International Journal of Scientific World, 2 (2) (2014) 42-47
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www.sciencepubco.com/index.php/IJSW
doi: 10.14419/ijsw.v2i2.2564
Research Paper

Bayesian estimation for the Kumaraswamy-inverse Rayleigh distribution based on progressive first failure censored samples

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Abstract

This paper considers Bayesian estimation of parameter and reliability function of Kumaraswamy-inverse Rayleigh distribution under the different loss functions with progressively first failure censored samples. We used squared error , minimum expected, weighted and Linex loss functions for obtaining the Bayes estimators of parameter and reliability function. Finally, Comparisons are made between Bayes estimators under different loss functions using simulation study.

Keywords: *Kumaraswamy-inverse Rayleigh distribution, Progressive first failure censoring, Bayesian estimation, reliability function.*

1. Introduction

The Kumaraswamy-inverse Weibull distribution was introduced by Shahbaz et. al [5]. This distribution is an extension of the Inverse Weibull distribution. The Kumaraswamy-inverse Weibull distribution has a cumulative distribution function of the form

$$F(x) = 1 - \left[1 - \exp\left(-\gamma\lambda x^{-\beta}\right)\right]^{\alpha} \tag{1}$$

where $(x, \gamma, \lambda, \alpha, \beta) \in \Re^{+}$. In cumulative distribution function (1), if $\gamma = 1$ and $\beta = 2$, the resulting distribution is called the Kumaraswamy-inverse Rayleigh distribution (Hussian [6]). The cumulative distribution function, probability density function and reliability function of Kumaraswamy-inverse Rayleigh distribution are given, respectively, by

$$F(x) = 1 - \left[1 - \exp\left(-\lambda x^{-2}\right)\right]^{\alpha} \tag{2}$$

$$f(x) = 2\alpha\lambda x^{-3} \exp\left(-\lambda x^{-2}\right)\left[1 - \exp\left(-\lambda x^{-2}\right)\right]^{\alpha-1} \tag{3}$$

$$R(x) = \left[1 - \exp\left(-\lambda x^{-2}\right)\right]^{\alpha} \tag{4}$$

where $\alpha > 0$ and $\lambda > 0$ are shape and scale parameters respectively. In this article, we consider progressive first-failure-censored samples from a Kumaraswamy-inverse Rayleigh distribution. we assumed that shape parameter β is known and used Bayesian estimation method to estimate parameter α and reliability function of Kumaraswamy-inverse Rayleigh distribution based on different symmetric and asymmetric loss functions.

2. A progressive first failure censoring scheme

In a life-testing experiment, Suppose that n independent groups with k items within each group are put in a life test. R_1 groups and the group in which the first failure is observed are randomly removed from the test as soon as the first failure ($X^{\mathbf{R}}_{1:m:n:k} = X_1$) has occurred, R_2 groups and the group in which the second failure is observed are randomly removed from the test as soon as the second failure ($X^{\mathbf{R}}_{2:m:n:k} = X_2$) has occurred, and finally $R_m (m \leq n)$ groups and the group in which the m-th failure is observed are randomly removed from the test as soon as the m-th failure ($X^{\mathbf{R}}_{m:m:n:k} = X_m$) has occurred. Then $X_1 < X_2 < \cdots < X_m$ are called progressively first-failure-censored order statistics with the progressive censoring scheme $\mathbf{R}$. It is clear that $n = m + R_1 + R_2 + \cdots R_m$. If the failure times of the $n \times k$ items originally in the test are from a continuous population with distribution function $F(x)$ and probability density function $f(x)$, the joint probability density function for $X_1, X_2, \cdots, X_m$ is given by

$$f(x_1, x_2, \cdots, x_m) = ck^m \prod_{i=1}^{m} f(x_i)\,(1 - F(x_i))^{k(R_i+1)-1} \qquad 0 < x_1 < x_2 < \cdots < x_m < \infty \tag{5}$$

where $c = n(n - R_1 - 1)(n - R_1 - R_2 - 2)\cdots(n - R_1 - R_2 - \cdots - R_{m-1} - m + 1)$. Note that if $k = 1$, the equation (5) reduces to the joint probability density function of progressively type II censored order statistics. If $\mathbf{R} = (0, \cdots, 0)$,equation (5) reduces to the joint probability density function of first-failure censored order statistics. If $k = 1$ and $R = (0, \cdots, 0)$, then $n = m$ which corresponds to the complete sample, and if $k = 1$ and $\mathbf{R} = (0, \cdots, 0, n - m)$, then type II censored order statistics are obtained (See Wu and kus [1] or Wu and Huang [2]).

