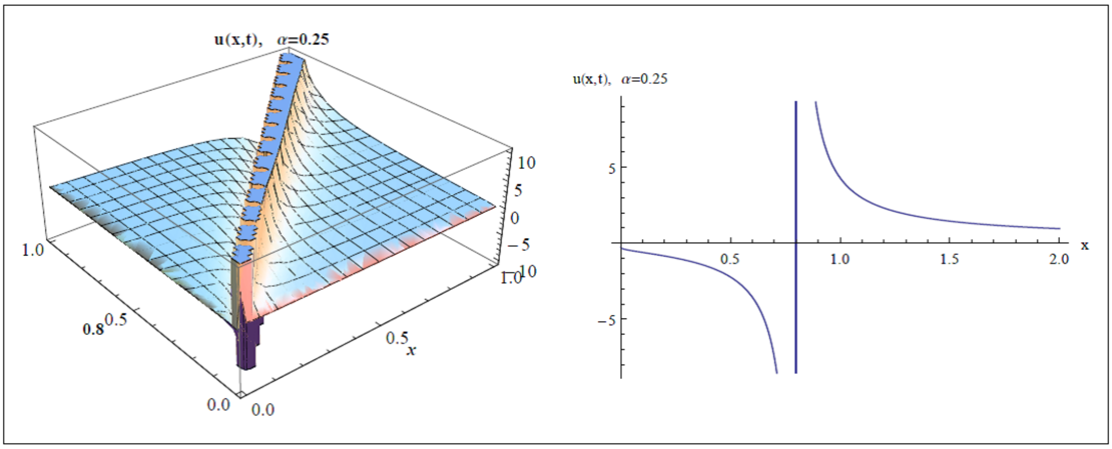

Fig. 6: Profiles of Eq. (21), Subsituting the Values of the Parameters $a_1 = 2.8, p = 3, q = 1, r = 0, \alpha = 0.25$ and $t = 0.25$ for 2D Graph.

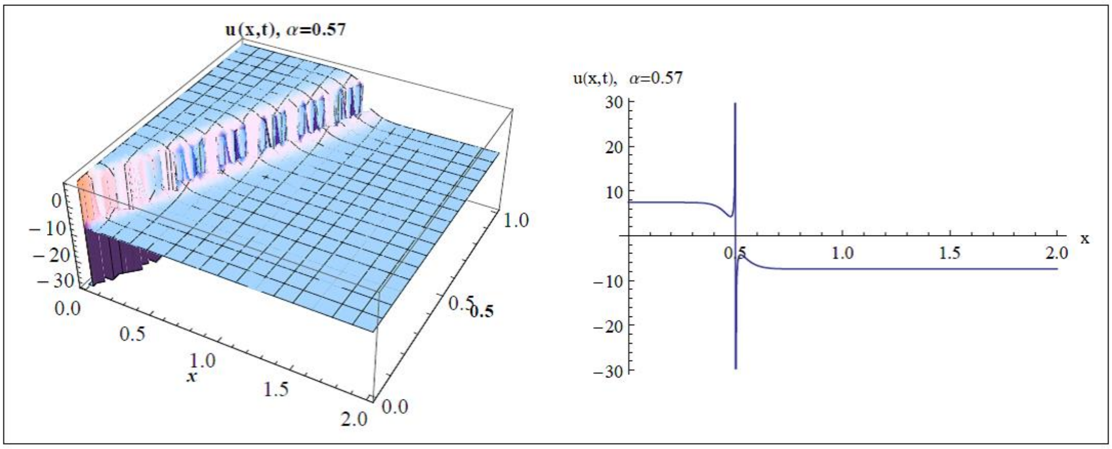

Fig. 7: Profiles of Eq. (22), Subsituting the Values of the Parameters $a_1 = 1.8, b_1 = 1.2, p = 2, q = 1, a = 1.2, \alpha = 0.57$ and $t = 2.4$ for the 2D Graph.

5. Results and discussion

This study effectively examines and constructs varieties of solutions for the conformable space-time fractional Benney-Luke equation consisting of exponential, trigonometric and hyperbolic function using two reliable methods of Kudryashov and modified extended than expansion methods, respectively. Various graphical illustrations are depicted using the Mathematica software ranging from Fig. 1 through Fig. 7 at various chosen values of α. different solitons solutions such the singular periodic wave's shapes, kink-type solution shapes and singular soliton solution shapes are obtained for the problem. Finally, it is worth mentioning that at $\alpha = 1$, the classical Benney-Luke equation is recovered, which shows that clearly these reliable methods would give some of the exact analytical solutions obtained in [3].

6. Conclusion

In conclusion, the conformable space-time fractional Benney-Luke equation is extensively examined in this study by employing two reliable analytical methods. The first method was the powerful Kudryashov method that gives exponential function solutions while the second method was the modified extended than expansion method with the Riccati differential equation and yields different trigonometric and hyperbolic function solutions, respectively. Thus, the method can be used in treating various nonlinear space-time fractional differential equations. Finally, the graphs of all solutions are depicted for visualization.

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