International Journal of Scientific World, *9 (1) (2023) 1-7*

International Journal of Scientific World

Website: www.sciencepubco.com/index.php/IJSW

Research paper

On the light deflection and perihelion precession by BBMB black holes in HPM approximation

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Abstract

In this paper, the homotopy perturbation method (HPM) is applied for calculating the weak deflection angle and the perihelion precession angle of planetary orbits in the gravitational field of Bocharova-Bronnikov-Melnikov-Bekenstein (BBMB) black hole. Follow the procedure of HPM, we obtain the approximate solutions for the null and time-like geodesics in the gravitational field of BBMB black hole. On the basis of these solutions and the general formulae for the angle of deflection and the perihelion precession angle, derived by the author earlier via HPM, the corresponding angles for the BBMB black hole are obtained and compared with the similar angles in Schwarzschild spacetime. Notably that if the deflection angle obtained in this article using the HPM confirms the results of other researchers, then the perihelion precession angle obtained here for the first time can be compared with those known for a classical non-rotating static black hole.

Keywords: *Homotopy Perturbation Method; BBMB Black Holes; Light Deflection; Perihelion Precession.*

1. Introduction

Both the deflection of light and the perihelion precession of orbits in the gravitational fields of compact objects are the first and key predictions of General Relativity (GR). Until now, these effects play a significant role in the study of problems related to the gravity and astrophysics, and continue to provide a powerful tool for studying many compact objects in our universe [1]-[6]. Since light deflection and perihelion precession have historically been associated with the Solar System [7, 8], in the recent decades, much more attention has been paid to the study of light deflection and perihelion precession in the gravitational fields of various compact astrophysical objects.

As the weak gravitational lensing provides a way to find the mass of astronomical objects without requiring about their composition or dynamical states, many authors study gravitational lensing by various astrophysical objects using various methods proposed recently. Let us mention just a few of the latest articles on this topic. For example, the equations of motion of the massive and massless particles in the Schwarzschild geometry is studied by using the Laplace-Adomian Decomposition Method in [9], that shows the obvious success of this method in obtaining series solutions to a wide range of strongly nonlinear differential equations.

Also, Gibbons and Werner proposed a new method to calculate weak deflection angle using the Gauss-Bonnet theorem [10]. This method was applied to the Rindler modified Schwarzschild black hole, and the weak deflection angle was obtained in [11]. In Ref. [12], the Gauss-Bonnet theorem is also applied to the study of light rays in a plasma medium in a static and spherically symmetric gravitational field and to the study of timelike geodesics followed for test massive particles in a spacetime with the same symmetries. The possibility of using the theorem follows from a correspondence between timelike curves followed by light rays in a plasma medium and spatial geodesics in an associated Riemannian optical metric. A similar correspondence follows for massive particles. The calculation of the bending angle using the trajectory equation based on geometric optics is also provided in [13].

Wormholes also cause gravitational lensing effects. Gravitational lensing by wormholes were investigated first by the authors of [14, 15]. The weak gravitational lensing for a black hole and wormhole as well in massive gravity has been studied in the paper [16]. A new analytic approximation describing light bending in Schwarzschild metric, when fast accurate calculations of light deflection are required, has been proposed in Ref.[17]. The features of gravitational lensing by spherically symmetric wormholes, when they are not symmetric with respect to their throats, were considered in [18].

The perihelion precession in GR and some modified theories of gravity were recently studied in [19]-[26]. The perihelion precession of planetary orbits is estimated for different gravity theories in string-inspired models by the authors of [27]. Moreover, a way to obtain information about higher dimensions from observations by studying a brane based spherically symmetric solution is considered for the classic tests of GR in [28]. The analytical computation of the Mercury perihelion precession in the frame of relativistic gravitational law and

