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Research paper

[0, 1]truncated fréchet-gamma and inverted gamma distributions

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Abstract

In this paper, we introduce a new family of continuous distributions based on [0, 1]] truncated Fréchet distribution. [0, 1]] Truncated Fréchet Gamma ([0, 1]] TFG) and truncated Fréchet inverted Gamma ([0, 1]] TFIG) distributions are discussed as special cases. The cumulative distribution function, the rth moment, the mean, the variance, the skewness, the kurtosis, the mode, the median, the characteristic function, the reliability function and the hazard rate function are obtained for the distributions under consideration. It is well known that an item fails when a stress to which it is subjected exceeds the corresponding strength. In this sense, strength can be viewed as "resistance to failure." Good design practice is such that the strength is always greater than the expected stress. The safety factor can be defined in terms of strength and stress as strength/stress. So, the [0, 1]] TFG strength-stress and the [0, 1]] TFIG strength-stress models with different parameters will be derived here. The Shannon entropy and Relative entropy will be derived also.

***Keywords**: [0, 1] 1] TFG; [0, 1] 1] TFIG; Stress-Strength Model; Shannon's Entropy; Relative Entropy.*

1. Introduction

Here, we proposed a distribution with the hope it would attract wider applicability in other fields. The generalization which is motivated by the work of Eugene et al. [1] will be our guide. [1] defined the beta G distribution from a quite arbitrary cumulative distribution function (cdf) , G(x) by

$$F(x) = (1/\beta(a,b)) \int_0^{G(x)} w^{a-1}(1-w)^{b-1}\, dw \qquad (1)$$

Where a >0 and b >0 are two additional parameters whose role is to introduce skewness and to vary tail weight and $\beta(a,b) = \int_0^1 w^{a-1}(1-w)^{b-1}\, dw$ is the beta function. The class of distributions (1) has an increased attention after the works by [1] and [2]. Application of $X = G^{-1}(V)$ to the random variable V following a beta distribution with parameters a and b, V ~B(a, b) say, yields X with cdf (1).[1] defined the beta normal (BN) distribution by taking G(x) to be the cdf of the normal distribution and derived some of its first moments. General expressions for the moments of the BN distribution were derived [3]. An extensive review of scientific literature on this subject is available in [4]. We can write (1) as,

$$F(x) = I_{G(x)}(a,b) \qquad (2)$$

Where, $I_y(a,b) = (1/B(a,b) \int_0^y w^{a-1}(1-w)^{b-1}\, dw$, denotes the incomplete beta function ratio, i.e., the cdf of the beta distribution with parameters aand b. For general a andb, we can express (2) in terms of the well-known hypergeometric function defined by,

$${}_2F_1(\alpha,\beta,\gamma;x) = \sum_{i=0}^{\infty} \frac{(\alpha)_i\,(\beta)_i}{(\gamma)_i\; i!} x^i$$

Where $(\alpha)_i = \alpha(\alpha+1)\ldots(\alpha+i-1)$denotes the ascending factorial. We obtain,

$$F(x) = \frac{G(x)^a}{a\,B(a,b)}\; {}_2F_1(a, 1-b, a+1; G(x))$$

The properties of the cdf, F(x) for any beta Gdistribution defined from a parent G(x) in (1), could, in principle, follow from the properties of the hypergeometric function which are well established in the literature; see, for example, Section 9.1 of [5]. The probability density function (pdf) corresponding to (1) can be written in the form,

$$f(x) = \frac{1}{B(a,b)} G(x)^{a-1}\left(1-G(x)\right)^{b-1} g(x) \qquad (3)$$

Where $g(x) = \partial G(x)/\partial x$is the pdf of the parent distribution.

Now, since the pdf and cdf of [0, 1] truncated Fréchet distribution are respectively,

$$h(x) = \frac{ab}{e^{-a}} x^{-(b+1)} e^{-ax^{-b}} \quad 0 < x < 1 \qquad (4)$$

$$H(x) = \frac{1}{e^{-a}} e^{-ax^{-b}} \qquad (5)$$

Graphs for some arbitrary parameters values of pdf and cdf are shown in figure (1) and figure (2) respectively,

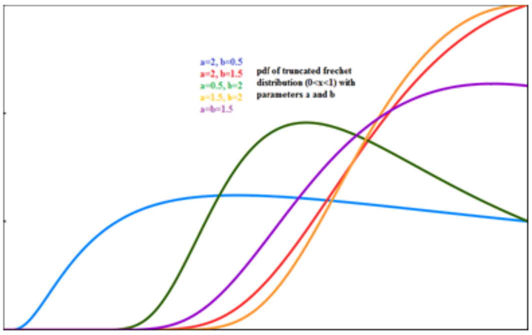

Fig. 1: PDF of (0, 1) Truncated Frechet Distribution.

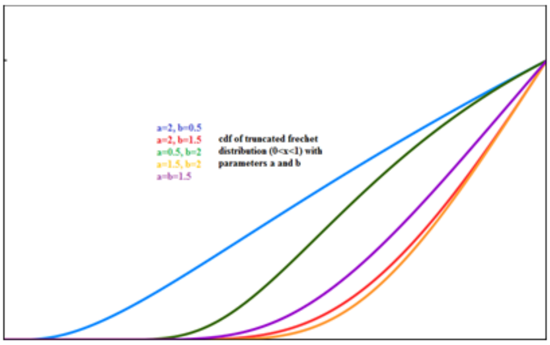

Fig. 2: CDF of (0, 1) Truncated Frechet Distribution.

Now, given two absolutely continuous cdfs, H and G, so that h and g are their corresponding pdfs. We suggest a new distribution F by composingH with G, so that $F(x) = H(G(x))$ is a CDF,

$$F(x) = \int_0^{G(x)} \frac{ab}{e^{-a}} t^{-(b+1)} e^{-at^{-b}} \, dt$$

$$= \frac{1}{e^{-a}} e^{-at^{-b}} \Big]_0^{G(x)} = \frac{1}{e^{-a}} e^{-aG(x)^{-b}} \tag{6}$$

With pdf,

$$f\ \ (x) = \frac{\partial}{\partial x} F(x) = \frac{\partial}{\partial x} \frac{e^{-aG(x)^{-b}}}{e^{-a}} \tag{7}$$

$$= \frac{ab}{e^{-a}} e^{-aG(x)^{-b}} \left(G(x)\right)^{-(b+1)} g(x)$$

With $G(x)$being a baseline distribution, we define in (6) and (7) above, a generalized class of distributions. Wewill name it the [0, 1] truncated Fréchet -G distribution.

In the following two sections, we will assume that G are Gamma and Inverted Gamma distributions respectively.

2. [0, 1] truncated fréchet gamma distribution

Assume that

$$g(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

And

$$G(x) = \gamma(\alpha, \beta x)/\Gamma(\alpha) \ (\ 0 < x)$$

where $\gamma(\alpha, \beta x)$ is the lower incomplete Gamma function are pdf and cdf of Gamma random variable respectively, then, by applying (6) and (7) above, we get the cdf and pdf of [0,1] TFG random variable as follows,

$$F(x) = \frac{1}{e^{-a}} e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}} \tag{8}$$

$$f(x) = \frac{ab\beta^{\alpha}}{e^{-a}\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(b+1)} e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}} \quad x \geq 0$$

$$f(x) = a\, b\, e^{a} \beta^{\alpha} \Gamma^{b}(\alpha) x^{\alpha-1} e^{-\beta x} \left(\gamma(\alpha,\beta x)\right)^{-(b+1)} e^{-a\left(\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)^{-b}} \quad x \geq 0 \tag{9}$$

By Leibnsiz integral rule

$$I = \int_{a(x)}^{b(x)} f(x)\, dx \implies \frac{\partial}{\partial x} I = f\left(x, b(x)\right). \frac{d}{dx} b(x) - f\left(x, a(x)\right). \frac{d}{dx} a(x) + \int_{a(x)}^{b(x)} f(x,t)\, dt$$

So, the reliability and hazard rate functions are respectively

$$R(x) = 1 - F(x) = 1 - \frac{e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}}}{e^{-a}}$$

$$= 1 - e^{-a\left[\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b} - 1\right]}$$

$$\lambda(x) = \frac{f(x)}{R(x)} = \frac{\frac{ab\beta^{\alpha}}{e^{-a}\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(b+1)} e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}}}{1 - e^{-a\left[\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b} - 1\right]}}$$

The rth raw moment can be derived as follows,

$$E(x^r) = \int_0^{\infty} x^r \frac{ab\beta^{\alpha}}{e^{-a}\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(b+1)} e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}} dx$$

Since, $e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^i \left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-bi}$,

Then,

$$E(x^r) = \frac{ab\beta^{\alpha}}{e^{-a}\Gamma(\alpha)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^i \int_0^{\infty} x^{r+\alpha-1}\, e^{-\beta x} \left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(b+1)} \left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-bi} dx$$

$$E(x^r) = \frac{b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \int_0^{\infty} x^{r+\alpha-1}\, e^{-\beta x} \left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)} dx$$

Since $\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)} = \left\{1 - \left(1 - \frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)\right\}^{-(bi+b+1)}$

by using$(1-z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{j!\Gamma(k)} z^j$and

$$(1-z)^b = \sum_{u=0}^{\infty} (-1)^u \frac{\Gamma(b+1)}{u!\Gamma(b-u+1)} z^u |z| < 1\, , k, b > 0 \tag{10}$$

Then,

$$\left\{1 - \left(1 - \frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)\right\}^{-(bi+b+1)} = \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \sum_{u=0}^{\infty} (-1)^u \frac{\Gamma(j+1)}{u!\Gamma(j-u+1)} \left(\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)^u ,$$

And then,

$$E(x^r) = \frac{b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)} \sum_{i,j=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \sum_{u=0}^{\infty} (-1)^u \frac{\Gamma(j+1)}{u!\Gamma(j-u+1)} \int_0^{\infty} x^{r+\alpha-1}\, e^{-\beta x} \left(\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)^u dx$$

Let, $y = \beta x \implies x = \beta^{-1} y \implies dx = \beta^{-1} dy$,

Then,

$$E(x^r) = \frac{b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)} \sum_{i,j=0}^{\infty} \sum_{u=0}^{\infty} \frac{(-1)^{i+u}}{i!u!} a^{i+1} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \frac{\Gamma(j+1)}{\Gamma(j-u+1)}$$

$$\int_0^\infty \beta^{-r-\alpha+1}\, y^{r+\alpha-1} e^{-y}\left(\frac{\gamma(\alpha,y)}{\Gamma(\alpha)}\right)^u \beta^{-1} dy$$

$$= \frac{be^a}{\beta^r\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\int_0^\infty y^{r+\alpha-1}\; e^{-y}\left(\frac{\gamma(\alpha,y)}{\Gamma(\alpha)}\right)^u dy$$

By using [6],

$$I(\alpha+r,u)=\int_0^\infty y^{r+\alpha-1}\; e^{-y}\left(\frac{\gamma(\alpha,y)}{\Gamma(\alpha)}\right)^u dy$$

$$=\alpha^{-u}\,\Gamma\big(r+\alpha(u+1)\big)\,F_A^{(u)}(r+\alpha(u+1);\alpha,\dots,\alpha;\alpha+1,\dots,\alpha+1;-1,\dots,-1) \quad (11)$$

Where, $F_A^{(u)}$ is the lauricella function of type A, then,

$$E(x^r)=\frac{be^a}{\beta^r\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\,I(\alpha+r,u) \quad (12)$$

And then, the characteristic function is

$$Q_x(t)=E\big(e^{ixt}\big)=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}E(x^r)\text{ , since } e^{ixt}=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}x^r$$

$$Q_x(x)=\frac{be^a}{\Gamma(\alpha)}\sum_{r=0}^{\infty}\frac{(it/\beta)^r}{r!}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+r,u)$$

So, the mean μ and variance σ^2 of the of [0,1] TFG random variable are,

$$\mu=E(x)=\frac{be^a}{\beta\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\,I(\alpha+1,u) \quad (13)$$

$$\sigma^2=E(x^2)-(Ex)^2$$
$$=\frac{be^a}{\beta^2\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+2,u)-\left\{\frac{be^a}{\beta\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+1,u)\right\}^2 \quad (14)$$

Since, $F(x)=\dfrac{e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}}}{e^{-a}}=\frac{1}{2}$, The median M_e can be calculated by solving the Nonlinear equation $\left(1+\frac{\ln(2)}{a}\right)^{\frac{-1}{b}}-\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}=0$ numerically.

The skewness of [0, 1] TFG random variable will be,

$$Sk=\frac{\mu_3}{\mu_2^{3/2}}=\frac{Ex^3-3\mu Ex^2+2\mu^3}{(\sigma^2)^{3/2}}$$

$$=\frac{\left\{\begin{array}{l}\frac{be^a}{\beta^3\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+3,u)-3\\ \left\{\frac{be^a}{\beta\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+1,u)\right\}\\ \left\{\frac{be^a}{\beta^2\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+2,u)\right\}+2\\ \left\{\left\{\frac{be^a}{\beta\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+1,u)\right\}\right\}^3\end{array}\right\}}{\left\{\begin{array}{l}\frac{be^a}{\beta^2\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+2,u)-\\ \left\{\frac{be^a}{\beta\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+1,u)\right\}^2\end{array}\right\}^{3/2}} \quad (15)$$

Also,the kurtosis is,

$$kr=\frac{\mu_4}{\mu_2^2}-3=\frac{Ex^4-4\mu Ex^3+6\mu^2Ex^2-4\mu^4}{(\sigma^2)^2}-3$$

$$kr=\frac{\left\{\begin{array}{l}\frac{be^a}{\beta^4\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+4,u)-4\\ \left\{\frac{be^a}{\beta\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+1,u)\right\}\\ \left\{\frac{be^a}{\beta^3\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+3,u)\right\}+6\\ \left\{\frac{be^a}{\beta\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+1,u)\right\}^2\\ \left\{\frac{be^a}{\beta^2\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+2,u)\right\}-3\\ \left\{\frac{be^a}{\beta\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+1,u)\right\}^4\end{array}\right\}}{\left\{\begin{array}{l}\frac{be^a}{\beta^2\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+2,u)-\\ \left\{\frac{be^a}{\beta\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+1,u)\right\}^2\end{array}\right\}^2}-3$$

$$=\frac{\left\{\begin{array}{l}\frac{(\Gamma(\alpha))^3}{b^3e^{3a}}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+4,u)-4\\ \frac{(\Gamma(\alpha))^2}{b^2e^{2a}}\left\{\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+1,u)\right\}\\ \left\{\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+3,u)\right\}+6\\ \frac{\Gamma(\alpha)}{be^a}\left\{\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+1,u)\right\}^2\\ \left\{\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+2,u)\right\}-3\\ \left\{\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+1,u)\right\}^4\end{array}\right\}}{\left\{\begin{array}{l}\frac{\Gamma(\alpha)}{be^a}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+2,u)-\\ \left\{\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+1,u)\right\}^2\end{array}\right\}^2}-3 \quad (16)$$

The quantile function x_q of [0, 1] TFG random variable can be derived as,

$$q=P\big(x\le x_q\big)=F\big(x_q\big)=\frac{e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}}}{e^{-a}}\quad 0<q<1, x_q>0$$

$$\Rightarrow \gamma(\alpha,\beta x)-\Gamma(\alpha)\left(1-\frac{\ln(q)}{a}\right)^{\frac{-1}{b}}=0 \quad (17)$$

So by using the inverse transform method, we can generate [0, 1] TFG random variable by solving

$$\gamma(\alpha,\beta x)-\Gamma(\alpha)\left(1-\frac{\ln(u)}{a}\right)^{\frac{-1}{b}}=0$$

Numerically, where u is a random number uniformly distributed in the unit interval [0, 1].

