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Research paper

General solution of second order fractional differential equations

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Abstract

Fractional differential equations are often seeming perplexing to solve. Therefore, finding comprehensive methods for solving them sounds of high importance. In this paper, a general method for solving second order fractional differential equations has been presented based on conformable fractional derivative. This method realizes on determining a general solution of homogeneous and a particular solution of a second order linear fractional differential equations. Furthermore, a general solution has been developed for fractional Euler's equation. For more explanation of each part, some examples have been solved.

Keywords: *Linear fractional differential equations; Conformable fractional derivative; Constant coefficients approach; Euler's equation; Variation of parameters; Lagrange method; Undetermined coefficients;*

1. Introduction

Fractional differential equations are studied in various fields of physics and engineering, specifically in signal processing, control engineering, electromagnetism, biosciences, fluid mechanics, electrochemistry, diffusion processes, dynamic of viscoelastic material, continuum and statistical mechanics and propagation of spherical flames. There are many fractional differential equations which can't be solved analytically. Due to this fact, finding an approximate solution of fractional differential equations is clearly an important task. In recent years, many effective methods have been proposed for the approximate solution fractional differential equations, such as Adomian decomposition method [3,4], homotopy perturbation method [5-8], homotopy analysis method [9,10], variational iteration method [11], generalized differential transform method [12] and other methods [13-29].

The organization of the paper is as follows: In Section 2, Basic definitions, such as conformable fractional derivative, and conformable fractional integral, will be presented. In Section 3, Basic theoretical of the method, will be described. In Section 4, the methods such as, use of a known solution to find another one or D'Alambert approach, homogeneous equation with constant coefficients, Euler's dimensional equation, will be expanded. In Section 5, the methods such as, variation of parameters or Lagrange technique, undetermined coefficients, will be explained. Finally, conclusion is presented in section 6.

2. Basic definitions

The purpose of this section is to recall some preliminaries of the proposed method.

2.1. Conformable fractional derivative

Given a function $f:[0,\infty)\to\mathbb{R}$. Then the conformable fractional derivative of f of order α is defined by

$$\mathrm{T}_\alpha(\mathrm{f})(\mathrm{x}) = \lim_{\varepsilon\to 0}\frac{\mathrm{f}(\mathrm{x}+\varepsilon\mathrm{x}^{1-\alpha})-\mathrm{f}(\mathrm{x})}{\varepsilon} \quad (1)$$

For all $x>0, \alpha\in(0,1)$. If f is α - differentiable in some $(0,a), a>0$, and provided that $\lim_{x\to 0^+}\mathrm{T}_\alpha(f)(x)$ exists, then define $\mathrm{T}_\alpha(f)(0)=\lim_{x\to 0^+}\mathrm{T}_\alpha(f)(x)$.

If the conformable derivative of f of order α exists, then we simply say that f is α- differentiable (see [1,2]).

One can easily show that T_α satisfies all the properties in the following properties (see [1]):

Let $\alpha\in(0,1]$ and f,g be α-differentiable at a point $x>0$, Then

A. For $a,b\in\mathbb{R}$ $\mathrm{T}_\alpha(af+bg)=a\,\mathrm{T}_\alpha(f)+b\,\mathrm{T}_\alpha(g)$,

B. For all $p\in\mathbb{R}$ $\mathrm{T}_\alpha(x^p)=px^{p-\alpha}$,

C. For all constant functions $f(x)=\lambda$, $\mathrm{T}_\alpha(\lambda)=0$,

D. $\mathrm{T}_\alpha(f.g)=g.\mathrm{T}_\alpha(f)+f.\mathrm{T}_\alpha(g)$,

E. $\mathrm{T}_\alpha\left(\frac{f}{g}\right)=\frac{g.\mathrm{T}_\alpha(f)-f.\mathrm{T}_\alpha(g)}{g^2}$,

F. $\mathrm{T}_\alpha(f)=x^{1-\alpha}\frac{df}{dx}$.

2.2. Conformable fractional integral

Given a function $f:[a,\infty)\to\mathbb{R},\ a\geq 0$. Then the conformable fractional integral of f is defined by

$$\mathrm{I}_\alpha^a(f)(x)=\int_a^x \frac{f(t)}{t^{1-\alpha}}dt \tag{2}$$

Where the integral is the usual Riemann improper integral, and $\alpha\in(0,1)$ (see [1,2]).
For the sake of simplicity, let's consider $\mathrm{I}_\alpha^0(f)(x)=\mathrm{I}_\alpha(f)(x)$.
One of the most useful results is the following (see [1]):
For all $x\geq a$, and any continuous function in the domain of I_α^a, we have $\mathrm{T}_\alpha\big(\mathrm{I}_\alpha^a f(x)\big)=f(x)$.