comparison with GR relativity is presented in [29]. In order to study the defection and gravitational lensing of null and timelike signals in the Kiselev spacetime in the weak field limit, the authors of [30] used a perturbative method previously developed for asymptotically flat spacetimes.
Recently, the authors of [6] derived the deflection angle of light in a plasma medium by BBMB black hole using the Gibbons and Werner method (Gauss-Bonnet method). They obtained the Gaussian optical curvature and implemented the Gauss-Bonnet theorem to explore the deflection angle for spherically symmetric spacetime of BBMB black hole [31, 32].
In the present paper, we investigate the perihelion precession and deflection of light in the spherically symmetric spacetime of BBMB black hole using the homotopy perturbation method [33]-[35], which is quite efficient in many nonlinear problems (see, e.g., [36]-[40]). In our woks [41, 42], by using HPM we have obtained formulas for calculating the angles of light deflection and perihelion precession from approximate solutions for geodesic equations, that give a better approximation than those often used in such problems.

2. Geodesic equations in BBMB spacetime

Here, we mostly follow Ref.[5] in representing the main equations of geodesic motion in a spherical symmetry spacetime. According to General Relativity [1, 5], in the case of general spherically symmetric spacetime, its stationary line element can be represented by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{h(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{1}$$

Since the perihelion precession and deflection of light are usually treated as the time-like and null geodesics in spacetime [1], respectively, let us consider the geodesics $\gamma(\tau)$ in the above spherically symmetric spacetime. We set the geodesic $\gamma(\tau)$ expressed in the spherical coordinates $x^\mu = (t, r, \theta, \varphi)$ as $x^\mu(\tau)$, which satisfy

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0.$$

The geodesic $\gamma(\tau)$ can be obtained by solving the above equation. However, taking into account the symmetry of spacetime (1), one could use the following simple way to obtain the geodesic $\gamma(\tau)$.
First, we can find that one component of the geodesic $\gamma(\tau)$ can always be chosen as $\theta(\tau) = \pi/2$, which means that the geodesic can always be chosen to lay in the equatorial plane of the spherically symmetric spacetime. Thus, $t = t(\tau),\ r = r(\tau),\ \theta = \pi/2,\ \varphi = \varphi(\tau)$. Let us denote the tangent vector of geodesic $\gamma(\tau)$ as $U^\mu \equiv dx^\mu/d\tau$. For the time-like geodesic, we chose τ to be the proper time . Hence, from (1) we can obtain

$$f(r)\left(\frac{dt}{d\tau}\right)^2 - h^{-1}(r)\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\varphi}{d\tau}\right)^2 = -k, \tag{2}$$

where we have used $\theta = \pi/2$, and $k = 1$ corresponds to the time-like geodesic, while $k = 0$ is the null geodesic.
Second, it could be noted that $\xi^a = (\partial/\partial t)^a$ and $\psi^a = (\partial/\partial\varphi)^a$ are two Killing vectors in the spherically symmetric spacetime (1). Therefore, there are two conserved quantities along the geodesic $\gamma(\tau)$, the total energy and the angular momentum per unit mass, as follows

$$E = -g_{ab}\xi^a U^b = f(r)\frac{dt}{d\tau}, \quad L = g_{ab}\psi^a U^b = r^2\frac{d\varphi}{d\tau}. \tag{3}$$

After inserting (3) into (2), one could obtain

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{h(r)}{f(r)}E^2 - h(r)\left(k + \frac{L^2}{r^2}\right). \tag{4}$$

This equation contains only one function $r(\tau)$, and it could be solved in principle. Then, after inserting the solved $r(\tau)$ into (3), the rest components $t(\tau)$ and $\varphi(\tau)$ of geodesic could be finally obtained.
However, it should be noted that the perihelion precession as well as the deflection of light are usually related to the geodesics orbits, i.e. $r = r(\varphi)$. Therefore, it is convenient to rewrite equation (4) with the help of (3) as

$$\left(\frac{dr}{d\varphi}\right)^2\left(\frac{L}{r^2}\right)^2 = \frac{h(r)}{f(r)}E^2 - h(r)\left(k + \frac{L^2}{r^2}\right). \tag{5}$$