2.1. Shannon and relative entropies

An entropy of a random variable X is a measure of variation of the uncertainty. The Shannon entropy of [0,1] $TFGG(a,b,\theta)$ random variable *X* can be found as follows,

$$H=-\int_0^\infty f(x)\ln\big(f(x)\big)\;dx$$

$$H=-\int_0^\infty f(x)\ln\left(\frac{ab\beta^\alpha}{e^{-a}\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(b+1)} e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}}\right)dx$$

$$H=\ln\left(\frac{e^{-a}\Gamma(\alpha)}{ab\beta^\alpha}\right)-(\alpha-1)E(\ln x)+\beta E(x)+(b+1)\,E\left(\ln\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}\right)+aE\left(\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}\right)$$

Let, $I_1=-(\alpha-1)E(\ln x)$

$$I_1=-(\alpha-1)\frac{ab\beta^\alpha}{e^{-a}\Gamma(\alpha)}\int_0^\infty \ln(x)\,x^{\alpha-1}e^{-\beta x}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(b+1)} e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}}dx$$

Since, $e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}} = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^i\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-bi}$,then,

$$I_1 = -(\alpha-1)\frac{b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}$$
$$\int_0^\infty \ln(x)\, x^{\alpha-1}e^{-\beta x}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(b+1)}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-bi}dx$$

$$= -(\alpha-1)\frac{b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\int_0^\infty \ln(x)\, x^{\alpha-1}e^{-\beta x}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)}dx$$

By using equation (10), we get,

$$\left\{1-\left(1-\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)\right\}^{-(bi+b+1)} = \sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{u=0}^{\infty}(-1)^u\frac{\Gamma(j+1)}{u!\Gamma(j-u+1)}\left(\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)^u,$$

And then,

$$I_1 = -(\alpha-1)\frac{b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{u=0}^{\infty}\frac{(-1)^u}{u!}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\int_0^\infty \ln(x)\, x^{\alpha-1}e^{-\beta x}\left(\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)^u dx$$

By using incomplete gamma function

$$\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)} = \frac{(\beta x)^\alpha}{\Gamma(\alpha)}\sum_{m=0}^{\infty}\frac{(-\beta x)^m}{(\alpha+m)m!} \tag{18}$$

$$I_1 = -(\alpha-1)\frac{b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{u=0}^{\infty}\frac{(-1)^u}{u!}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\int_0^\infty \ln(x)\, x^{\alpha-1}e^{-\beta x}\left[\frac{(\beta x)^\alpha}{\Gamma(\alpha)}\sum_{m=0}^{\infty}\frac{(-\beta x)^m}{(\alpha+m)m!}\right]^u dx$$

By application of an equation in section 0.314 of [5] for power series raised to power, we obtain for any u positive integer

$$\left[\sum_{m=0}^{\infty}a_m(\beta x)^m\right]^u = \sum_{m=0}^{\infty}C_{u,m}(\beta x)^m$$

, Where the coefficient $C_{u,m}$(for $m = 1,2,\ldots$) satisfy the recurrence relation

$$C_{u,m} = (ma_0)^{-1}\sum_{p=1}^{m}(up-m+p)\,a_pC_{u,m-p}\,,\ C_{u,0} = a_0^u \text{ and } a_p = \frac{(-1)^p}{(\alpha+p)p!}$$

We get,

$$\left(\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)^u = \frac{1}{(\Gamma(\alpha))^u}(\beta x)^{\alpha u}\sum_{m=0}^{\infty}C_{u,m}(\beta x)^m$$

We get,

$$I_1 = -(\alpha-1)\frac{b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}$$

$$\sum_{u=0}^{\infty}\frac{(-1)^u}{u!}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\int_0^\infty \ln(x)$$

$$x^{\alpha-1}e^{-\beta x}\frac{1}{(\Gamma(\alpha))^u}(\beta x)^{\alpha u}\sum_{m=0}^{\infty}C_{u,m}(\beta x)^m\,dx$$

$$= -(\alpha-1)\frac{b\beta^{\alpha+\alpha u+m}}{e^{-a}(\Gamma(\alpha))^{u+1}}\sum_{i,j=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{u=0}^{\infty}\frac{(-1)^u}{u!}$$

$$\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\sum_{m=0}^{\infty}C_{u,m}\int_0^\infty \ln(x)\, x^{\alpha+\alpha u+m-1}e^{-\beta x}\,dx$$

since $\int_0^\infty x^{s-1}e^{-mx}\ln(x)\,dx = m^{-s}\Gamma(s)\{\Psi(s)-\ln(m)\}$, where $s = \alpha+\alpha u+m$ and $m = \beta$

Then,

$$\int_0^\infty \ln(x)\, x^{\alpha+\alpha u+m-1}e^{-\beta x}\,dx = \beta^{-\alpha-\alpha u-m}\Gamma(\alpha(1+u)+m)\{\Psi(\alpha(1+u)+m)-\ln(\beta)\}$$

$$I_1 = -(\alpha-1)\frac{be^a}{(\Gamma(\alpha))^{u+1}}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\sum_{m=0}^{\infty}C_{u,m}\,\Gamma(\alpha(1+u)+m)\{\Psi(\alpha(1+u)+m)-\ln(\beta)\}$$

And, $I_2 = \beta E(x)$

$$= \frac{\beta be^a}{\beta\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}a^{i+1}}{i!u!}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+1,u)$$

$$= \frac{be^a}{\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+1,u)$$

And,

$$I_3 = (b+1)E\left(\ln\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}\right)$$

$$I_3 = (b+1)\int_0^\infty \ln\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}\frac{ab\beta^\alpha}{e^{-a}\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(b+1)}e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}}dx$$

Since,

$$e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}} = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^i\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-bi}$$

Then,

$$I_3 = \frac{(b+1)b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\int_0^\infty \ln\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}x^{\alpha-1}e^{-\beta x}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)}dx$$

since $\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)} = \left\{1-\left(1-\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)\right\}^{-(bi+b+1)}$

By using equation (10), we get,

$$\left\{1-\left(1-\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)\right\}^{-(bi+b+1)} = \sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{u=0}^{\infty}\frac{(-1)^u\Gamma(j+1)}{u!\Gamma(j-u+1)}\left(\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)^u$$

Then,

$$I_3 = \frac{(b+1)b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\int_0^\infty \ln\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}$$

$$x^{\alpha-1}e^{-\beta x}\left(\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)^u dx$$

Using expansion incomplete gamma function

$$\gamma(\theta,x) = x^\theta\,\Gamma(\theta)e^{-x}\sum_{m=0}^{\infty}\frac{x^m}{\Gamma(\theta+m+1)}$$

We get,

$$\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)} = (\beta x)^\alpha e^{-\beta x}\sum_{m=0}^{\infty}\frac{(\beta x)^m}{\Gamma(\alpha+m+1)} \text{ and } \Gamma(\alpha+m+1) = (\alpha+m)!$$

Then,

$$I_3 = \frac{(b+1)b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}$$

$$\int_0^\infty \ln\left\{(\beta x)^\alpha e^{-\beta x}\sum_{m=0}^\infty \frac{(\beta x)^m}{(\alpha+m)!}\right\} x^{\alpha-1}e^{-\beta x}$$

$$\left[(\beta x)^\alpha e^{-\beta x}\sum_{m=0}^\infty \frac{(\beta x)^m}{(\alpha+m)!}\right]^u dx$$

$$=\frac{(b+1)b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i,j=0}^\infty\sum_{u=0}^\infty \frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}$$
$$\sum_{m_1=0}^\infty\cdots\sum_{m_u=0}^\infty \frac{\beta^{\alpha u+m_1+\cdots+m_u}}{(\alpha+m_1)!\dots(\alpha+m_u)!}$$

$$\int_0^\infty \ln\left\{(\beta x)^\alpha e^{-\beta x}\sum_{m=0}^\infty \frac{(\beta x)^m}{(\alpha+m)!}\right\} x^{\alpha(u+1)+m_1+\cdots+m_u-1}e^{-\beta(u+1)x}\,dx$$

Since,

$$\ln\left\{(\beta x)^\alpha e^{-\beta x}\sum_{m=0}^\infty \frac{(\beta x)^m}{(\alpha+m)!}\right\}=\alpha\ln(\beta x)-\beta x+\ln\left(\sum_{m=0}^\infty \frac{(\beta x)^m}{(\alpha+m)!}\right)$$

Then,

$$I_3=\frac{(b+1)b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i,j=0}^\infty\sum_{u=0}^\infty \frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}$$

$$\sum_{m_1=0}^\infty\cdots\sum_{m_u=0}^\infty \frac{\beta^{\alpha u+m_1+\cdots+m_u}}{(\alpha+m_1)!\dots(\alpha+m_u)!}$$
$$\int_0^\infty\left\{\alpha\ln(\beta x)-\beta x+\ln\left(\sum_{m=0}^\infty \frac{(\beta x)^m}{(\alpha+m)!}\right)\right\}x^{\alpha(u+1)+m_1+\cdots+m_u-1}e^{-\beta(u+1)x}\,dx$$

$$I_{31}=\alpha\int_0^\infty \ln(\beta x)\,x^{\alpha(u+1)+m_1+\cdots+m_u-1}e^{-\beta(u+1)x}\,dx$$

let $y=\beta x \implies x=\beta^{-1}y \implies dx=\beta^{-1}dy$

Then,

$$I_{31}=\alpha\int_0^\infty \ln(y)\,\beta^{-\alpha u-\alpha-(m_1+\cdots+m_u)+1}y^{\alpha(u+1)+m_1+\cdots+m_u-1}e^{-(u+1)y}\beta^{-1}\,dy$$

$$=\alpha\beta^{-\alpha u-\alpha-(m_1+\cdots+m_u)}\int_0^\infty \ln(y)\,y^{\alpha(u+1)+m_1+\cdots+m_u-1}e^{-(u+1)y}\,dy$$

$$=\alpha\beta^{-\alpha u-\alpha-(m_1+\cdots+m_u)}(u+1)^{-(\alpha(u+1)+m_1+\cdots+m_u)}$$

$$\Gamma(\alpha(u+1)+m_1+\cdots+m_u)\{\Psi(\alpha(u+1)+m_1+\cdots+m_u)-\ln(u+1)\}$$

$$I_{32}=-\beta\int_0^\infty x\,x^{\alpha(u+1)+m_1+\cdots+m_u-1}e^{-\beta(u+1)x}\,dx$$

$$=-\beta\int_0^\infty x^{\alpha(u+1)+m_1+\cdots+m_u}\,e^{-\beta(u+1)x}\,dx$$

$$=-\frac{\Gamma(\alpha(u+1)+m_1+\cdots+m_u+1)}{\beta^{\alpha u+\alpha+m_1+\cdots+m_u}(u+1)^{\alpha(u+1)+m_1+\cdots+m_u+1}}$$

Now, since,

$$\eta(\tau,\alpha,k,m,d_1,\dots,d_m)=\int_0^\infty \ln\left(\sum_{d=0}^\infty \frac{\left(\frac{x}{\alpha}\right)^{\tau d}}{\Gamma(k+d+1)}\right)\left(\frac{x}{\alpha}\right)^{\tau k(m+1)+\tau(d_1+\cdots+d_m)-1}e^{-(m+1)\left(\frac{x}{\alpha}\right)^\tau}dx$$

Then,

$$I_{33}=\int_0^\infty \ln\left(\sum_{m=0}^\infty \frac{(\beta x)^m}{(\alpha+m)!}\right)x^{\alpha(u+1)+m_1+\cdots+m_u-1}e^{-\beta(u+1)x}\,dx$$

$$=\eta(1,1/\beta,\alpha,u,m_1,\dots,m_u)$$

$$I_3=\frac{(b+1)be^a}{\Gamma(\alpha)}\sum_{i,j=0}^\infty\sum_{u=0}^\infty \frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}$$
$$\sum_{m_1=0}^\infty\cdots\sum_{m_u=0}^\infty \frac{1}{(\alpha+m_1)!\dots(\alpha+m_u)!}$$

$$\left\{\begin{matrix}\alpha\frac{\Gamma(\alpha(u+1)+m_1+\cdots+m_u)}{(u+1)^{(\alpha(u+1)+m_1+\cdots+m_u)}}\left\{\begin{matrix}\Psi(\alpha(u+1)+m_1+\cdots+m_u)-\\ \ln(u+1)\end{matrix}\right\}-\\ \frac{\Gamma(\alpha(u+1)+m_1+\cdots+m_u+1)}{(u+1)^{\alpha(u+1)+m_1+\cdots+m_u+1}}+\beta^{\alpha u+\alpha+m_1+\cdots+m_u}\\ \eta(1,1/\beta,\alpha,u,m_1,\dots,m_u)\end{matrix}\right\}$$

And, $I_4=aE\left(\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}\right)$

$$=\frac{a^2b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\int_0^\infty\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}x^{\alpha-1}e^{-\beta x}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(b+1)}e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}}dx$$

Since

$$e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}}=\sum_{i=0}^\infty \frac{(-1)^i}{i!}a^i\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-bi}$$