3. Basic theoretical of the method

Let's consider the general second order linear fractional differential equation based on conformable fractional derivative as follows

$$\mathrm{T}_\alpha\mathrm{T}_\alpha\big(u(x)\big)+P(x)\mathrm{T}_\alpha\big(u(x)\big)+Q(x)u(x)=R(x), \tag{3}$$

Where $P(x)$, $Q(x)$, and $R(x)$ are $\alpha-$ differentiable functions and $u(x)$ is an unknown function. If $R(x)$ is identically zero, then fractional equation (3) reduces to the homogeneous fractional equation

$$\mathrm{T}_\alpha\mathrm{T}_\alpha\big(u(x)\big)+P(x)\mathrm{T}_\alpha\big(u(x)\big)+Q(x)u(x)=0. \tag{4}$$

Theorem 3.1. If $u_h(x,C_1,C_2)$ is the general solution of fractional equation (4) and $u_p(x)$ is any particular solution of fractional equation (3), then $u_h(x,C_1,C_2)+u_p(x)$ is a general solution of fractional equation (3).
Proof. Suppose that $u(x)$ is a solution of (3), since $u_p(x)$ is any particular solution of fractional equation (3), then an easy calculation shows that $u(x)-u_p(x)$ is a solution of (4):

$$\mathrm{T}_\alpha\mathrm{T}_\alpha\big(u(x)-u_p(x)\big)+P(x)\mathrm{T}_\alpha\big(u(x)-u_p(x)\big)+$$
$$Q(x)\left(u(x)-u_p(x)\right)=(\mathrm{T}_\alpha\mathrm{T}_\alpha(u(x))+P(x)\mathrm{T}_\alpha(u(x))$$
$$-(\mathrm{T}_\alpha\mathrm{T}_\alpha\left(u_p(x)\right)+P(x)\mathrm{T}_\alpha\left(u_p(x)\right)+Q(x)u_p(x))=$$
$$R(x)-R(x)=0.$$

Since $u_h(x,C_1,C_2)$ is a general solution to (4), it results that $u(x)-u_p(x)=u_h(x,C_1,C_2)$ or $u(x)=u_h(x,C_1,C_2)+u_p(x)$, for a suitable choice of the constants C_1 and C_2(see [23]).

Theorem 3.2. If $u_1(x)$ and $u_2(x)$ are any two solutions of fractional equation (4), then $C_1u_1(x)+C_2u_2(x)$ is also a solution for any constants $C_1$and C_2.
Proof. The statement follows immediately from the fact that

$$\mathrm{T}_\alpha\mathrm{T}_\alpha(C_1u_1(x)+C_2u_2(x))+P(x)\mathrm{T}_\alpha(C_1u_1(x)+$$
$$C_2u_2(x))+Q(x)(C_1u_1(x)+C_2u_2(x))=$$
$$C_1(\mathrm{T}_\alpha\mathrm{T}_\alpha(u_1(x))+P(x)\mathrm{T}_\alpha(u_1(x))+Q(x)u_1(x))+$$
$$C_2(\mathrm{T}_\alpha\mathrm{T}_\alpha(u_2(x))+P(x)\mathrm{T}_\alpha(u_2(x))+Q(x)u_2(x))=$$
$$C_1.0+C_2.0=0.$$

Since by assumption, $u_1(x)$ and $u_2(x)$ are solutions of (4) (see [23]).

Definition. The fractional Wronskian of two functions $f(x)$ and $g(x)$, is defined by (see [22,23]),

$$W_\alpha\big(f(x),g(x)\big)=\begin{vmatrix} f(x) & g(x)\\ \mathrm{T}_\alpha\big(f(x)\big) & \mathrm{T}_\alpha\big(g(x)\big)\end{vmatrix}$$
$$=f(x).\mathrm{T}_\alpha\big(g(x)\big)-g(x).\mathrm{T}_\alpha\big(f(x)\big).$$

Theorem 3.3. If $u_1(x)$ and $u_2(x)$ are any two solutions of fractional equation (4) on an interval $[a,b]$, then their fractional Wronskian $W=W_\alpha(u_1(x),u_2(x))$ is either identically zero or never zero on $[a,b]$.
Proof. We begin by observing that

$$\mathrm{T}_\alpha(W)=u_1\mathrm{T}_\alpha\mathrm{T}_\alpha(u_2)-u_2\mathrm{T}_\alpha\mathrm{T}_\alpha(u_2)$$

Next, since $u_1(x)$ and $u_2(x)$ are both solutions of fractional equation (4), we have

$$\mathrm{T}_\alpha\mathrm{T}_\alpha(u_1(x))+P(x)\mathrm{T}_\alpha(u_1(x))+Q(x)u_1(x)=0$$
$$\mathrm{T}_\alpha\mathrm{T}_\alpha(u_2(x))+P(x)\mathrm{T}_\alpha(u_2(x))+Q(x)u_2(x)=0.$$

First equation multiplying by u_2 subtract to the second equation by u_1 result in

$$(u_1\mathrm{T}_\alpha\mathrm{T}_\alpha(u_2)-u_2\mathrm{T}_\alpha\mathrm{T}_\alpha(u_2))+P(x)(u_1\mathrm{T}_\alpha(u_2)-u_2\mathrm{T}_\alpha(u_1))=0$$

or

$$\mathrm{T}_\alpha(W)+P(x)W=0.$$

The general solution of this first order fractional differential equation based on conformable fractional derivative is (see [17])

$$W=W_\alpha(x_0)e^{-\mathrm{I}_\alpha(P(x))},$$

and since the exponential factor is never zero, the proof is completed (see [23]).