It is well known that the coordinate $u \equiv 1/r$ is more convenient than r to study the geodesic equations in the spherically symmetric gravitational fields. Thus, the main equation could be simply obtained from equation (5) by converting r into u:

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{h(u)}{f(u)}\left(\frac{E}{L}\right)^2 - h(u)\left(\frac{k}{L^2} + u^2\right).$$

At last, differentiating this equation with respect to φ, we get the second-order geodesic equation in the following form

$$\frac{d^2u}{d\varphi^2} = \frac{E^2}{2L^2}\frac{d}{du}\left[\frac{h(u)}{f(u)}\right] - h(u)u - \frac{1}{2}\left(\frac{k}{L^2} + u^2\right)\frac{dh(u)}{du}. \tag{6}$$

The BBMB black hole in a static and spherically symmetric form is given as (1) with the following metric functions [6]

$$f(r) = h(r) = 1 - \frac{2M}{r} + \frac{M^2}{r^2}, \tag{7}$$

where M is a mass of black hole. According to the latter, $f(u) = h(u) = 1 - 2Mu + M^2u^2$ and the geodesic equation (6) takes the following form:

$$\frac{d^2u}{d\varphi^2} + u = k\frac{M}{L^2}(1 - Mu) + 3Mu^2 - 2M^2u^3. \tag{8}$$

Thus, first of all we should approximately solve this equation in the case of $k = 0$ for the null-geodesic, and for the case of time-like trajectory, when $k = 1$. It is easy to see that the BBMB space-time metric coefficients defined by Eq.(7) can be formally obtained from the Reissner-Nordström metric $f(r) = h(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$ by replacing $Q \to M$. This fact allows to use the results of our works [41, 42] to investigate the problems stated above for the BBMB black hole.

3. On HPM briefly

In the next sections, we are going to apply HPM for studying the propagation of a light ray and the orbital motion in the BBMB space-time. For briefness, since the HPM has now become standard, we recall here only the basic ideas of the HPM [33, 34]. Considering equation (8) as the specific case of the following non-linear equation

$$L(u) + N(u) = 0$$

for the function $u(\varphi)$, where $\varphi \in \Phi$, L and N are the linear and non-linear terms, we construct a homotopy $u(\varphi, p) : \Phi \times [0,1] \to I\!R$ as follows

$$H(u, p) = (1 - p)[L(u) - L(u_0)] + p[L(u) + N(u)] = 0,$$

where $p \in [0,1]$ is an embedding parameter, and $u_0 = u_0(\varphi)$ is an initial approximation. Hence, one can see that changing p from 0 to 1 is the same as changing $H(u, p)$ from $L(u) - L(u_0)$ to $L(u) + N(u)$, which are called homotopic. By applying the perturbation procedure, we assume that the solution of (8) can be expressed as a series in p, as follows:

$$u(\varphi) = u_0(\varphi) + pu_1(\varphi) + p^2u_2(\varphi) + \ldots. \tag{9}$$

When we put $p \to 1$, then equation $L(u) + N(u) = 0$ corresponds to (8), and (9) becomes the approximate solution of (8), that is $u(\varphi) = \lim_{p\to 1} u = u_0(\varphi) + u_1(\varphi) + u_2(\varphi) + \ldots$.

It should be noted that the series (9) is convergent for most cases. However, the convergent rate depends upon the nonlinear operator $A(u) = L(u) + N(u)$. Sometimes, even the first approximation is sufficient to obtain the exact solution [33]. As it is emphasized in [34] and [35], the second derivative of $N(u)$ with respect to u must be small, because the parameter p may be relatively large, i.e. $p \to 1$, and the norm of $L^{-1}\partial N/\partial u$ must be smaller than one, in order that the series converges.