Then,

$$I_4=\frac{a^2b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i=0}^\infty \frac{(-1)^i}{i!}a^i\int_0^\infty x^{\alpha-1}e^{-\beta x}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(2b+1)}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-bi}dx$$

$$=\frac{b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i=0}^\infty \frac{(-1)^i}{i!}a^{i+2}\int_0^\infty x^{\alpha-1}e^{-\beta x}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(bi+2b+1)}dx$$

$$\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(bi+2b+1)}=\sum_{j=0}^\infty\sum_{u=0}^\infty \frac{(-1)^u}{u!}\frac{\Gamma([b(i+2)+1]+j)}{j!\Gamma([b(i+2)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\left(\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)^u$$

Then,

$$I_4=\frac{b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i,j=0}^\infty\sum_{u=0}^\infty \frac{(-1)^{i+u}}{i!u!}a^{i+2}\frac{\Gamma([b(i+2)+1]+j)}{j!\Gamma([b(i+2)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\int_0^\infty x^{\alpha-1}e^{-\beta x}\left(\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)^u dx$$

let $y=\beta x \implies x=\beta^{-1}y \implies dx=\beta^{-1}dy$,then,

$$I_4=\frac{b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i,j=0}^\infty\sum_{u=0}^\infty \frac{(-1)^{i+u}}{i!u!}a^{i+2}\frac{\Gamma([b(i+2)+1]+j)}{j!\Gamma([b(i+2)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\int_0^\infty \beta^{-\alpha+1}y^{\alpha-1}e^{-y}\left(\frac{\gamma(\alpha,y)}{\Gamma(\alpha)}\right)^u\beta^{-1}dy$$

$$I_4=\frac{be^a}{\Gamma(\alpha)}\sum_{i,j=0}^\infty\sum_{u=0}^\infty \frac{(-1)^{i+u}}{i!u!}a^{i+2}\frac{\Gamma([b(i+2)+1]+j)}{j!\Gamma([b(i+2)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\int_0^\infty y^{\alpha-1}e^{-y}\left(\frac{\gamma(\alpha,y)}{\Gamma(\alpha)}\right)^u dy$$

By using equation (11) we get,

$$I_4=\frac{be^a}{\Gamma(\alpha)}\sum_{i,j=0}^\infty\sum_{u=0}^\infty \frac{(-1)^{i+u}}{i!u!}a^{i+2}\frac{\Gamma([b(i+2)+1]+j)}{j!\Gamma([b(i+2)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha,u)$$

$$H=\ln\left(\frac{e^{-a}\Gamma(\alpha)}{ab\beta^\alpha}\right)-(\alpha-1)\frac{be^a}{(\Gamma(\alpha))^{u+1}}\sum_{i,j=0}^\infty\sum_{u=0}^\infty \frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}$$

$$\sum_{m=0}^\infty C_{u,m}\,\Gamma(\alpha(1+u)+m)\{\Psi(\alpha(u+1)+m)-\ln(\beta)\}+\frac{be^a}{\Gamma(\alpha)}\sum_{i,j=0}^\infty\sum_{u=0}^\infty \frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+1,u)$$

$$+\frac{(b+1)be^a}{\Gamma(\alpha)}\sum_{i,j=0}^\infty\sum_{u=0}^\infty \frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}$$

$$\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\sum_{m_1=0}^\infty\cdots\sum_{m_u=0}^\infty \frac{1}{(\alpha+m_1)!\dots(\alpha+m_u)!}$$

$$\left\{\begin{matrix}\alpha\frac{\Gamma(\alpha(u+1)+m_1+\cdots+m_u)}{(u+1)^{(\alpha(u+1)+m_1+\cdots+m_u)}}\left\{\begin{matrix}\Psi(\alpha(u+1)+m_1+\cdots+m_u)\\ -\ln(u+1)\end{matrix}\right\}\\ -\frac{\Gamma(\alpha(u+1)+m_1+\cdots+m_u+1)}{(u+1)^{\alpha(u+1)+m_1+\cdots+m_u+1}}+\beta^{\alpha u+\alpha+m_1+\cdots+m_u}\\ \eta(1,1/\beta,\alpha,u,m_1,\dots,m_u)\end{matrix}\right\}+$$

$$\frac{be^{a}}{\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+2}\frac{\Gamma([b(i+2)+1]+j)}{j!\Gamma([b(i+2)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha,u) \tag{19}$$

The relative entropy (or the Kullback–Leibler divergence) is a measure of the difference between two probability distributions F and F^* It is not symmetric in F and F^* In applicationsF typically represents the "true" distribution of data, observations, or a precisely calculated theoretical distribution, while F^* typically represents a theory, model, description, or approximation of F. Specifically, the Kullback–Leibler divergence of F^* from F, denoted D_{KL} $(F\|F^*)$, is a measure of the information gained when one revises ones beliefs from the prior probability distribution F^* to the posterior probability distribution F. More exactly, it is the amount of information that is lost when F^* is used to approximate F,defined operationally as the expected extra number of bits required to code samples from F using a code optimized for F^* rather than the code optimized for F.

The relative entropy $Dkl(F\|F^*)$ for a random variable [0, 1] TFG (a,b,α,β) can be found as follows,

$$\frac{f(x)}{f^*(x)}=\frac{\frac{ab\beta^{\alpha}}{e^{-a}\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(b+1)}e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}}}{\frac{a_1b_1\beta_1^{\alpha_1}}{e^{-a_1}\Gamma(\alpha_1)}x^{\alpha_1-1}e^{-\beta_1 x}\left\{\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right\}^{-(b_1+1)}e^{-a_1\left\{\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right\}^{-b_1}}}$$

$$=\int_0^{\infty}f(x)\ln\left(\frac{ab\beta^{\alpha}e^{-a_1}\Gamma(\alpha_1)\,x^{\alpha-1}e^{-\beta x}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(b+1)}e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}}}{a_1b_1\beta_1^{\alpha_1}e^{-a}\Gamma(\alpha)x^{\alpha_1-1}e^{-\beta_1 x}\left\{\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right\}^{-(b_1+1)}e^{-a_1\left\{\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right\}^{-b_1}}}\right)dx$$

$$=\int_0^{\infty}f(x)\left[\begin{array}{c}\ln\left(\frac{ab\beta^{\alpha}e^{-a_1}\Gamma(\alpha_1)}{a_1b_1\beta_1^{\alpha_1}e^{-a}\Gamma(\alpha)}\right)\\ +(\alpha-\alpha_1)\ln(x)+(\beta_1-\beta)x-(b+1)\ln\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}\\ -a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}\\ +(b_1+1)\ln\left\{\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right\}+a_1\left\{\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right\}^{-b_1}\end{array}\right]dx$$

Let, $I_1=(\alpha-\alpha_1)\int_0^{\infty}\ln(x)\ f(x)\,dx$

$$I_1=(\alpha-\alpha_1)\int_0^{\infty}\ln(x)\frac{ab\beta^{\alpha}}{e^{-a}\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(b+1)}e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}}dx$$

$$=(\alpha-\alpha_1)\frac{be^{a}}{(\Gamma(\alpha))^{u+1}}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\sum_{m=0}^{\infty}C_{u,m}$$

$$\Gamma(\alpha(1+u)+m)\{\Psi(\alpha(1+u)+m)-\ln(\beta)\}$$

And, $I_2=(\beta_1-\beta)\int_0^{\infty}x\ f(x)dx$

$$=(\beta_1-\beta)\int_0^{\infty}x\frac{ab\beta^{\alpha}}{e^{-a}\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(b+1)}e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}}dx$$

$$=\frac{(\beta_1-\beta)be^{a}}{\beta\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha+1,u)$$

And, $I_3=-(b+1)\int_0^{\infty}\ln\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}f(x)\,dx$

$$=-(b+1)\int_0^{\infty}\ln\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}\frac{ab\beta^{\alpha}}{e^{-a}\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(b+1)}e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}}dx$$

$$=\frac{-(b+1)be^{a}}{\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}$$

$$\sum_{m_1=0}^{\infty}\cdots\sum_{m_u=0}^{\infty}\frac{1}{(\alpha+m_1)!\ldots(\alpha+m_u)!}$$

$$\left\{\begin{array}{c}\alpha\frac{\Gamma(\alpha(u+1)+m_1+\cdots+m_u)}{(u+1)^{(\alpha(u+1)+m_1+\cdots+m_u)}}\left\{\begin{array}{c}\Psi(\alpha(u+1)+m_1+\cdots+m_u)\\-\ln(u+1)\end{array}\right\}\\-\frac{\Gamma(\alpha(u+1)+m_1+\cdots+m_u+1)}{(u+1)^{\alpha(u+1)+m_1+\cdots+m_u+1}}+\beta^{\alpha u+\alpha+m_1+\cdots+m_u}\\ \eta(1,1/\beta,\alpha,u,m_1,\ldots,m_u)\end{array}\right\}$$

And, $I_4=-a\int_0^{\infty}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}f(x)dx$

$$=-a\int_0^{\infty}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}\frac{ab\beta^{\alpha}}{e^{-a}\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-\binom{b}{+1}}e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}}dx$$

$$=\frac{-be^{a}}{\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{j}\frac{(-1)^{i+u}}{i!u!}a^{i+2}\frac{\Gamma([b(i+2)+1]+j)}{j!\Gamma([b(i+2)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha,u)$$

And, $I_5=(b_1+1)\int_0^{\infty}\ln\left\{\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right\}f(x)\,dx$

$$=\frac{(b_1+1)ab\beta^{\alpha}}{e^{-a}\Gamma(\alpha)}\int_0^{\infty}\ln\left\{\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right\}x^{\alpha-1}e^{-\beta x}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(b+1)}e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}}dx$$

since $e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}}=\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}a^{i}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-bi}$

Then,

$$I_5=(b_1+1)\frac{b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)}\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}a^{i+1}\int_0^{\infty}\ln\left\{\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right\}x^{\alpha-1}e^{-\beta x}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)}dx$$

since $\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)}=\left\{1-\left(1-\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)\right\}^{-(bi+b+1)}$

Then,

By using equation (10), we get,

$$\left\{1-\left(1-\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)\right\}^{-(bi+b+1)}=\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{u=0}^{\infty}(-1)^{u}\frac{\Gamma(j+1)}{u!\Gamma(j-u+1)}\left(\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)^{u}$$

And then,

$$I_5=\frac{(b_1+1)b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\frac{(-1)^{i}}{i!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{u=0}^{\infty}\frac{(-1)^{u}}{u!}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}$$

$$\int_0^{\infty}\ln\left\{\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right\}x^{\alpha-1}e^{-\beta x}\left(\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)^{u}dx$$

since $\gamma(\alpha_1,\beta_1 x)=(\beta_1 x)^{\alpha_1}\Gamma(\alpha_1)e^{-\beta_1 x}\sum_{k=0}^{\infty}\frac{(\beta_1 x)^{k}}{\Gamma(\alpha_1+k+1)}$

And

$$\gamma(\alpha,\beta x)=(\beta x)^{\alpha}\,\Gamma(\alpha)e^{-\beta x}\sum_{m=0}^{\infty}\frac{(\beta x)^{m}}{\Gamma(\alpha+m+1)}$$

Then,

$$I_5=\frac{(b_1+1)b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\frac{(-1)^{i}}{i!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{u=0}^{\infty}\frac{(-1)^{u}}{u!}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}$$

$$\int_0^\infty \ln\left\{\frac{(\beta_1 x)^{\alpha_1} e^{-\beta_1 x}}{\sum_{k=0}^\infty \frac{(\beta_1 x)^k}{(\alpha_1+k)!}}\right\} x^{\alpha-1}e^{-\beta x}\left[\frac{(\beta x)^\alpha e^{-\beta x}}{\sum_{m=0}^\infty \frac{(\beta x)^m}{(\alpha+m)!}}\right]^u dx$$

$$I_5 = (b_1+1)\frac{b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i,j=0}^\infty\sum_{u=0}^\infty \frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}$$

$$\sum_{m_1=0}^\infty\dots\sum_{m_u=0}^\infty\frac{1}{(\alpha+m_1)!\dots(\alpha+m_u)!}$$

$$\int_0^\infty \ln\left\{\frac{(\beta_1 x)^{\alpha_1} e^{-\beta_1 x}}{\sum_{k=0}^\infty \frac{(\beta_1 x)^k}{(\alpha_1+k)!}}\right\} x^{\alpha-1}e^{-\beta(u+1)x}$$

$$(\beta x)^{\alpha u+m_1+\dots+m_u}\,dx$$

since, $\ln\left\{(\beta_1 x)^{\alpha_1}e^{-\beta_1 x}\sum_{k=0}^\infty\frac{(\beta_1 x)^k}{(\alpha_1+k)!}\right\} = \alpha_1\ln(\beta_1 x) - \beta_1 x + \ln\left(\sum_{k=0}^\infty\frac{(\beta_1 x)^k}{(\alpha_1+k)!}\right)$

Then,

$$I_5 = (b_1+1)\frac{b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i,j=0}^\infty\sum_{u=0}^\infty \frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\sum_{m_1=0}^\infty\dots\sum_{m_u=0}^\infty\frac{1}{(\alpha+m_1)!\dots(\alpha+m_u)!}$$

$$\int_0^\infty\left\{\alpha_1\ln(\beta_1 x)-\beta_1 x+\ln\left(\sum_{k=0}^\infty\frac{(\beta_1 x)^k}{(\alpha_1+k)!}\right)\right\} x^{\alpha-1}e^{-\beta(u+1)x}(\beta x)^{\alpha u+m_1+\dots+m_u}\,dx$$

$$I_{51} = \alpha_1\int_0^\infty \ln(\beta_1 x)\,x^{\alpha-1}e^{-\beta(u+1)x}(\beta x)^{\alpha u+m_1+\dots+m_u}\,dx$$

let $y=\beta_1 x \Rightarrow x=\beta_1^{-1}y \Rightarrow dx=\beta_1^{-1}dy$

$$I_{51} = \alpha_1\int_0^\infty \ln(y)\,\beta_1^{-\alpha+1}y^{\alpha-1}e^{\frac{-\beta}{\beta_1}(u+1)y}\left(\frac{\beta}{\beta_1}\right)^{\alpha u+m_1+\dots+m_u}y^{\alpha u+m_1+\dots+m_u}\beta_1^{-1}\,dy$$

$$I_{51} = \alpha_1\beta^{\alpha u+m_1+\dots+m_u}\beta_1^{-(\alpha u+\alpha+m_1+\dots+m_u)}\beta_1^{\alpha u+\alpha+m_1+\dots+m_1}$$

$$\frac{\Gamma(\alpha(u+1)+m_1+\dots+m_u)}{\beta^{\alpha u+\alpha+m_1+\dots+m_1}(u+1)^{\alpha u+\alpha+m_1+\dots+m_u}}\left\{\Psi(\alpha(u+1)+m_1+\dots+m_u)-\ln\left(\frac{\beta}{\beta_1}(u+1)\right)\right\}$$