3. Likelihood, prior and posterior

Suppose $X_1 < X_2 < \cdots < X_m$ under the progressive censoring scheme $\mathbf{R} = (R_1, R_2, \cdots, R_m)$ are progressively first-failure-censored samples samples from the Kumaraswamy-inverse Rayleigh distribution which has probability density function (3) then the likelihood function for the observed m samples can be written as follows:

$$L(\alpha, \mathbf{X}) \propto \alpha^m e^{-\alpha u(x_i)} \tag{6}$$

where $u(x_i) = -k \sum_{i=1}^{m} (R_i + 1) \ln\left[1 - \exp\left(-\lambda x_i^{-2}\right)\right]$. When parameter λ is known, we consider the natural conjugate family of prior densities for parameter a as the following

$$\pi(\alpha) = \frac{b^a}{\Gamma(a)} \alpha^{a-1} e^{-b\alpha}, \quad a, b > 0 \tag{7}$$

From equations (6) and (7), the posterior distribution of α given the data $\mathbf{X} = (x_1, \cdots, x_m)$is is thus

$$\pi(\alpha|x) = \frac{(b + u(x_i))^{m+a}}{\Gamma(m + a)} \alpha^{m+a-1} e^{-\alpha(b+u(x_i))} \tag{8}$$

4. Bayesian Estimation

4.1. Bayes estimators under squared error loss function

The Bayes estimator of α, under the squared error loss function, is the posterior mean of the pdf given in Equation (8) and can be found as,

$$\hat{\alpha}_{s.e} = \frac{m + a}{b + u(x_i)} \tag{9}$$

Another problem of interest is that of estimating reliability function $R(t)$, with fixed $t > 0$. The Bayes estimator of the reliability function $R(t)$, under squared error loss function, is given by

$$\hat{S}_{s.e} = \left(\frac{b + u(x_i)}{b + u(x_i) - \ln w(t)}\right)^{(m+a)} \tag{10}$$

where $w(t) = 1 - \exp(-\lambda t^{-2})$.

4.2. Bayes estimators under minimum expected loss function

In Bayesian estimation, widely used loss function is a quadratic loss function given by

$$L(\lambda, \hat{\lambda}) = w\left(\hat{\lambda} - \lambda\right)^2$$

If $w = 1$, it reduces to squared error loss function and for $w = \lambda^{-2}$,it becomes

$$L(\lambda, \hat{\lambda}) = \lambda^{-2}\left(\hat{\lambda} - \lambda\right)^2$$

known as minimum expected loss function introduced by Rao Tummala and Sathe [4]. based on minimum expected loss function The Bayesian estimator of α is given by

$$\hat{\alpha}_{m.e} = \frac{E(\lambda^{-1}|\mathbf{x})}{E(\lambda^{-2}|\mathbf{x})}$$

Therefore we obtain Bayes estimator of the parameter α as the following form

$$\hat{\alpha}_{m.e} = \frac{m + a - 2}{b + u(x_i)}. \tag{11}$$

and based on minimum expected loss function The the Bayesian estimator of the reliability function $R(t)$ is:

$$\hat{S}_{m.e} = \left(\frac{b + u(x_i) + 2\ln w(t)}{b + u(x_i) + \ln w(t)}\right)^{(m+a)} \tag{12}$$

4.3. Bayes estimators under weighted loss function

The weighted loss function is defined as

$$L(\hat{\alpha} - \alpha) = \frac{(\hat{\alpha} - \alpha)^2}{\alpha}, \qquad \hat{\alpha} = \frac{1}{E(\alpha|x)}$$

The Bayes estimator of parameter α under weighted loss function is given by

$$\hat{\alpha}_{w.l} = \frac{m + a - 1}{b + u(x_i)} \tag{13}$$

Similarly, the Bayes estimator of the reliability function $R(t)$ under weighted loss function, is given by

$$\hat{S}_{w.l} = \left(\frac{b + u(x_i)}{b + u(x_i) + \ln w(t)}\right)^{-(m+a)} \tag{14}$$