4. Computation of deflection angle in BBMB spacetime using HPM

First of all, we intend to obtain an approximate solution of equation (8) for a light beam using HPM. The null-geodesic equation follows from Eq. (8) fodr $k = 0$ in the form

$$\frac{d^2u}{d\varphi^2} + u = 3Mu^2 - 2M^2u^3. \tag{10}$$

In the absence of mass ($M = 0$), the obvious analytic solution for (10) is a straight line expressed in polar coordinates,

$$u_0(\varphi) = \frac{1}{b}\sin\varphi, \tag{11}$$

where b is a constant impact parameter.

Assuming that the unperturbed equation (10) should have solution (11), consider the following homotopy

$$u'' + u - p\left(3Mu^2 - 2M^2u^3\right) = 0, \tag{12}$$

where $p \in [0,1]$.

By substituting (9) into equation (10) and collecting together all terms with the same degree of the embedding parameter p, one can transform this equation into another series in p. Equating each coefficient of this series equal to zero yields a set of linear ordinary differential equations for $u_0(\varphi), u_1(\varphi), u_2(\varphi)$, etc.:

$$\left.\begin{aligned}
p^0 &: \quad u_0'' + u_0 = 0, \\
p^1 &: \quad u_1'' + u_1 - 3Mu_0^2 + 2M^2u_0^3 = 0, \\
p^2 &: \quad u_2'' + u_2 - 6Mu_0u_1 + 6M^2u_0^2u_1 = 0, \\
&\qquad \ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{13}$$

According to equation (11), the initial conditions for $u_0(0)$ and $u_i(0)$ can be chosen as follows

$$u_0(0) = 0, \;\; u_0'(0) = \frac{1}{b}, \quad u_i(0) = u_i'(0) = 0, \tag{14}$$

where $i \geq 1$.
The system of linear equations (13) subject to initial conditions (14) can be easily solved, giving

$$u(\varphi) = \frac{1}{b}\sin\varphi + \frac{M}{b^2}\Big(1-\cos\varphi\Big)^2 - \frac{M^2}{4b^3}\Big[(\cos^2\varphi+2)\sin\varphi - 3\varphi\cos\varphi\Big] \tag{15}$$

for the simplest approximation of solution $u \approx u_0 + u_1$.
One can see that solution (15) satisfies the initial condition $u(0) = 0$. Therefore, the deflection angle of light β can be obtained from the equation $u(\pi+\beta) = 0$, using the small angle approximation

$$\sin(\pi+\beta) \approx -\beta, \qquad \cos(\pi+\beta) \approx -1. \tag{16}$$

Using the approximation (16) in formula (15), one can obtain the following expression for the deflection angle in BBMB space-time:

$$\beta \approx \frac{4M}{b} - \frac{3\pi M^2}{4b^2}. \tag{17}$$

We should note that the recent article [6] contains the investigation of deflection angle by BBMB black hole in plasma medium by using Gauss-Bonnet theorem. The deflection angle obtained there for photon beams moving in a homogeneous plasma medium is equal to

$$\beta \simeq \frac{4M}{b} + \frac{2M\omega_e^2}{b\omega_\infty^2} - \frac{3\pi M^2}{4b^2}, \tag{18}$$

where ω_e and ω_∞ are electron plasma frequency and light frequency calculated by an observer at infinity respectively. Since the effect of plasma can be removed if $(\omega_e = 0)$, or $(\beta = \omega_e/\omega_\infty \to 0)$, the angle (18) reduces to vacuum case, and it is coincides with our result (17).
Note that the accuracy of the approximation formula for the deflection angle given by Eq. (17) is not very high, since a rather rough approximation (16) was used in its derivation. In our work [42], we have derived the formula for finding the light deflection angle using HPM.
For this end, let us consider equation (15) as $u(\varphi) = (1/l)\sin\varphi + (1/l)U(\varphi)$, where the first term is the straight path of light without disturbing by gravity, that is $u_0(\varphi)$, and $U(\varphi) = l\,[u_1(\varphi) + u_2(\varphi) + ...]$. Then, the deflection equation, $u(\varphi) = 0$, becomes as follows $\sin\varphi + U(\varphi) = 0$. By solving this equation for $\varphi = \pi + \beta$ via HPM in [42], the deflection angle was obtained as follows:

$$\beta_{HPM} = U(\pi)\left(1 + U'(\pi) + U'^2(\pi) + \frac{U(\pi)U''(\pi)}{2} + \frac{U^2(\pi)}{6}\right). \tag{19}$$