Now, $I_{51} = \alpha_1\frac{\Gamma(\alpha(u+1)+m_1+\dots+m_u)}{\beta^\alpha(u+1)^{\alpha u+\alpha+m_1+\dots+m_u}}\left\{\Psi(\alpha(u+1)+m_1+\dots+m_u)-\ln\left(\frac{\beta}{\beta_1}(u+1)\right)\right\}$

Now, $I_{52} = -\beta_1\int_0^\infty x\,x^{\alpha-1}e^{-\beta(u+1)x}(\beta x)^{\alpha u+m_1+\dots+m_u}\,dx$

$$= -\beta_1\beta^{\alpha u+m_1+\dots+m_u}\int_0^\infty x^{\alpha(u+1)+m_1+\dots+m_u}e^{-\beta(u+1)x}\,dx$$

$$= -\beta_1\beta^{\alpha u+m_1+\dots+m_u}\frac{\Gamma(\alpha(u+1)+m_1+\dots+m_u+1)}{(\beta(u+1))^{\alpha(u+1)+m_1+\dots+m_u+1}}$$

$$= -\frac{\beta_1}{\beta^\alpha\,\beta}\frac{\Gamma(\alpha(u+1)+m_1+\dots+m_u+1)}{(u+1)^{\alpha(u+1)+m_1+\dots+m_u+1}}$$

Since,

$$\eta^*(\tau,\alpha,k,m,\tau_1,\alpha_1,k_1,d_1,\dots,d_m) = \int_0^\infty\ln\left(\sum_{s=0}^\infty\frac{\left(\frac{x}{\alpha_1}\right)^{\tau_1 s}}{\Gamma(k_1+s+1)}\right)\left(\frac{x}{\alpha}\right)^{\tau k+\tau k m+\tau(d_1+\dots+d_m)-1}e^{-(m+1)\left(\frac{x}{\alpha}\right)^\tau}dx$$

Then,

$$I_{53} = \int_0^\infty\ln\left(\sum_{k=0}^\infty\frac{(\beta_1 x)^k}{\Gamma(\alpha_1+k+1)}\right)x^{\alpha-1}e^{-\beta(u+1)x}(\beta x)^{\alpha u+m_1+\dots+m_u}\,dx$$

$$= \beta^{\alpha u+m_1+\dots+m_u}\int_0^\infty\ln\left(\sum_{k=0}^\infty\frac{(\beta_1 x)^k}{\Gamma(\alpha_1+k+1)}\right)x^{\alpha(u+1)+m_1+\dots+m_u-1}e^{-\beta(u+1)x}\,dx$$

$$= \beta^{\alpha u+m_1+\dots+m_u}\eta^*(1,1/\beta,\alpha,u,1,1/\beta_1,\alpha_1,m_1,\dots,m_u)$$

$$I_5 = (b_1+1)\frac{be^a}{\Gamma(\alpha)}\sum_{i,j=0}^\infty\sum_{u=0}^\infty \frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\sum_{m_1=0}^\infty\dots\sum_{m_u=0}^\infty\frac{1}{(\alpha+m_1)!\dots(\alpha+m_u)!}$$

$$\left\{\begin{array}{c}\alpha_1\frac{\Gamma(\alpha(u+1)+m_1+\dots+m_u)}{(u+1)^{\alpha u+\alpha+m_1+\dots+m_u}}\left\{\begin{array}{c}\Psi(\alpha(u+1)+m_1+\dots+m_u)\\ -\ln\left(\frac{\beta}{\beta_1}(u+1)\right)\end{array}\right\}\\ -\frac{\beta_1}{\beta}\frac{\Gamma(\alpha(u+1)+m_1+\dots+m_u+1)}{(u+1)^{\alpha(u+1)+m_1+\dots+m_u+1}}+\beta^{\alpha u+\alpha+m_1+\dots+m_u}\\ \eta^*(1,1/\beta,\alpha,u,1,1/\beta_1,\alpha_1,m_1,\dots,m_u)\end{array}\right\}$$

And

$$I_6 = a_1\int_0^\infty\left\{\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right\}^{-b_1}f(x)\,dx$$

$$I_6 = a_1\int_0^\infty\left\{\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right\}^{-b_1}\frac{ab\beta^\alpha}{e^{-a}\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(b+1)}e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}}dx$$

since $e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}} = \sum_{i=0}^\infty\frac{(-1)^i}{i!}a^i\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-bi}$

Then,

$$I_6 = \frac{a_1 b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i=0}^\infty\frac{(-1)^i}{i!}a^{i+1}\int_0^\infty x^{\alpha-1}e^{-\beta x}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)}\left\{\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right\}^{-b_1}dx$$

since $\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)} = \left\{1-\left(1-\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)\right\}^{-(bi+b+1)}$

Then, by using equation (10) we get,

$$\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)} = \sum_{j=0}^\infty\sum_{u=0}^\infty\frac{(-1)^u}{u!}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\left(\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)^u$$

And then,

$$I_6 = \frac{a_1 b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i,j=0}^\infty\sum_{u=0}^\infty\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\int_0^\infty x^{\alpha-1}e^{-\beta x}$$

$$\left(\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)^u\left\{\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right\}^{-b_1}dx$$

since $\left\{\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right\}^{-b_1} = \left\{1-\left(1-\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right)\right\}^{-b_1}$

Also by using equation (10) we get,

$$\left\{\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right\}^{-b_1} = \sum_{l=0}^\infty\sum_{s=0}^\infty\frac{(-1)^s}{s!}\frac{\Gamma(b_1+l)}{l!\Gamma(b_1)}\frac{\Gamma(l+1)}{\Gamma(l-s+1)}\left(\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right)^s$$

$$I_6 = \frac{a_1 b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i,j=0}^\infty\sum_{u=0}^\infty\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\sum_{l=0}^\infty\frac{\Gamma(b_1+l)}{l!\Gamma(b_1)}$$

$$\sum_{s=0}^\infty\frac{(-1)^s}{s!}\frac{\Gamma(l+1)}{\Gamma(l-s+1)}\int_0^\infty x^{\alpha-1}e^{-\beta x}\left(\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)^u\left(\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right)^s dx$$

By using equation (18) expansion incomplete gamma function we get,

$$I_6=\frac{a_1 b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{i+u+s}}{i!u!s!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\sum_{l=0}^{\infty}\frac{\Gamma(b_1+l)}{l!\Gamma(b_1)}$$

$$\frac{\Gamma(l+1)}{\Gamma(l-s+1)}\int_0^{\infty}x^{\alpha-1}e^{-\beta x}\left[\frac{(\beta x)^{\alpha}}{\Gamma(\alpha)}\sum_{m=0}^{\infty}\frac{(-\beta x)^m}{(\alpha+m)m!}\right]^u\left[\frac{(\beta_1 x)^{\alpha_1}}{\Gamma(\alpha_1)}\sum_{k=0}^{\infty}\frac{(-\beta_1 x)^k}{(\alpha_1+k)k!}\right]^s dx$$

$$=\frac{a_1 b\beta^{\alpha}}{e^{-a}(\Gamma(\alpha))^{u+1}}\sum_{i,j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{i+u+s}}{i!u!s!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\frac{\Gamma(b_1+l)}{l!\Gamma(b_1)}\frac{\Gamma(l+1)}{\Gamma(l-s+1)}$$

$$\frac{1}{(\Gamma(\alpha_1))^s}\sum_{m_1=0}^{\infty}\cdots\sum_{m_u=0}^{\infty}\frac{(-1)^{m_1+\cdots+m_u}\beta^{\alpha u+m_1+\cdots+m_u}}{(\alpha+m_1)\ldots(\alpha+m_u)m_1!\ldots m_u!}$$

$$\sum_{k_1=0}^{\infty}\cdots\sum_{k_s=0}^{\infty}\frac{(-1)^{k_1+\cdots+k_s}\beta_1^{\alpha_1 s+k_1+\cdots+k_s}}{(\alpha_1+k_1)\ldots(\alpha_1+k_s)k_1!\ldots k_s!}\int_0^{\infty}x^{\alpha_1 s+\alpha(u+1)+m_1+\cdots+m_u+k_1+\cdots+k_s-1}e^{-\beta x}dx$$

$$=\frac{a_1 b e^{a}}{(\Gamma(\alpha))^{u+1}}\frac{1}{(\Gamma(\alpha_1))^s}\sum_{i,j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{i+u+s}}{i!u!s!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\frac{\Gamma(b_1+l)}{l!\Gamma(b_1)}\frac{\Gamma(l+1)}{\Gamma(l-s+1)}$$

$$\sum_{m_1=0}^{\infty}\cdots\sum_{m_u=0}^{\infty}\frac{(-1)^{m_1+\cdots+m_u}}{(\alpha+m_1)\ldots(\alpha+m_u)m_1!\ldots m_u!}\sum_{k_1=0}^{\infty}\cdots\sum_{k_s=0}^{\infty}\frac{(-1)^{k_1+\cdots+k_s}\beta_1^{\alpha_1 s+k_1+\cdots+k_s}}{(\alpha_1+k_1)\ldots(\alpha_1+k_s)k_1!\ldots k_s!}$$

$$\frac{\Gamma(\alpha_1 s+\alpha(u+1)+m_1+\cdots+m_u+k_1+\cdots+k_s)}{\beta^{\alpha_1 s+k_1+\cdots+k_s}}$$

Then,

$$DKL(F\|F^*)=\ln\left(\frac{ab\beta^{\alpha}e^{-a_1}\Gamma(\alpha_1)}{a_1 b_1\beta_1^{\alpha_1}e^{-a}\Gamma(\alpha)}\right)+(\alpha-\alpha_1)\frac{be^{a}}{(\Gamma(\alpha))^{u+1}}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}$$

$$\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\sum_{m=0}^{\infty}C_{u,m}\,\Gamma(\alpha(1+u)+m)\{\Psi(\alpha(1+u)+m)-\ln(\beta)\}+$$

$$\frac{(\beta_1-\beta)be^{a}}{\beta\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}$$

$$I(\alpha+1,u)-\frac{(b+1)be^{a}}{\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}$$

$$\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\sum_{m_1=0}^{\infty}\cdots\sum_{m_u=0}^{\infty}\frac{1}{(\alpha+m_1)!\ldots(\alpha+m_u)!}$$

$$\left\{\begin{array}{c}\alpha\frac{\Gamma(\alpha(u+1)+m_1+\cdots+m_u)}{(u+1)^{(\alpha(u+1)+m_1+\cdots+m_u)}}\left\{\begin{array}{c}\Psi(\alpha(u+1)+m_1+\cdots+m_u)-\\ \ln(u+1)\end{array}\right\}-\\ \frac{\Gamma(\alpha(u+1)+m_1+\cdots+m_u+1)}{(u+1)^{\alpha(u+1)+m_1+\cdots+m_u+1}}+\beta^{\alpha u+\alpha+m_1+\cdots+m_u}\\ \eta(1,1/\beta,\alpha,u,m_1,\ldots,m_u)\end{array}\right\}-$$

$$\frac{be^{a}}{\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+2}\frac{\Gamma([b(i+2)+1]+j)}{j!\Gamma([b(i+2)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}I(\alpha,u)$$

$$+(b_1+1)\frac{be^{a}}{\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}$$

$$\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\sum_{m_1=0}^{\infty}\cdots\sum_{m_u=0}^{\infty}\frac{1}{(\alpha+m_1)!\ldots(\alpha+m_u)!}$$

$$\left\{\begin{array}{c}\alpha_1\frac{\Gamma(\alpha(u+1)+m_1+\cdots+m_u)}{(u+1)^{\alpha u+\alpha+m_1+\cdots+m_u}}\left\{\begin{array}{c}\Psi(\alpha(u+1)+m_1+\cdots+m_u)\\ -\ln\left(\frac{\beta}{\beta_1}(u+1)\right)\end{array}\right\}\\ -\frac{\beta_1}{\beta}\frac{\Gamma(\alpha(u+1)+m_1+\cdots+m_u+1)}{(u+1)^{\alpha(u+1)+m_1+\cdots+m_u+1}}+\beta^{\alpha u+\alpha+m_1+\cdots+m_u}\\ \eta^*(1,1/\beta,\alpha,u,1,1/\beta_1,\alpha_1,m_1,\ldots,m_u)\end{array}\right\}$$

$$+\frac{a_1 b e^{a}(\beta_1/\beta)^{\alpha_1 s}}{(\Gamma(\alpha))^{u+1}(\Gamma(\alpha_1))^s}\sum_{i,j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{i+u+s}}{i!u!s!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\frac{\Gamma(b_1+l)}{l!\Gamma(b_1)}\frac{\Gamma(l+1)}{\Gamma(l-s+1)}$$

$$\sum_{m_1=0}^{\infty}\cdots\sum_{m_u=0}^{\infty}\frac{(-1)^{m_1+\cdots+m_u}}{(\alpha+m_1)\ldots(\alpha+m_u)m_1!\ldots m_u!}\sum_{k_1=0}^{\infty}\cdots\sum_{k_s=0}^{\infty}\frac{(-1)^{k_1+\cdots+k_s}(\beta_1/\beta)^{\alpha_1 s+k_1+\cdots+k_s}}{(\alpha_1+k_1)\ldots(\alpha_1+k_s)k_1!\ldots k_s!}$$

$$\Gamma(\alpha_1 s+\alpha(u+1)+m_1+\cdots+m_u+k_1+\cdots+k_s)\qquad(20)$$

2.3. Stress-strength reliability

Let Y and X be the stress and the strength random variables, independent of each other, follow respectively [01] $TFG(a,b,\alpha,\beta)$ and [0, 1] $TFG(a_1,b_1,\alpha_1,\beta_1)$, then,

$$R=P(Y<X)=\int_0^{\infty}f_x(x)\,F_y(x)\,dx$$

$$R=\int_0^{\infty}\frac{ab\beta^{\alpha}}{e^{-a}\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(b+1)}e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}}\frac{e^{-a_1\left\{\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right\}^{-b_1}}}{e^{-a_1}}dx$$

Since

$$e^{-a\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-b}}=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^i\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-bi}$$

Then,

$$R=\frac{ab\beta^{\alpha}}{e^{-a}\Gamma(\alpha)e^{-a_1}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^i\int_0^{\infty}x^{\alpha-1}e^{-\beta x}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(b+1)}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-bi}e^{-a_1\left\{\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right\}^{-b_1}}dx$$

$$R=\frac{b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)e^{-a_1}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\int_0^{\infty}x^{\alpha-1}e^{-\beta x}\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)}e^{-a_1\left\{\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right\}^{-b_1}}dx$$