4.4. Bayes estimators under Linex loss function

The LINEX loss function for α can be expressed as the following proportional

$$L(\Delta) \propto \exp\gamma\Delta) - c\Delta - 1; \qquad \gamma \neq 0$$

where $\Delta = (\hat{\alpha} - \alpha)$ and $\hat{\alpha}$ is an estimate of α. The Bayes estimator of α, under Linex loss function is given by (Zellner [7])

$$\alpha_{\hat{Linex}} = -\frac{1}{\gamma} \ln E_{\alpha}[\exp(-\gamma\alpha)] \tag{15}$$

we obtain Bayesian estimator of the parameter α, and reliability function $R(t)$, from (15) and (8) as the following forms

$$\hat{\alpha}_{li} = -\frac{m+a}{\gamma} \ln\left[\frac{b+u(x_i)}{b+u(x_i)+\gamma}\right] \tag{16}$$

$$\hat{S}_{li} = -\frac{1}{\gamma} \ln\left[\sum_{j=0}^{\infty} \frac{(-\gamma)^j}{j!} \left(\frac{b+u(x_i)}{b+u(x_i)-j\ln w(t)}\right)^{(m+a)}\right] \tag{17}$$

5. Simulation study

In order to compare the estimators of parameter and reliability function of Kumaraswamy-inverse Rayleigh distribution , Monte Carlo simulations were performed utilizing 2000 progressively first-failure-censored samples for each simulations. The mean square error (MSE) is used to compare the estimators. Using the algorithm presented in Balakrishnan and Sandhu [3], we generated progressively first failure censored samples from the Kumaraswamy-inverse Rayleigh distribution with parameters $\alpha = \lambda = 2$. The true value of $R(t)$, at $t = 1$ is $R(1) = 0.7476451$ and the prior parameters are $a = 4$, $b = 1$. The simulation results are summarized in tables 1 and 2.

6. Conclusions

From Tables 1 and 2, it is clear that the Bayes estimates of parameter α and reliability function $S = R(t)$, under the LINEX loss function have the smallest estimated MSEs as compared with the estimates under squared error loss, minimum expected and weighted loss functions. It is immediate to note that MSEs decrease as sample size n and m increases.

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Table 1: Estimated means and MSEs of various estimators of α

k	n	m	$\mathbf{R}=(R_1,...,R_m)$	$\hat{\alpha}_{s.e}$ MSE($\hat{\alpha}_{s.e}$)	$\hat{\alpha}_{m.e}$ MSE($\hat{\alpha}_{m.e}$)	$\hat{\alpha}_{w.l}$ MSE ($\hat{\alpha}_{w.l}$)	$\hat{\alpha}_{li}$ MSE ($\hat{\alpha}_{li}$)
1	10	5	(3,2,0,0,0)	2.847335	2.214594	2.530964	2.173908
				1.5246083	0.5340129	0.9192619	0.3156078
		7	(3,0*6)	2.642686	2.162198	2.402442	2.132708
				1.0486233	0.4517774	0.6872304	0.2913873
		10	(0,...,0)	2.524530	2.163883	2.344207	2.137421
				0.7685117	0.3893408	0.5438924	0.2705353
4	10	5	(3,2,0,0,0)	2.841031	2.209691	2.525361	2.168933
				1.5425498	0.5492249	0.9359284	0.3233388
		7	(3,0*6)	2.639475	2.159570	2.399522	2.131072
				1.0312187	0.4420375	0.6739081	0.2851338
		10	(0,...,0)	2.498519	2.141588	2.320053	2.119100
				0.7196066	0.3661507	0.5086252	0.2539655
7	10	5	(3,2,0,0,0)	2.818108	2.191862	2.504985	2.155225
				1.4931381	0.5351818	0.9059430	0.3152019
		7	(3,0*6)	2.647923	2.166483	2.407203	2.137603
				1.0231730	0.4316242	0.6644658	0.2778630
		10	(0,...,0)	2.511919	2.153074	2.332496	2.128032
				0.7573781	0.3873378	0.5376383	0.2695625
1	20	10	(1,1,...,1)	2.506057	2.148049	2.327053	2.124063
				0.7390961	0.3767774	0.5234300	0.2643520
		15	(2,2,1,0*12)	2.355479	2.107534	2.231506	2.094320
				0.4256996	0.2511968	0.3222498	0.1954880
		20	(0,...,0)	2.280428	2.090392	2.185410	2.081388
				0.2996177	0.1938536	0.2373236	0.1594534
4	20	10	(1,1,...,1)	2.501284	2.143957	2.322621	2.121721
				0.7016720	0.3516201	0.4924276	0.2477882
		15	(2,2,1,0*12)	2.352608	2.104965	2.228787	2.091625
				0.4375895	0.2617968	0.3334937	0.2017639
		20	(0,...,0)	2.280359	2.090329	2.185344	2.081086
				0.3083763	0.2012343	0.2453785	0.1651029
7	20	10	(1,1,...,1)	2.492238	2.136204	2.314221	2.113378
				0.7410525	0.3849834	0.5287833	0.2678319
		15	(2,2,1,0*12)	2.353605	2.105857	2.229731	2.092719
				0.4282152	0.2539167	0.3248814	0.1967879
		20	(0,...,0)	2.288809	2.098075	2.193442	2.087768
				0.3283650	0.2154483	0.2623865	0.1760092