Taking the order of accuracy represented by equation (15), let us specify the magnitude of the deflection angle in the case of BBMB black hole. For the approximate solution (15), we can write that

$$U(\varphi) = \frac{M}{b}\Big(1-\cos\varphi\Big)^2 - \frac{M^2}{4b^2}\Big[(\cos^2\varphi+2)\sin\varphi - 3\varphi\cos\varphi\Big]. \tag{20}$$

Substituting $\varphi = \pi$ into $U(\varphi), U'(\varphi)$ and $U''(\varphi)$, we get

$$U(\pi) = -U''(\pi) = \frac{4M}{b} - \frac{3\pi M^2}{4b^2}, \quad U'(\pi) = 0, \tag{21}$$

according to equation (20). Using equations (19) and (21), we get the following angle of deflection in the gravitational field of BBMB black hole

$$\beta_{BBMB} = \left(\frac{4M}{b} - \frac{3\pi M^2}{4b^2}\right) \times \left[1 - \frac{16M^2}{3b^2} + \frac{2\pi M^3}{b^3} - \frac{3\pi^2 M^4}{16b^4}\right]. \tag{22}$$

At the same time, the best approximation for the deflection angle in Schwarzschild spacetime obtain via HPM earlier in [42] is given by

$$\beta_{Schw} = \left(\frac{4M}{b} + \frac{15\pi M^2}{4b^2}\right) \times \left[1 + \frac{8M^2}{3b^2} - \frac{10\pi M^3}{b^3} + \left(64 - \frac{75\pi^2}{16}\right)\frac{M^4}{b^4}\right]. \tag{23}$$

It can be seen from Fig.1 that the angle of light deflection by the BBMB black hole β_{BBMB} determined by equation (22) increases with increasing M/b, but more slowly than it increase according to equation (17). The angle β_{BBMB} grows even more slowly compared to β_{Schw} and compared to the simplest approximation $\beta_0 = 4M/b$.

5. Computation of precession angle in BBMB spacetime using HPM

For the case of time-like trajectory in BBMB space-time, when $k = 1$, the geodesic equation (8) becomes as follows

$$\frac{d^2u}{d\varphi^2} + \Big(1 + \frac{M^2}{L^2}\Big)u = \frac{M}{L^2} + 3u^2M - 2M^2u^3. \tag{24}$$

Note that the corresponding equation in the case of Newton's gravity of a point mass M,

$$\frac{d^2u_0}{d\varphi^2} + u_0 = \frac{M}{L^2}, \tag{25}$$

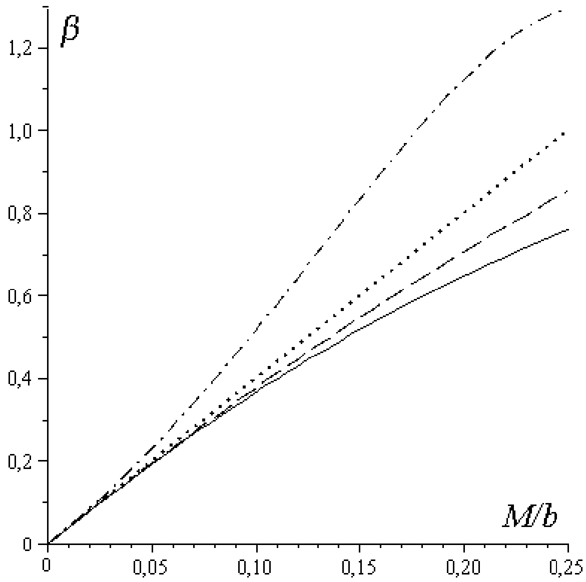

Figure 1: Plots of the deflection angle β_{BBMB} versus the parameter M/b given by Eq.(22) (solid line) and Eq.(17) (dash line), compared to β_{Schw} in Eq.(23) (dash-dot line) and the simplest $\beta_0 = 4M/b$ (dot line).