By using equation (10) we get,

$$\left\{\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)}=\sum_{j=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^u}{u!}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\left(\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)^u$$

$$R=\frac{b\beta^{\alpha}}{e^{-a}e^{-a_1}\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{u=0}^{\infty}\frac{(-1)^u}{u!}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}\int_0^{\infty}x^{\alpha-1}e^{-\beta x}\left(\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)^u e^{-a_1\left\{\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right\}^{-b_1}}dx$$

Since

$$e^{-a_1\left\{\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right\}^{-b_1}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(a_1)^n\left\{\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right\}^{-nb_1}$$

Then,

$$R=\frac{b\beta^{\alpha}}{e^{-a}e^{-a_1}\Gamma(\alpha)}\sum_{i,j=0}^{\infty}\sum_{u=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{i+u+n}}{i!u!n!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}(a_1)^n$$

$$\int_0^\infty x^{\alpha-1}\, e^{-\beta x}\left(\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)^u\left(\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right)^{-nb_1}\, dx$$

Also,by using equation (10) we get,

$$since\ \left\{1-\left(1-\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right)\right\}^{-nb_1} = \sum_{v=0}^\infty\sum_{s=0}^\infty \frac{(-1)^s}{s!}\frac{\Gamma(nb_1+v)}{v!\Gamma(nb_1)}\frac{\Gamma(v+1)}{\Gamma(v-s+1)}\left(\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right)^s$$

$$R=\frac{b\beta^\alpha}{e^{-a}e^{-a_1}\Gamma(\alpha)}\sum_{i=0}^\infty\sum_{u=0}^\infty\sum_{n=0}^\infty\frac{(-1)^{i+u+n}}{i!u!n!}a^{i+1}\sum_{j=0}^\infty\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}(a_1)^n$$

$$\sum_{v=0}^\infty\frac{\Gamma(nb_1+v)}{v!\Gamma(nb_1)}\sum_{s=0}^\infty\frac{(-1)^s}{s!}\frac{\Gamma(v+1)}{\Gamma(v-s+1)}\int_0^\infty x^{\alpha-1}\, e^{-\beta x}\left(\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right)^u\left(\frac{\gamma(\alpha_1,\beta_1 x)}{\Gamma(\alpha_1)}\right)^s dx$$

$$=\frac{b\beta^\alpha}{e^{-a}e^{-a_1}\Gamma(\alpha)}\sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{v=0}^\infty\sum_{u=0}^\infty\sum_{n=0}^\infty\sum_{s=0}^\infty\frac{(-1)^{i+u+n+s}}{i!u!n!s!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}$$

$$(a_1)^n\frac{\Gamma(nb_1+v)}{v!\Gamma(nb_1)}\frac{\Gamma(v+1)}{\Gamma(v-s+1)}\int_0^\infty x^{\alpha-1}\, e^{-\beta x}\left[\frac{(\beta x)^\alpha}{\Gamma(\alpha)}\sum_{m=0}^\infty\frac{(-\beta x)^m}{(\alpha+m)m!}\right]^u\left[\frac{(\beta_1 x)^{\alpha_1}}{\Gamma(\alpha_1)}\sum_{k=0}^\infty\frac{(-\beta_1 x)^k}{(\alpha_1+k)k!}\right]^s dx$$

$$=\frac{b\beta^\alpha e^{a+a_1}}{(\Gamma(\alpha))^{u+1}(\Gamma(\alpha_1))^s}\sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{v=0}^\infty\sum_{u=0}^\infty\sum_{n=0}^\infty\sum_{s=0}^\infty\frac{(-1)^{i+u+n+s}}{i!u!n!s!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(nb_1+v)}{v!\Gamma(nb_1)}$$

$$\frac{\Gamma(j+1)}{\Gamma(j-u+1)}(a_1)^n\frac{\Gamma(v+1)}{\Gamma(v-s+1)}\int_0^\infty x^{\alpha-1}\, e^{-\beta x}\left[\sum_{m=0}^\infty\frac{(-1)^m(\beta x)^{\alpha+m}}{(\alpha+m)m!}\right]^u\left[\sum_{k=0}^\infty\frac{(-1)^k(\beta_1 x)^{\alpha_1+k}}{(\alpha_1+k)k!}\right]^s dx$$

$$=\frac{b\beta^\alpha e^{a+a_1}}{(\Gamma(\alpha))^{u+1}(\Gamma(\alpha_1))^s}\sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{v=0}^\infty\sum_{u=0}^\infty\sum_{n=0}^\infty\sum_{s=0}^\infty\frac{(-1)^{i+u+n+s}}{i!u!n!s!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}$$

$$(a_1)^n\frac{\Gamma(nb_1+v)}{v!\Gamma(nb_1)}\frac{\Gamma(v+1)}{\Gamma(v-s+1)}\sum_{m_1=0}^\infty\cdots\sum_{m_u=0}^\infty\frac{(-1)^{m_1+\cdots+m_u}\beta^{\alpha u+m_1+\cdots+m_u}}{(\alpha+m_1)\ldots(\alpha+m_u)m_1!\ldots.m_u!}$$

$$\sum_{k_1=0}^\infty\cdots\sum_{k_s=0}^\infty\frac{(-1)^{k_1+\cdots+k_s}\beta_1^{\alpha_1 s+k_1+\cdots+k_s}}{(\alpha_1+k_1)\ldots(\alpha_1+k_s)k_1!\ldots k_s!}\int_0^\infty x^{\alpha_1 s+\alpha(u+1)+m_1+\cdots+m_u+k_1+\cdots+k_s-1}\, e^{-\beta x}\, dx$$

$$=\frac{b\beta^\alpha e^{a+a_1}}{(\Gamma(\alpha))^{u+1}(\Gamma(\alpha_1))^s}\sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{v=0}^\infty\sum_{u=0}^\infty\sum_{n=0}^\infty\sum_{s=0}^\infty\frac{(-1)^{i+u+n+s}}{i!u!n!s!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}$$

$$(a_1)^n\frac{\Gamma(nb_1+v)}{v!\Gamma(nb_1)}\frac{\Gamma(v+1)}{\Gamma(v-s+1)}\sum_{m_1=0}^\infty\cdots\sum_{m_u=0}^\infty\frac{(-1)^{m_1+\cdots+m_u}\beta^{\alpha u+m_1+\cdots+m_u}}{(\alpha+m_1)\ldots(\alpha+m_u)m_1!\ldots m_u!}$$

$$\sum_{k_1=0}^\infty\cdots\sum_{k_s=0}^\infty\frac{(-1)^{k_1+\cdots+k_s}\beta_1^{\alpha_1 s+k_1+\cdots+k_s}}{(\alpha_1+k_1)\ldots(\alpha_1+k_s)k_1!\ldots k_s!}\frac{\Gamma(\alpha_1 s+\alpha(u+1)+m_1+\cdots+m_u+k_1+\cdots+k_s)}{\beta^{\alpha_1 s+\alpha(u+1)+m_1+\cdots+m_u+k_1+\cdots+k_s}}$$

$$=\frac{be^{a+a_1}}{(\Gamma(\alpha))^{u+1}(\Gamma(\alpha_1))^s}\sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{v=0}^\infty\sum_{u=0}^\infty\sum_{n=0}^\infty\sum_{s=0}^\infty\frac{(-1)^{i+u+n+s}}{i!u!n!s!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-u+1)}$$

$$(a_1)^n\frac{\Gamma(nb_1+v)}{v!\Gamma(nb_1)}\frac{\Gamma(v+1)}{\Gamma(v-s+1)}\sum_{m_1=0}^\infty\cdots\sum_{m_u=0}^\infty\frac{(-1)^{m_1+\cdots+m_u}}{(\alpha+m_1)\ldots(\alpha+m_u)m_1!\ldots m_u!}$$

$$\sum_{k_1=0}^\infty\cdots\sum_{k_s=0}^\infty\frac{(-1)^{k_1+\cdots+k_s}\beta_1^{\alpha_1 s+m_1+\cdots+m_s}}{(\alpha_1+k_1)\ldots(\alpha_1+k_s)k_1!\ldots k_s!}\frac{\Gamma(\alpha_1 s+\alpha(u+1)+m_1+\cdots+m_u+k_1+\cdots+k_s)}{\beta^{\alpha_1 s+k_1+\cdots+k_s}} \quad (21)$$

3. [0, 1] truncated fréchet-inverted gamma distribution

Assume that

$$g(x)=\beta^\alpha/\Gamma(\alpha)\, x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}$$

And

$$G(x)=\Gamma\left(\alpha,\frac{\beta}{x}\right)/\Gamma(\alpha)\ (0<x)$$

Are pdf and cdf of inverted random variable respectively, then, by applying (6) and (7) above, we get the cdf and pdf of [0, 1] TFIG random variable as follows,

$$F(x)=\frac{e^{-a\left\{\Gamma\left(\alpha,\frac{\beta}{x}\right)/\Gamma(\alpha)\right\}^{-b}}}{e^{-a}} \quad (22)$$

$$f(x)=\frac{ab\beta^\alpha}{e^{-a}\Gamma(\alpha)}x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left\{\Gamma\left(\alpha,\frac{\beta}{x}\right)/\Gamma(\alpha)\right\}^{-(b+1)} e^{-a\left\{\Gamma\left(\alpha,\frac{\beta}{x}\right)/\Gamma(\alpha)\right\}^{-b}}\quad x\geq 0 \quad (23)$$

So, the reliability $R(x)$and hazard rate $\lambda(x)$ function are respectively

$$R(x)=1-\frac{e^{-a\left\{\Gamma\left(\alpha,\frac{\beta}{x}\right)/\Gamma(\alpha)\right\}^{-b}}}{e^{-a}}=1-e^{-a\left[\left\{\Gamma\left(\alpha,\frac{\beta}{x}\right)/\Gamma(\alpha)\right\}^{-b}-1\right]}$$

$$\lambda(x)=\frac{f_1(x)}{R(x)}=\frac{\frac{ab\beta^\alpha}{e^{-a}\Gamma(\alpha)}x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(b+1)}e^{-a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}}}{1-e^{-a\left[\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}-1\right]}}$$

The rth raw moment can be derived as follows,

$$E(x)=\int_0^\infty x^r\frac{ab\beta^\alpha}{e^{-a}\Gamma(\alpha)}x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(b+1)}e^{-a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}}dx$$

Since

$$e^{-a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}}=\sum_{i=0}^\infty\frac{(-1)^i}{i!}a^i\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-bi}$$

Then,

$$E(x^r)=\frac{ab\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i=0}^\infty\frac{(-1)^i}{i!}a^i\int_0^\infty x^{-(\alpha-r+1)}\, e^{-\left(\frac{\beta}{x}\right)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(b+1)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-bi}dx$$

$$E(x^r)=\frac{b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i=0}^\infty\frac{(-1)^i}{i!}a^{i+1}\int_0^\infty x^{-(\alpha-r+1)}\, e^{-\left(\frac{\beta}{x}\right)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)}dx$$

Since

$$\Gamma(s,\lambda)+\gamma(s,\lambda)=\Gamma(s)\Longrightarrow\Gamma(s,\lambda)=\Gamma(s)-\gamma(s,\lambda)$$

Then,

$$= \frac{b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \int_0^{\infty} x^{-(\alpha-r+1)}\, e^{-\left(\frac{\beta}{x}\right)} \left\{\frac{\Gamma(\alpha)-\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)} dx$$

$$E(x^r) = \frac{b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \int_0^{\infty} x^{-(\alpha-r+1)}\, e^{-\left(\frac{\beta}{x}\right)} \left\{1 - \frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)} dx$$

By using

$$(1-z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{j!\Gamma(k)} z^j |z| < 1\,, k > 0$$

We get,

$$\left\{1 - \frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)} = \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \left(\frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right)^j$$

And then,

$$E(x^r) = \frac{b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \int_0^{\infty} x^{-(\alpha-r+1)}\, e^{-\left(\frac{\beta}{x}\right)} \left(\frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right)^j dx$$

By using expansion incomplete gamma function in equation (18) we get,

$$E(x^r) = \frac{b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \int_0^{\infty} x^{-(\alpha-r+1)}\, e^{-\left(\frac{\beta}{x}\right)} \left[\frac{\left(\frac{\beta}{x}\right)^{\alpha}}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{\left(\frac{-\beta}{x}\right)^m}{(\alpha+m)m!}\right]^j dx$$

By application of an equation in section 0.314 of [5] for power series to power we obtain

$$\left[\sum_{m=0}^{\infty} a_m \left(\frac{\beta}{x}\right)^m\right]^j = \sum_{m=0}^{\infty} C_{j,m} \left(\frac{\beta}{x}\right)^m$$

Where the coefficients $C_{j,m}$(for $m = 1,2,\ldots\ldots$) satisfy the recurrence relation

$$C_{j,m} = (ma_0)^{-1} \sum_{p=1}^{m} (jp-m+p)\, a_p C_{j,m-p}\,, C_{j,0} = a_0^j \text{ and } a_p = \frac{(-1)^p}{(\alpha+p)p!}$$

$$E(x^r) = \frac{b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \sum_{m=0}^{\infty} C_{j,m} \int_0^{\infty} x^{-(\alpha-r+1)}\, e^{-\left(\frac{\beta}{x}\right)}$$

$$\frac{1}{(\Gamma(\alpha))^j} \left(\frac{\beta}{x}\right)^{\alpha j} \left(\frac{\beta}{x}\right)^m dx$$

$$E(x^r) = \frac{b\beta^{\alpha}}{e^{-a}(\Gamma(\alpha))^{j+1}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \sum_{m=0}^{\infty} C_{j,m}\, \beta^{\alpha j+m}$$

$$\int_0^{\infty} x^{-(m+\alpha(j+1)-r+1)}\, e^{-\left(\frac{\beta}{x}\right)} dx$$

$$E(x^r) = \frac{\beta^r b e^a}{(\Gamma(\alpha))^{j+1}} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} C_{j,m} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\, \Gamma(m+\alpha(j+1)-r) \quad (24)$$

And then, the characteristic function is

$$Q_x(t) = E\left(e^{ixt}\right) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} E(x^r)$$

Since

$$e^{ixt} = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} x^r$$

$$Q_x(t) = \frac{be^a}{(\Gamma(\alpha))^{j+1}} \sum_{r=0}^{\infty} \frac{(it\beta)^r}{r!} \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} C_{j,m} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\, \Gamma(m+\alpha(j+1)-r)$$