Table 2: Estimated means and MSEs of various estimators of reliability function $R(t)$

k	n	m	$\mathbf{R}=(R_1,...,R_m)$	$\hat{S}_{s.e}$ MSE($\hat{S}_{s.e}$)	$\hat{S}_{m.e}$ MSE($\hat{S}_{m.e}$)	$\hat{S}_{w.l}$ MSE ($\hat{S}_{w.l}$)	$\hat{S}_{li}$ MSE ($\hat{S}_{li}$)
1	10	5	(3,2,0,0,0)	0.6727000	0.6458794	0.6598265	0.6858566
				0.04327193	0.05656659	0.04941108	0.04121359
		7	(3,0*6)	0.6899662	0.6706404	0.6806021	0.7070712
				0.03597317	0.04456047	0.04000882	0.03280836
		10	(0,...,0)	0.6995957	0.6856391	0.6927793	0.7188201
				0.03165453	0.03733389	0.03436473	0.02775752
4	10	5	(3,2,0,0,0)	0.6734790	0.6467316	0.6606412	0.6869254
				0.04318256	0.05648195	0.04932247	0.04111850
		7	(3,0*6)	0.6901789	0.6709014	0.6808375	0.7072558
				0.03578172	0.04430803	0.03978939	0.03259602
		10	(0,...,0)	0.7020048	0.6882935	0.6953067	0.7215884
				0.03065364	0.03613722	0.03327118	0.02663498
7	10	5	(3,2,0,0,0)	0.6755406	0.6491352	0.6628641	0.6893640
				0.04231564	0.05532197	0.04832181	0.04015553
		7	(3,0*6)	0.6892219	0.6698657	0.6798422	0.7059074
				0.03593251	0.04448401	0.03995239	0.03271775
		10	(0,...,0)	0.7008612	0.6870123	0.6940970	0.7204047
				0.03126968	0.03687913	0.03394666	0.02734437
1	20	10	(1,1,...,1)	0.7013698	0.6875728	0.6946306	0.7210210
				0.03104680	0.03660561	0.03369999	0.02711539
		15	(2,2,1,0*12)	0.7143932	0.7053184	0.7099280	0.7366496
				0.02552196	0.02872237	0.02707156	0.02086443
		20	(0,...,0)	0.7210782	0.7143165	0.7177384	0.7447080
				0.02287209	0.02508610	0.02395220	0.01787462
4	20	10	(1,1,...,1)	0.7016120	0.6878760	0.6949017	0.7210937
				0.03069341	0.03616738	0.03330689	0.02669613
		15	(2,2,1,0*12)	0.7147744	0.7057150	0.7103168	0.7371668
				0.02549235	0.02869704	0.02704381	0.02083327
		20	(0,...,0)	0.7211487	0.7143839	0.7178074	0.7448684
				0.02293052	0.02515304	0.02401471	0.01794961
7	20	10	(1,1,...,1)	0.7028376	0.6891592	0.6961560	0.7228813
				0.03065798	0.03615785	0.03328276	0.02670291
		15	(2,2,1,0*12)	0.7146062	0.7055433	0.7101468	0.7369199
				0.02547412	0.02867081	0.02702186	0.02081412
		20	(0,...,0)	0.7203858	0.7135744	0.7170217	0.7440087
				0.02328193	0.02554478	0.02438564	0.01832605