the analytical elliptical solution of which has already been known as

$$u_0(\varphi) = \frac{M}{L^2}(1 + e\cos\varphi), \tag{26}$$

where $e \in (0,1)$ is the orbital eccentricity.
Now consider the HPM solution of the equation (24). For this purpose, we suppose the following homotopy

$$u'' + u - \frac{M}{L^2} + p\left(\frac{M^2}{L^2}u - 3Mu^2 + 2M^2u^3\right) = 0, \tag{27}$$

where $p \in [0,1]$. According to (25) and (26), the initial conditions for $u_0(0)$ and $u_i(0)$ can be chosen as follows

$$u_0(0) = \frac{M}{L^2}(1+e),\ u_0'(0) = 0, \quad u_i(0) = u_i'(0) = 0, \tag{28}$$

where $i \geq 1$. Substituting (9) into equation (27), we get

$$p^0 \ : u_0'' + u_0 - \frac{M}{L^2} = 0, \tag{29}$$

$$p^1 \ : u_1'' + u_1 + \frac{M^2}{L^2}u_0 - 3Mu_0^2 + 2M^2u_0^3 = 0, \tag{30}$$

$$\cdots\cdots\cdots\cdots$$

where the simplest approximation is taken. For the approximate solution $u(\varphi) = u_0(\varphi) + u_1(\varphi)$, the set of linear equations (29), (30) with the initial conditions (28), can be easily solved, giving

$$u(\varphi) = \frac{M}{L^2}(1 + e\cos\varphi) + \frac{M}{L^2}U(\varphi), \tag{31}$$

where

$$\begin{aligned} U(\varphi) = \frac{M^2}{L^2}\Bigg[&2 + 2e^2 - 2(1+2e^2)\frac{M^2}{L^2} + \left(\frac{7}{6} - (4+e^2)\frac{M^2}{4L^2}\right)3e\varphi\sin\varphi \\ &-\left(2 + e^2 + (e^3 - 8e^2 - 8)\frac{M^2}{4L^2}\right)\cos\varphi - \left(1 - 2\frac{M^2}{L^2}\right)e^2\cos^2\varphi + \frac{M^2}{4L^2}e^3\cos^3\varphi\Bigg] \end{aligned} \tag{32}$$

is the differentiable correction function to the Keplerian orbit (26).
By using HPM, the formula for the precession angle is obtained in [41] from the maximum condition $u'(\varphi) = 0$ in the position of perihelion. Applying this condition to equation (31), one can see that the following equation

$$\sin\varphi - e^{-1}U'(\varphi) = 0 \tag{33}$$

must be solved subject to the unperturbed solution $\varphi = 2\pi$.
It is clear that this equation could be approximately solved using the assumption that the precession angle α is much smaller compared to 2π, i.e. using the approximate equalities $\sin(2\pi+\alpha) \approx \alpha$, $\cos(2\pi+\alpha) \approx 1$ and the similar ones.