So, the mean μ and variance σ^2 of the of [0, 1] TFIGrandom variable are,

$$E(x) = \frac{\beta b e^a}{(\Gamma(\alpha))^{j+1}} \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} C_{j,m} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\, \Gamma(m+\alpha(j+1)-1) \quad (25)$$

$$\sigma^2 = E(x^2) - (Ex)^2$$

$$\sigma^2 = \frac{\beta^2 b e^a}{(\Gamma(\alpha))^{j+1}} \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} C_{j,m} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\, \Gamma(m+\alpha(j+1)-2) - \left\{\frac{\beta b e^a}{(\Gamma(\alpha))^{j+1}} \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} C_{j,m} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\, \Gamma(m+\alpha(j+1)-1)\right\}^2 \quad (26)$$

The median M_e can be calculated numerically, since

$$F(x) = \frac{e^{-a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}}}{e^{-a}} = \frac{1}{2}$$

By solving the nonlinear equation

$$\Gamma\left(\alpha,\frac{\beta}{x}\right) - \Gamma(\alpha)\left(1 + \frac{\ln(2)}{a}\right)^{\frac{-1}{b}} = 0.$$

The skewness of [0, 1] TFIG random variable will be,

$$sk = \frac{\mu_3}{\mu_2^{3/2}} = \frac{Ex^3 - 3\mu Ex^2 + 2\mu^3}{(\sigma^2)^{3/2}}$$

$$= \frac{\left\{\begin{array}{c} \frac{\beta^3 b e^a}{(\Gamma(\alpha))^{j+1}} \sum_{i,m=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} C_{j,m} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \Gamma(m+\alpha(j+1)-3) \\ -3\left\{\begin{array}{c} \frac{\beta b e^a}{(\Gamma(\alpha))^{j+1}} \sum_{i,m=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} C_{j,m} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \\ \Gamma(m+\alpha(j+1)-1) \end{array}\right\} \\ \left\{\frac{\beta^2 b e^a}{(\Gamma(\alpha))^{j+1}} \sum_{i,m=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} C_{j,m} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \Gamma(m+\alpha(j+1)-2)\right\} \\ +2\left\{\begin{array}{c} \frac{\beta b e^a}{(\Gamma(\alpha))^{j+1}} \sum_{i,m=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} C_{j,m} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \\ \Gamma(m+\alpha(j+1)-1) \end{array}\right\}^3 \end{array}\right\}}{\left\{\begin{array}{c} \frac{\beta^2 b e^a}{(\Gamma(\alpha))^{j+1}} \sum_{i,m=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} C_{j,m} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \\ \Gamma(m+\alpha(j+1)-2) - \left\{\begin{array}{c} \frac{\beta b e^a}{(\Gamma(\alpha))^{j+1}} \sum_{i,m=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} C_{j,m} \\ \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \Gamma(m+\alpha(j+1)-1) \end{array}\right\}^2 \end{array}\right\}^{3/2}} \quad (27)$$

Also, the kurtosis is,

$$kr = \frac{\mu_4}{\mu_2^2} - 3 = \frac{Ex^4 - 4\mu Ex^3 + 6\mu^2 Ex^2 - 3\mu^4}{(\sigma^2)^2} - 3$$

$$=\frac{\left\{\begin{array}{c}\frac{\beta^4be^a}{(\Gamma(\alpha))^{j+1}}\sum_{i,m=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}C_{j,m}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\\ \Gamma(m+\alpha(j+1)-4)-4\left\{\begin{array}{c}\frac{\beta be^a}{(\Gamma(\alpha))^{j+1}}\sum_{i,m=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}C_{j,m}\\ \sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\\ \Gamma(m+\alpha(j+1)-1)\end{array}\right\}\\ \left\{\begin{array}{c}\frac{\beta^3be^a}{(\Gamma(\alpha))^{j+1}}\sum_{i,m=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}C_{j,m}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\\ \Gamma(m+\alpha(j+1)-3)\end{array}\right\}+6\\ \left\{\begin{array}{c}\frac{\beta be^a}{(\Gamma(\alpha))^{j+1}}\sum_{i,m=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}C_{j,m}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\\ \Gamma(m+\alpha(j+1)-1)\end{array}\right\}^2\\ \left\{\begin{array}{c}\frac{\beta^2be^a}{(\Gamma(\alpha))^{j+1}}\sum_{i,m=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}C_{j,m}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\\ \Gamma(m+\alpha(j+1)-2)\end{array}\right\}-3\\ \left\{\begin{array}{c}\frac{\beta be^a}{(\Gamma(\alpha))^{j+1}}\sum_{i,m=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}C_{j,m}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\\ \Gamma(m+\alpha(j+1)-1)\end{array}\right\}^4\end{array}\right\}}{\left\{\begin{array}{c}\frac{\beta^2be^a}{(\Gamma(\alpha))^{j+1}}\sum_{i,m=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}C_{j,m}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\\ \Gamma(m+\alpha(j+1)-2)-\left\{\begin{array}{c}\frac{\beta be^a}{(\Gamma(\alpha))^{j+1}}\sum_{i,m=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}C_{j,m}\\ \sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\\ \Gamma(m+\alpha(j+1)-1)\end{array}\right\}^2\end{array}\right\}^2}-3$$

$$kr=\frac{\left\{\begin{array}{c}\frac{(\Gamma(\alpha))^{3(j+1)}}{b^3e^{3a}}\sum_{i,m=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}C_{j,m}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\\ \Gamma(m+\alpha(j+1)-4)-4\frac{(\Gamma(\alpha))^{2(j+1)}}{b^2e^{2a}}\\ \left\{\begin{array}{c}\sum_{i,m=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}C_{j,m}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\\ \Gamma(m+\alpha(j+1)-1)\end{array}\right\}\\ \left\{\begin{array}{c}\sum_{i,m=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}C_{j,m}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\\ \Gamma(m+\alpha(j+1)-3)\end{array}\right\}+6\\ \frac{(\Gamma(\alpha))^{j+1}}{be^a}\left\{\begin{array}{c}\sum_{i,m=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}C_{j,m}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\\ \Gamma(m+\alpha(j+1)-1)\end{array}\right\}^2\\ \left\{\begin{array}{c}\sum_{i,m=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}C_{j,m}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\\ \Gamma(m+\alpha(j+1)-2)\end{array}\right\}-3\\ \left\{\begin{array}{c}\sum_{i,m=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}C_{j,m}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\\ \Gamma(m+\alpha(j+1)-1)\end{array}\right\}^4\end{array}\right\}}{\left\{\begin{array}{c}\frac{(\Gamma(\alpha))^{j+1}}{be^a}\sum_{i,m=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}C_{j,m}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\\ \Gamma(m+\alpha(j+1)-2)-\left\{\begin{array}{c}\sum_{i,m=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}C_{j,m}\\ \sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\\ \Gamma(m+\alpha(j+1)-1)\end{array}\right\}^2\end{array}\right\}^2}-3 \qquad (28)$$

The quantile function x_q of [0, 1] TFIG random variable can be obtained as,

$$q=P(x\le x_q)=F_x(x_q)=\frac{e^{-a\left\{\Gamma\left(\alpha,\frac{\beta}{x}\right)/\Gamma(\alpha)\right\}^{-b}}}{e^{-a}}\ ,0<q<1\ ,x_q>0$$

By solving the nonlinear equation

$$\Gamma\left(\alpha,\frac{\beta}{x}\right)-\Gamma(\alpha)\left[1-\frac{\ln(q)}{a}\right]^{\frac{-1}{b}}=0 \qquad (29)$$

So by using the inverse transform method one can generate the random variable

$$\Gamma\left(\alpha,\frac{\beta}{x}\right)-\Gamma(\alpha)\left[1-\frac{\ln(u)}{a}\right]^{\frac{-1}{b}}=0$$

Where u is a random number distribution uniformly in the unit interval [0, 1].

3.1. Shannon and relative entropies

The Shannon entropy of [0, 1] $\mathrm{TFIG}(a,b,\alpha,\beta)$ random variable X can be found as follows,

$$H=-\int_0^{\infty}f(x)\ln\big(f(x)\big)\ dx$$

$$=-\int_0^{\infty}f(x)\ln\left(\begin{array}{c}\frac{ab\beta^{\alpha}}{e^{-a}\Gamma(\alpha)}x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(b+1)}\\ e^{-a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}}\end{array}\right)dx$$

$$=\int_0^{\infty}f_1(x)\left[\begin{array}{c}-\ln\left(\frac{ab\beta^{\alpha}}{e^{-a}\Gamma(\alpha)}\right)+(\alpha+1)\ln(x)\\ +\frac{\beta}{x}+(b+1)\ln\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}+\\ a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}\end{array}\right]dx$$

$$H=\ln\left(\frac{e^{-a}\Gamma(\alpha)}{ab\beta^{\alpha}}\right)+(\alpha+1)E(\ln(x))+\beta E\left(\frac{1}{x}\right)+(b+1)\,E\left(\ln\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}\right)+aE\left(\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}\right)$$

Let, $I_1=(\alpha+1)E(\ln(x))$

$$I_1=(\alpha+1)\int_0^{\infty}\ln(x)\frac{ab\beta^{\alpha}}{e^{-a}\Gamma(\alpha)}x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(b+1)}e^{-a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}}\ dx$$

Since

$$e^{-a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}}=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^i\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-bi}$$

Then,

$$I_1=(\alpha+1)\frac{b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\int_0^{\infty}\ln(x)\,x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(b+1)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-bi}dx$$

$$=(\alpha+1)\frac{b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\int_0^{\infty}\ln(x)\,x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)}dx$$

Since

$$\Gamma\left(\alpha,\frac{\beta}{x}\right)=\Gamma(\alpha)-\gamma\left(\alpha,\frac{\beta}{x}\right)$$

Then,

$$I_1=(\alpha+1)\frac{b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\int_0^{\infty}\ln(x)\,x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left\{1-\frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)}dx$$

By using equation (10) we get,

$$\left\{1-\frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)}=\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\left(\frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right)^j$$

$$I_1=(\alpha+1)\frac{b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\int_0^{\infty}\ln(x)\,x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left(\frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right)^j dx$$

By using equation (18) and

$$\left[\sum_{m=0}^{\infty} a_m \left(\frac{\beta}{x}\right)^m\right]^j = \sum_{m=0}^{\infty} C_{j,m} \left(\frac{\beta}{x}\right)^m$$

We get,

$$\left(\frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right)^j = \frac{1}{(\Gamma(\alpha))^j}\left(\frac{\beta}{x}\right)^{\alpha j}\sum_{m=0}^{\infty} C_{j,m}\left(\frac{\beta}{x}\right)^m$$

where $C_{j,m} = (ma_0)^{-1}\sum_{p=1}^{m}(jp - m + p)\, a_p C_{j,m-p}$, $C_{j,0} = a_0^j$ and $a_p = \frac{(-1)^p}{(\alpha+p)p!}$

$$I_1 = (\alpha+1)\frac{b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{m=0}^{\infty} C_{j,m}$$

$$\int_0^\infty \ln(x)\, x^{-(\alpha+1)} e^{-\left(\frac{\beta}{x}\right)}$$

$$\frac{1}{(\Gamma(\alpha))^j}\left(\frac{\beta}{x}\right)^{\alpha j}\left(\frac{\beta}{x}\right)^m dx$$

$$= (\alpha+1)\frac{b\beta^{\alpha+\alpha j+m}}{e^{-a}(\Gamma(\alpha))^{j+1}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{m=0}^{\infty} C_{j,m}$$

$$\int_0^\infty \ln(x)\, x^{-(m+\alpha(j+1)+1)} e^{-\left(\frac{\beta}{x}\right)} dx$$

$$= \frac{-(\alpha+1)b\beta^{\alpha+\alpha j+m}}{e^{-a}(\Gamma(\alpha))^{j+1}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{m=0}^{\infty} C_{j,m}$$

$$\int_0^\infty \ln\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)^{m+\alpha(j+1)+1} e^{-\left(\frac{\beta}{x}\right)} dx$$

let $y = \frac{1}{x} \Rightarrow x = y^{-1} \Rightarrow dx = -y^{-2}dy$

Then,

$$I_1 = \frac{-(\alpha+1)b\beta^{\alpha+\alpha j+m}}{e^{-a}(\Gamma(\alpha))^{j+1}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{m=0}^{\infty} C_{j,m}$$

$$\int_0^\infty \ln(y)\, y^{m+\alpha(j+1)+1} e^{-\beta y} y^{-2} dy$$

$$= \frac{-(\alpha+1)b\beta^{\alpha+\alpha j+m}}{e^{-a}(\Gamma(\alpha))^{j+1}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{m=0}^{\infty} C_{j,m}$$

$$\int_0^\infty \ln(y)\, y^{m+\alpha(j+1)-1} e^{-\beta y} dy$$

$$= -(\alpha+1)\frac{be^a}{(\Gamma(\alpha))^{j+1}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{m=0}^{\infty} C_{j,m}$$

$$\Gamma\big(m+\alpha(j+1)\big)\left\{\Psi\big(m+\alpha(j+1)\big) - \ln(\beta)\right\}$$

And

$$I_2 = \beta E\left(\frac{1}{x}\right)$$

$$= \beta\frac{ab\beta^\alpha}{e^{-a}\Gamma(\alpha)}\int_0^\infty x^{-1}x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(b+1)} e^{-a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}} dx$$

Since

$$e^{-a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}} = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^i\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-bi}$$

Then,

$$I_2 = \frac{\beta ab\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^i$$

$$\int_0^\infty x^{-(\alpha+2)} e^{-\left(\frac{\beta}{x}\right)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(b+1)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-bi} dx$$

$$= \frac{\beta b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}$$

$$\int_0^\infty x^{-(\alpha+2)} e^{-\left(\frac{\beta}{x}\right)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)} dx$$

Since

$$\Gamma\left(\alpha,\frac{\beta}{x}\right) = \Gamma(\alpha) - \gamma\left(\alpha,\frac{\beta}{x}\right)$$

Then,

$$I_2 = \frac{b\beta^{\alpha+1}}{e^{-a}\Gamma(\alpha)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\int_0^\infty x^{-(\alpha+2)} e^{-\left(\frac{\beta}{x}\right)}\left\{1 - \frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)} dx$$

By using equation (10) we get,

$$\left\{1 - \frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)} = \sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\left(\frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right)^j$$

$$I_2 = \frac{b\beta^{\alpha+1}}{e^{-a}\Gamma(\alpha)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}$$

$$\int_0^\infty x^{-(\alpha+2)} e^{-\left(\frac{\beta}{x}\right)}\left(\frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right)^j dx$$