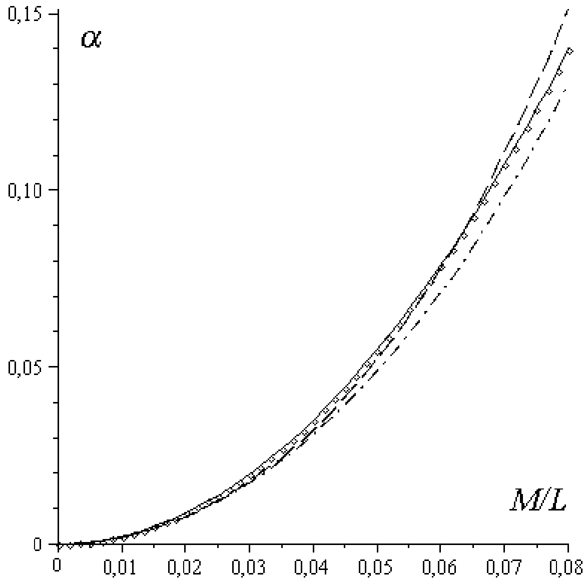

Figure 2: Plots of the precession angle α_{BBMB} versus the parameter M/L given by Eq.(36) with $e = 0.09$ (solid line), and $e = 0.99$ (point line), compared to α_{Schw} given by Eq.(37) with $e = 0.09$ (dash line) and $e = 0.99$ (dash-dot line).

Nevertheless, when the need the greater accuracy, one has to take into account not only linear terms but also the higher degrees of α in expansion of $U'(2\pi+\alpha)$ in the power series. In our paper [41], using HPM, we derived the following formula

$$\alpha_{HPM} = e^{-1}U'(2\pi)\left(1+e^{-1}U''(2\pi)+e^{-2}\left[U''^2(2\pi)+\frac{U'(2\pi)U'''(2\pi)}{2}+\frac{U'^2(2\pi)}{6}\right]\right). \tag{34}$$

Using the approximate solution (31) and (32), one can get all terms in equation (34), for example,

$$U'(2\pi) = 7\pi e\frac{M^2}{L^2} - 3\pi e(4+e^2)\frac{M^4}{2L^4}. \tag{35}$$

With the help of (34) and (35), one can easily obtain the angle of the orbit precession per revolution in the minimum degree of approximation as

$$\alpha_{BBMB} = 7\pi\frac{M^2}{L^2} - 3\pi(4+e^2)\frac{M^4}{2L^4}. \tag{36}$$

Note that in [41] we obtained the following angle of precession by Schwarzschild black hole in HPM approximation:

$$\alpha_{Schw} = 6\pi\frac{M^2}{L^2}\left[1+\frac{3(1+e)^2}{e}\frac{M^2}{L^2}\right]. \tag{37}$$

It is seen from Fig.2 that the precession angle α_{BBMB} in the space-time of BBMB black hole monotonically increases with the growth of the parameter M/L. Moreover, the dependence of α_{BBMB} on the orbital eccentricity given by equation (36) is rather weak. This is illustrated for $e = 0.09$ and $e = 0.99$. Furthermore, α_{BBMB} is compared to the precession angle α_{Schw} given by equation (37) with the same values of eccentricity.

6. Conclusions

Thus, in this work we have provided the analytical computation of the deflection of light and the perihelion precession in the gravitational field of BBMB black hole spacetime with the help of HPM. First, we have applied HPM for solving the geodesic equations in the spacetime of BBMB black hole. More specific, we have followed the simple procedure of HPM obtained the approximate solutions for the null and time-like geodesics in the gravitational field of BBMB black hole.

Then, on the basis of the obtained solutions and the general formulae for the angle of deflection and the perihelion precession angle, derived by the author earlier via HPM, the corresponding angles for the BBMB black hole are obtained and compared with the similar angles in Schwarzschild spacetime. As a result, in the spacetime of BBMB black hole we have obtained the light deflection angle β_{BBMB} given by Eq. (22) and the perihelion shift per revolution α_{BBMB} given by Eq. (36). Additionally, we have also demonstrated the graphical behavior of deflection angle and the perihelion precession by BBMB black hole.

In conclusion, it could be noted that formulas (19) and (34) for the approximate calculation of the deflection and precession angles, respectively, are better to applied to the exact particle trajectories which could be obtained by some method (see, for example, [43, 44] and references therein). Unfortunately, it is most often not possible to find suitable exact solutions of geodesic equations in functions, the analysis of which is rather simple. However, the study of this issue is already beyond the scope of this article.

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