By using eq. expansion incomplete gamma (18) and

$$\left[\sum_{m=0}^{\infty} a_m \left(\frac{\beta}{x}\right)^m\right]^j = \sum_{m=0}^{\infty} C_{j,m} \left(\frac{\beta}{x}\right)^m$$

We get,

$$\left(\frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right)^j = \frac{1}{(\Gamma(\alpha))^j}\left(\frac{\beta}{x}\right)^{\alpha j}\sum_{m=0}^{\infty} C_{j,m}\left(\frac{\beta}{x}\right)^m$$

Where,

$C_{j,m} = (ma_0)^{-1}\sum_{p=1}^{m}(jp - m + p)\, a_p C_{j,m-p}$, $C_{j,0} = a_0^j$ and $a_p = \frac{(-1)^p}{(\alpha+p)p!}$

And then,

$$I_2 = \frac{\beta^{\alpha+\alpha j+m+1}b}{e^{-a}(\Gamma(\alpha))^{j+1}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{m=0}^{\infty} C_{j,m}$$

$$\int_0^\infty x^{-(m+\alpha(j+1)+2)} e^{-\left(\frac{\beta}{x}\right)} dx$$

$$= \frac{be^a}{(\Gamma(\alpha))^{j+1}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{m=0}^{\infty} C_{j,m}\,\Gamma(m+\alpha(j+1)+1)$$

And

$$I_3 = (b+1)E\left(\ln\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}\right)$$

$$I_3 = (b+1)\frac{ab\beta^\alpha}{e^{-a}\Gamma(\alpha)}$$

$$\int_0^\infty \ln\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\} x^{-(\alpha+1)} e^{-\left(\frac{\beta}{x}\right)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(b+1)} e^{-a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}} dx$$

$$= (b+1)\frac{b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}$$

$$\int_0^\infty ln\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\} x^{-(\alpha+1)} e^{-\left(\frac{\beta}{x}\right)} \left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)} dx$$

Since

$$\Gamma\left(\alpha,\frac{\beta}{x}\right) = \Gamma(\alpha) - \gamma\left(\alpha,\frac{\beta}{x}\right)$$

Then,

$$I_3 = \frac{(b+1)b\beta^\alpha}{e^{-a}\Gamma(\alpha)} \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+1} \int_0^\infty ln\left\{1 - \frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\} x^{-(\alpha+1)} e^{-\left(\frac{\beta}{x}\right)} \left\{1 - \frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)} dx$$

By using

$$ln(1-x) = -\sum_{n=1}^\infty \frac{x^n}{n} \quad -1 < x < 1$$

We get

$$ln\left\{1 - \frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\} = \frac{-1}{n} \sum_{n=1}^\infty \left(\frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right)^n$$

And then,

$$I_3 = -(b+1) \frac{b\beta^\alpha}{e^{-a}\Gamma(\alpha)} \sum_{n=1}^\infty \frac{1}{n} \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^\infty \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \int_0^\infty \left(\frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right)^n x^{-(\alpha+1)} e^{-\left(\frac{\beta}{x}\right)} \left(\frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right)^j dx$$

$$= \frac{-(b+1)b\beta^\alpha}{e^{-a}\Gamma(\alpha)} \sum_{n=1}^\infty \frac{1}{n} \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^\infty \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \int_0^\infty \left(\frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right)^{n+j} x^{-(\alpha+1)} e^{-\left(\frac{\beta}{x}\right)} dx$$

By using equation (18) and

$$\left[\sum_{m=0}^\infty a_m \left(\frac{\beta}{x}\right)^m\right]^j = \sum_{m=0}^\infty C_{j,m} \left(\frac{\beta}{x}\right)^m$$

We get,

$$\left(\frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right)^{n+j} = \frac{1}{(\Gamma(\alpha))^{n+j}} \left(\frac{\beta}{x}\right)^{\alpha(n+j)} \sum_{m=0}^\infty C_{n+j,m} \left(\frac{\beta}{x}\right)^m$$

Where

$C_{n+j,m} = (ma_0)^{-1} \sum_{p=1}^m \big((n+j)p - m + p\big) a_p C_{n+j,m-p}, C_{n+j,0} = a_0^{n+j}$ and $a_p = \frac{(-1)^p}{(\alpha+p)p!}$.

$$I_3 = -(b+1) \frac{b\beta^{\alpha+\alpha n+\alpha j+m}}{e^{-a}(\Gamma(\alpha))^{n+j+1}} \sum_{n=1}^\infty \frac{1}{n} \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^\infty \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \sum_{m=0}^\infty C_{n+j,m} \int_0^\infty x^{-(m+\alpha(n+j+1)+1)} e^{-\left(\frac{\beta}{x}\right)} dx$$

$$= -(b+1) \frac{be^a}{(\Gamma(\alpha))^{n+j+1}} \sum_{n=1}^\infty \frac{1}{n} \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^\infty \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \sum_{m=0}^\infty c_{n+j,m} \Gamma\big(m+\alpha(n+j+1)\big)$$

And

$$I_4 = aE\left(\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}\right)$$

$$= \frac{a^2 b\beta^\alpha}{e^{-a}\Gamma(\alpha)} \int_0^\infty \left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b} x^{-(\alpha+1)} e^{-\left(\frac{\beta}{x}\right)} \left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(b+1)} e^{-a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}} dx$$

$$= \frac{a^2 b\beta^\alpha}{e^{-a}\Gamma(\alpha)} \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^i \int_0^\infty x^{-(\alpha+1)} e^{-\left(\frac{\beta}{x}\right)} \left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(2b+1)} \left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-bi} dx$$

$$= \frac{b\beta^\alpha}{e^{-a}\Gamma(\alpha)} \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+2} \int_0^\infty x^{-(\alpha+1)} e^{-\left(\frac{\beta}{x}\right)} \left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(bi+2b+1)} dx$$

$$= \frac{b\beta^\alpha}{e^{-a}\Gamma(\alpha)} \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+2} \int_0^\infty x^{-(\alpha+1)} e^{-\left(\frac{\beta}{x}\right)} \left\{1 - \frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(bi+2b+1)} dx$$

$$\left\{1 - \frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(bi+2b+1)} = \sum_{j=0}^\infty \frac{\Gamma([b(i+2)+1]+j)}{j!\Gamma([b(i+2)+1])} \left(\frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right)^j$$

Then,

$$I_4 = \frac{b\beta^\alpha}{e^{-a}\Gamma(\alpha)} \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+2} \sum_{j=0}^\infty \frac{\Gamma([b(i+2)+1]+j)}{j!\Gamma([b(i+2)+1])} \int_0^\infty x^{-(\alpha+1)} e^{-\left(\frac{\beta}{x}\right)} \left(\frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right)^j dx$$

Since

$$\left(\frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right)^j = \frac{1}{(\Gamma(\alpha))^j} \left(\frac{\beta}{x}\right)^{\alpha j} \sum_{m=0}^\infty C_{j,m} \left(\frac{\beta}{x}\right)^m$$

Then,

$$I_4 = \frac{b\beta^\alpha e^a}{(\Gamma(\alpha))^{j+1}} \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+2} \sum_{j=0}^\infty \frac{\Gamma([b(i+2)+1]+j)}{j!\Gamma([b(i+2)]+1)} \sum_{m=0}^\infty C_{j,m} \int_0^\infty \beta^{\alpha j+m} x^{-(m+\alpha(j+1)+1)} e^{-\left(\frac{\beta}{x}\right)} dx$$

$$= \frac{be^a}{(\Gamma(\alpha))^{j+1}} \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+2} \sum_{j=0}^\infty \frac{\Gamma([b(i+2)+1]+j)}{j!\Gamma([b(i+2)+1])} \sum_{m=0}^\infty C_{j,m} \Gamma\big(m+\alpha(j+1)\big)$$

$$H = ln\left(\frac{e^{-a}\Gamma(\alpha)}{ab\beta^\alpha}\right) - (\alpha+1) \frac{be^a}{(\Gamma(\alpha))^{j+1}} \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^\infty \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \sum_{m=0}^\infty C_{j,m} \Gamma\big(m+\alpha(j+1)\big)\big\{\Psi\big(m+\alpha(j+1)\big) - ln(\beta)\big\} + \frac{be^a}{(\Gamma(\alpha))^{j+1}} \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^\infty \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}$$

$$\sum_{m=0}^\infty C_{j,m} \Gamma(m+\alpha(j+1)+1) - (b+1) \frac{be^a}{(\Gamma(\alpha))^{n+j+1}} \sum_{n=1}^\infty \frac{1}{n} \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+1}$$

$$\sum_{j=0}^\infty \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \sum_{m=0}^\infty C_{n+j,m} \Gamma\big(m+\alpha(n+j+1)\big) + \frac{be^a}{(\Gamma(\alpha))^{j+1}}$$

$$\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+2}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+2)+1]+j)}{j!\Gamma([b(i+2)+1])}\sum_{m=0}^{\infty}C_{j,m}\,\Gamma\binom{m}{+\alpha(j+1)} \quad (30)$$

The relative entropy DKL$(F\|F^*)$ for a random variable [0, 1] TFIG(a,b,α,β) can be found as follows,

$$\frac{f(x)}{f^*(x)}=\frac{\frac{ab\beta^\alpha}{e^{-a}\Gamma(\alpha)}x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(b+1)}e^{-a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}}}{\frac{a_1b_1\beta_1{}^{\alpha_1}}{e^{-a_1}\Gamma(\alpha_1)}x^{-(\alpha_1+1)}e^{-\left(\frac{\beta_1}{x}\right)}\left\{\frac{\Gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\}^{-(b_1+1)}e^{-a_1\left\{\frac{\Gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\}^{-b_1}}}$$

$$DKL=\int_0^\infty f(x)\,ln\left(\frac{ab\beta^\alpha e^{-a_1}\Gamma(\alpha_1)x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(b+1)}e^{-a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}}}{a_1b_1\beta_1^{\alpha_1}e^{-a}\Gamma(\alpha)x^{-(\alpha_1+1)}e^{-\left(\frac{\beta_1}{x}\right)}\left\{\frac{\Gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\}^{-(b_1+1)}e^{-a_1\left\{\frac{\Gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\}^{-(b_1)}}}\right)dx$$

$$=\int_0^\infty f(x)\begin{bmatrix} ln\left(\frac{ab\beta^\alpha e^{-a_1}\Gamma(\alpha_1)}{a_1b_1\beta_1^{\alpha_1}e^{-a}\Gamma(\alpha)}\right)+(\alpha_1-\alpha)\,ln(x)\\ +(\beta_1-\beta)\frac{1}{x}-(b+1)\,ln\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}\\ -a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}\\ +(b_1+1)\,ln\left\{\frac{\Gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\}+a_1\left\{\frac{\Gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\}^{-b_1}\end{bmatrix}dx$$

$$=ln\left(\frac{ab\beta^\alpha e^{-a_1}\Gamma(\alpha_1)}{a_1b_1\beta_1^{\alpha_1}e^{-a}\Gamma(\alpha)}\right)+(\alpha_1-\alpha)E(ln(x))+(\beta_1-\beta)E\left(\frac{1}{x}\right)-(b+1)E\left(ln\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}\right)-$$

$$aE\left(\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}\right)+(b_1+1)E\left(ln\left\{\frac{\Gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\}\right)+a_1E\left(\left\{\frac{\Gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\}^{-b_1}\right)$$

Let, $I_1=(\alpha_1-\alpha)E(ln(x))$

$$=\frac{(\alpha_1-\alpha)ab\beta^\alpha}{e^{-a}\Gamma(\alpha)}\int_0^\infty ln(x)\,x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(b+1)}e^{-a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}}dx$$

$$=-(\alpha_1-\alpha)\frac{be^a}{(\Gamma(\alpha))^{j+1}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{m=0}^{\infty}C_{j,m}\,\Gamma\big(m+\alpha(j+1)\big)\{\Psi\big(m+\alpha(j+1)\big)-ln(\beta)\}$$

And

$$I_2=(\beta_1-\beta)E\left(\frac{1}{x}\right)$$

$$=\frac{(\beta_1-\beta)ab\beta^\alpha}{e^{-a}\Gamma(\alpha)}\int_0^\infty x^{-1}\,x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(b+1)}e^{-a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}}dx$$

$$=(\beta_1-\beta)\frac{be^a}{\beta(\Gamma(\alpha))^{j+1}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{m=0}^{\infty}C_{j,m}\,\Gamma(m+\alpha(j+1)+1)$$

And

$$I_3=-(b+1)E\left(ln\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}\right)$$

$$=-(b+1)\frac{ab\beta^\alpha}{e^{-a}\Gamma(\alpha)}\int_0^\infty ln\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(b+1)}e^{-a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}}dx$$

$$=(b+1)\frac{be^a}{(\Gamma(\alpha))^{n+j+1}}\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{m=0}^{\infty}C_{n+j,m}\,\Gamma\big(m+\alpha(n+j+1)\big)$$

And

$$I_4=-\,aE\left(\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}\right)$$

$$=\frac{-a^2b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\int_0^\infty\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(b+1)}e^{-a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}}dx$$

$$=\frac{-be^a}{(\Gamma(\alpha))^{j+1}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+2}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+2)+1]+j)}{j!\Gamma([b(i+2)+1])}\sum_{m=0}^{\infty}C_{j,m}\,\Gamma\big(m+\alpha(j+1)\big)$$

And

$$I_5=(b_1+1)E\left(ln\left\{\frac{\Gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\}\right)$$

$$=(b_1+1)\frac{ab\beta^\alpha}{e^{-a}\Gamma(\alpha)}\int_0^\infty ln\left\{\frac{\Gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\}x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(b+1)}e^{-a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}}dx$$

Since

$$e^{-a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}}=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^i\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-bi}$$

Then,

$$=(b_1+1)\frac{b\beta^\alpha}{e^{-a}\Gamma(\alpha)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\int_0^\infty ln\left\{\frac{\Gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\}x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)}dx$$

Since

$$\Gamma\left(\alpha,\frac{\beta}{x}\right)=\Gamma(\alpha)-\gamma\left(\alpha,\frac{\beta}{x}\right)$$

And

$$\Gamma\left(\alpha_1,\frac{\beta_1}{x}\right)=\Gamma(\alpha_1)-\gamma\left(\alpha_1,\frac{\beta_1}{x}\right)$$

Then,

$$= \frac{(b_1+1)b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \int_0^{\infty} ln\left\{1 - \frac{\gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\} x^{-(\alpha+1)} e^{-\left(\frac{\beta}{x}\right)} \left\{1 - \frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)} dx$$

By using equation (10) we get,

$$I_5 = \frac{(b_1+1)b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}$$

$$\int_0^{\infty} ln\left\{1 - \frac{\gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\} x^{-(\alpha+1)} e^{-\left(\frac{\beta}{x}\right)} \left(\frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right)^j dx$$

By using

$$ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \quad |x| < 1$$

We get

$$ln\left\{1 - \frac{\gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\} = -\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right)^n$$

And then,

$$I_5 = -(b_1+1) \frac{b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \int_0^{\infty} \left(\frac{\gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right)^n x^{-(\alpha+1)}$$

$$e^{-\left(\frac{\beta}{x}\right)} \left(\frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right)^j dx$$

By using equation (18) and

$$\left[\sum_{m=0}^{\infty} a_m \left(\frac{\beta}{x}\right)^m\right]^j = \sum_{m=0}^{\infty} C_{j,m} \left(\frac{\beta}{x}\right)^m$$

We get,

$$I_5 = \frac{-(b_1+1)b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \frac{1}{(\Gamma(\alpha_1))^n}$$

$$\sum_{d=0}^{\infty} C_{n,d} \frac{1}{(\Gamma(\alpha))^j} \sum_{m=0}^{\infty} C_{j,m} \int_0^{\infty} \left(\frac{\beta_1}{x}\right)^{\alpha_1 n} \left(\frac{\beta_1}{x}\right)^d$$

$$x^{-(\alpha+1)} e^{-\left(\frac{\beta}{x}\right)} \left(\frac{\beta}{x}\right)^{\alpha j} \left(\frac{\beta}{x}\right)^m dx$$

$$= -(b_1+1) \frac{b\beta^{\alpha}\beta_1^{\alpha_1 n+d}}{e^{-a}(\Gamma(\alpha))^{j+1}(\Gamma(\alpha_1))^n}$$

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}$$

$$\sum_{d=0}^{\infty} C_{n,d} \sum_{m=0}^{\infty} C_{j,m}\, \beta^{\alpha j+m} \int_0^{\infty} x^{-(m+d+\alpha_1 n+\alpha(j+1)+1)} e^{-\left(\frac{\beta}{x}\right)} dx$$

$$= \frac{-(b_1+1)b(\beta_1/\beta)^{\alpha_1 n+d} e^a}{(\Gamma(\alpha))^{j+1}(\Gamma(\alpha_1))^n} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}$$

$$\sum_{d=0}^{\infty} C_{n,d} \sum_{m=0}^{\infty} C_{j,m}\, \Gamma\big(m+d+\alpha_1 n+\alpha(j+1)\big)$$

And

$$I_6 = a_1 E\left(\left\{\frac{\Gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\}^{-b_1}\right)$$

$$= \frac{a_1 a b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)} \int_0^{\infty} \left\{\frac{\Gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\}^{-b_1} x^{-(\alpha+1)} e^{-\left(\frac{\beta}{x}\right)} \left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(b+1)} e^{-a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}} dx$$

$$= \frac{a_1 b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1}$$

$$\int_0^{\infty} \left\{\frac{\Gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\}^{-b_1} x^{-(\alpha+1)} e^{-\left(\frac{\beta}{x}\right)} \left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)} dx$$

$$= \frac{a_1 b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \int_0^{\infty} \left\{1 - \frac{\gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\}^{-b_1} x^{-(\alpha+1)} e^{-\left(\frac{\beta}{x}\right)} \left\{1 - \frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)} dx$$

By using equation (10) we get,

$$I_6 = \frac{a_1 b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}$$

$$\sum_{l=0}^{\infty} \frac{\Gamma(b_1+l)}{l!\Gamma(b_1)} \int_0^{\infty} \left(\frac{\gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right)^l x^{-(\alpha+1)}$$

$$e^{-\left(\frac{\beta}{x}\right)} \left(\frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right)^j dx$$

By using (18) and

$$\left[\sum_{m=0}^{\infty} a_m \left(\frac{\beta}{x}\right)^m\right]^j = \sum_{m=0}^{\infty} C_{j,m} \left(\frac{\beta}{x}\right)^m$$

We get,

$$I_6 = \frac{a_1 b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}$$

$$\sum_{l=0}^{\infty} \frac{\Gamma(b_1+l)}{l!\Gamma(b_1)} \int_0^{\infty} \frac{1}{(\Gamma(\alpha_1))^l} \left(\frac{\beta_1}{x}\right)^{\alpha_1 l} \sum_{d=0}^{\infty} C_{l,d}$$

$$\left(\frac{\beta_1}{x}\right)^d x^{-(\alpha+1)} e^{-\left(\frac{\beta}{x}\right)} \frac{1}{(\Gamma(\alpha))^j} \left(\frac{\beta}{x}\right)^{\alpha j} \sum_{m=0}^{\infty} C_{j,m} \left(\frac{\beta}{x}\right)^m dx$$

$$= \frac{a_1 b\beta^{\alpha}}{e^{-a}\Gamma(\alpha)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}$$

$$\sum_{l=0}^{\infty} \frac{\Gamma(b_1+l)}{l!\Gamma(b_1)} \sum_{d=0}^{\infty} C_{l,d} \sum_{m=0}^{\infty} C_{j,m}\, \beta_1^{\alpha_1 l+d}$$

$$\frac{1}{(\Gamma(\alpha_1))^l} \frac{1}{(\Gamma(\alpha))^j} \beta^{\alpha j+m} \int_0^{\infty} x^{-(d+\alpha_1 l+\alpha(j+1)+m+1)} e^{-\left(\frac{\beta}{x}\right)} dx$$

$$= \frac{a_1 b(\beta_1/\beta)^{\alpha_1 l+d} e^a}{(\Gamma(\alpha_1))^l (\Gamma(\alpha))^{j+1}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1}$$

$$\sum_{i=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \sum_{l=0}^{\infty} \frac{\Gamma(b_1+l)}{l!\Gamma(b_1)} \sum_{d=0}^{\infty} C_{l,d}$$

$$\sum_{m=0}^{\infty} C_{j,m}\, \Gamma(d+\alpha_1 l+\alpha(j+1)+m)$$

$$Dkl(F\|F^*) = ln\left(\frac{ab\beta^{\alpha} e^{-a_1}\Gamma(\alpha_1)}{a_1 b_1 \beta_1^{\alpha_1} e^{-a}\Gamma(\alpha)}\right) - (\alpha_1 - \alpha) \frac{be^a}{(\Gamma(\alpha))^{j+1}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}$$

$$\sum_{m=0}^{\infty} C_{j,m}\, \Gamma\big(m+\alpha(j+1)\big)\left\{\Psi\big(m+\alpha(j+1)\big) - ln(\beta)\right\} + (\beta_1 - \beta) \frac{be^a}{\beta(\Gamma(\alpha))^{j+1}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1}$$

$$\sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \sum_{m=0}^{\infty} C_{j,m}\, \Gamma(m+\alpha(j+1)+1) + (b+1) \frac{be^a}{(\Gamma(\alpha))^{n+j+1}} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}$$

$$a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{m=0}^{\infty}C_{n+j,m}\,\Gamma\big(m+\alpha(n+j+1)\big)-\frac{be^{a}}{(\Gamma(\alpha))^{j+1}}\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}a^{i+2}$$

$$\sum_{j=0}^{\infty}\frac{\Gamma([b(i+2)+1]+j)}{j!\Gamma([b(i+2)+1])}\sum_{m=0}^{\infty}C_{j,m}\,\Gamma\big(m+\alpha(j+1)\big)-(b_1+1)\frac{b(\beta_1/\beta)^{\alpha_1 n+d}e^{a}}{(\Gamma(\alpha))^{j+1}(\Gamma(\alpha_1))^{n}}\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}a^{i+1}\sum_{n=1}^{\infty}\frac{1}{n}$$

$$\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{d=0}^{\infty}C_{n,d}\sum_{m=0}^{\infty}C_{j,m}\,\Gamma\big(m+d+\alpha_1 n+\alpha(j+1)\big)+\frac{a_1 b(\beta_1/\beta)^{\alpha_1 l+d}e^{a}}{(\Gamma(\alpha_1))^{l}(\Gamma(\alpha))^{j+1}}$$

$$\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{l=0}^{\infty}\frac{\Gamma(b_1+l)}{l!\Gamma(b_1)}\sum_{d=0}^{\infty}C_{l,d}\sum_{m=0}^{\infty}C_{j,m}\,\Gamma(d+\alpha_1 l+\alpha(j+1)+m) \quad (31)$$

3.2 Stress-strength reliability

Let Y and X be the stress and the strength random variable, independent of each other, follow respectively [0,1] TFIG(a,b,α,β) and [0,1] TFIG$(a_1,b_1,\alpha_1,\beta_1)$,then,

$$R=P(Y<X)=\int_0^{\infty}f_x(x)F_Y(x)\,dx$$

$$R=\int_0^{\infty}\frac{ab\beta^{\alpha}}{e^{-a}\Gamma(\alpha)}x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(b+1)} e^{-a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}}\frac{e^{-a_1\left\{\frac{\Gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\}^{-b_1}}}{e^{-a_1}}\,dx$$

Since

$$e^{-a\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-b}}=\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}a^{i}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-bi}$$

Then,

$$R=\frac{b\beta^{\alpha}}{e^{-(a_1+a)}\Gamma(\alpha)}\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}a^{i+1}\int_0^{\infty}x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)} e^{-a_1\left\{\frac{\Gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\}^{-b_1}}\,dx$$

Also by using,

$$e^{-a_1\left\{\frac{\Gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\}^{-b_1}}=\sum_{u=0}^{\infty}\frac{(-1)^{u}}{u!}(a_1)^{u}\left\{\frac{\Gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\}^{-b_1 u}$$

We get,

$$R=\frac{b\beta^{\alpha}e^{a_1+a}}{\Gamma(\alpha)}\sum_{i,u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}(a_1)^{u}\int_0^{\infty}x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left\{\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)}\left\{\frac{\Gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\}^{-b_1 u}dx$$

$$since\ \Gamma\left(\alpha,\frac{\beta}{x}\right)=\Gamma(\alpha)-\gamma\left(\alpha,\frac{\beta}{x}\right),\ \Gamma\left(\alpha_1,\frac{\beta_1}{x}\right)=\Gamma(\alpha_1)-\gamma\left(\alpha_1,\frac{\beta_1}{x}\right)$$

Then,

$$R=\frac{b\beta^{\alpha}e^{a_1+a}}{\Gamma(\alpha)}\sum_{i,u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}(a_1)^{u}\int_0^{\infty}x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left\{1-\frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right\}^{-(bi+b+1)}\left\{1-\frac{\gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right\}^{-b_1 u}dx$$

By using equation (10) we get,

$$R=\frac{b\beta^{\alpha}e^{a_1+a}}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}(a_1)^{u}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{l=0}^{\infty}\frac{\Gamma(b_1u+l)}{l!\Gamma(b_1u)}$$

$$\int_0^{\infty}x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left(\frac{\gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}\right)^{j}\left(\frac{\gamma\left(\alpha_1,\frac{\beta_1}{x}\right)}{\Gamma(\alpha_1)}\right)^{l}dx$$

By using equation (18) and

$$\left[\sum_{m=0}^{\infty}a_m\left(\frac{\beta}{x}\right)^{m}\right]^{j}=\sum_{m=0}^{\infty}C_{j,m}\left(\frac{\beta}{x}\right)^{m}$$

We get,

$$R=\frac{b\beta^{\alpha}e^{a_1+a}}{(\Gamma(\alpha))^{j+1}(\Gamma(\alpha_1))^{l}}\sum_{i,u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}(a_1)^{u}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{l=0}^{\infty}\frac{\Gamma(b_1u+l)}{l!\Gamma(b_1u)}$$

$$\sum_{m=0}^{\infty}C_{j,m}\sum_{d=0}^{\infty}C_{l,d}\int_0^{\infty}x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)}\left(\frac{\beta}{x}\right)^{\alpha j}\left(\frac{\beta}{x}\right)^{m}\left(\frac{\beta_1}{x}\right)^{\alpha_1 l}\left(\frac{\beta_1}{x}\right)^{d}dx$$

$$=\frac{b\beta^{\alpha}e^{a_1+a}}{(\Gamma(\alpha))^{j+1}(\Gamma(\alpha_1))^{l}}\sum_{i,u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}(a_1)^{u}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{l=0}^{\infty}\frac{\Gamma(b_1u+l)}{l!\Gamma(b_1u)}\sum_{m=0}^{\infty}C_{j,m}$$

$$\sum_{d=0}^{\infty}C_{l,d}\,\beta_1^{\alpha_1 l+d}\beta^{\alpha j+m}\int_0^{\infty}x^{-(\alpha_1 l+d+m+\alpha(j+1)+1)}e^{-\left(\frac{\beta}{x}\right)}dx$$

$$=\frac{b(\beta_1/\beta)^{\alpha_1 l+d}e^{a_1+a}}{(\Gamma(\alpha))^{j+1}(\Gamma(\alpha_1))^{l}}\sum_{i,u=0}^{\infty}\frac{(-1)^{i+u}}{i!u!}a^{i+1}(a_1)^{u}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\sum_{l=0}^{\infty}\frac{\Gamma(b_1u+l)}{l!\Gamma(b_1u)}$$

$$\sum_{m=0}^{\infty}C_{j,m}\sum_{d=0}^{\infty}C_{l,d}\,\Gamma\big(\alpha_1 l+d+m+\alpha(j+1)\big) \quad (32)$$

4. Summary and conclusions

In a statistical analysis a lot of distributions are used to represent set(s) data. Recently. New distributions are derived to extend some of the well-known families of distributions, such that the new distributions are more flexible than the others to model real data. The composing of some distributions with each other's in some way has been in the foreword of data modeling.

In this paper, we presented a new family of continuous distributions based on [0, 1]] truncated Fréchet distribution. [0, 1]] Truncated Fréchet Gamma ([0, 1]] TFG) and [0, 1 truncated Fréchet inverted Gamma ([0, 1]] TFIG) distributions are discussed as special cases. Properties of [0, 1] TFG 1] TFG and [0, 1] TFIG 1] TFIG is derived. We provide forms for characteristic function, rth raw moment, mean, variance, skewness, kurtosis, mode, median, reliability function, hazard rate function, Shannon's entropy function and Relative entropy function.

This paper deals also with the determination of stress-strength reliability R=p[y < x] when x (strength), and y (stress) are two independent [0, 1]] TFG ([0, 1] TFIG) distributions 1] TFIG) distributions with different parameters.

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