Theorem 3.4. If $u_1(x)$ and $u_2(x)$ are any two solutions of fractional equation (4) on an interval $[a,b]$, then they are linearly dependent on this interval if and only if their fractional Wronskian $W=W_\alpha(u_1(x),u_2(x))$, is identically zero.

Proof. We begin by assuming that $u_1(x)$ and $u_2(x)$ are linearly dependent, and we show

$$W_\alpha\big(u_1(x),u_2(x)\big)=0.$$

First, if either of the functions is identically zero on $[a,b]$, then the conclusion is clear. Therefore, we may therefor assume without loss of generality, that neither of them is identically zero, and their linear dependence result that each of those is a constant multiple of the other one. Accordingly, we have $u_2=Cu_1$, for some constant, so $\mathrm{T}_\alpha(u_2)=C\mathrm{T}_\alpha(u_1)$.
By elimination C, from this equation, we obtain
$u_1\mathrm{T}_\alpha(u_2)-u_2\mathrm{T}_\alpha(u_1)=0$,
which proves this half of the theorem. We now assume that the $W_\alpha\big(u_1(x),u_2(x)\big)=0$, and prove linearly dependent. If $u_1(x)$ is identically zero on $[a,b]$, then the functions are linearly dependent. We may therefore assume that $u_1(x)$, does not vanish identically on $[a,b]$, from which it follows by continuity that $u_1(x)$ does not vanish at all on some subinterval $[c,d]$ of $[a,b]$. Since the Wronskian is identically zero on $[a,b]$, we can divide it by u_1^2 to get
$\frac{u_1\mathrm{T}_\alpha(u_2)-u_2\mathrm{T}_\alpha(u_1)}{u_1^2}=0$, on $[c,d]$.
This can be written in the form $\mathrm{T}_\alpha\left(\frac{u_2}{u_1}\right)=0$, and by conformable fractional integrating we obtain $\frac{u_2}{u_1}=C$, or $u_2=Cu_1$, for some constant C, and all x, in $[c,d]$. Finally, since u_2, and Cu_1, have equal value in $[c,d]$, they have equal conformable fractional derivative, so $u_2=Cu_1$, all x, in $[a,b]$, which concludes the argument (see [23]).

Theorem 3.5. Let $u_1(x)$ and $u_2(x)$, be linearly dependent of the homogeneous fractional equation (4), on the interval $[a,b]$. Then $C_1u_1(x)+C_2u_2(x)$, is the general solution of the fractional equation (4) on this interval.

Proof. Let $u(x)$, be any solution of (4) on $[a,b]$. We must show that constant C_1,C_2, can be found so that $u(x)=C_1u_1(x)+C_2u_2(x)$, for all x in $[a,b]$. Since $C_1u_1(x)+C_2u_2(x)$, and $u(x)$ are both solutions of (4) on $[a,b]$, it suffices to show that for some point x_0, in $[a,b]$, we can find C_1,C_2 so that
$C_1u_1(x_0)+C_2u_2(x_0)=u(x_0)$, and
$C_1\mathrm{T}_\alpha\big(u_1(x_0)\big)+C_2\mathrm{T}_\alpha\big(u_2(x_0)\big)=\mathrm{T}_\alpha(u(x_0))$.
For this system to be solvable for C_1,C_2, it suffices that the following determinant be non-zero.

$$\begin{vmatrix} u_1(x_0) & u_2(x_0)\\ \mathrm{T}_\alpha(u_1(x_0)) & \mathrm{T}_\alpha(u_2(x_0))\end{vmatrix}$$
$$=u_1(x_0).\mathrm{T}_\alpha(u_2(x_0))-u_2(x_0).\mathrm{T}_\alpha(u_1(x_0))$$

This leads us to investigate the function Wronskian of $u_1(x),u_2(x)$ at x_0. According to theorems 3.3 and 3.4 it is clear that $W_\alpha(u_1(x_0), u_2(x_0))$, have a value different from zero (see [23]).

4. Determining a general solution of a homogeneous fractional equation

This section is motivated to obtain general solution of homogeneous fractional differential equations and then to solve Euler's equation.

4.1. The use of a known solution to find another one or D'Alambert approach

We assume that $u_1(x)$, is a known nonzero solution of Eq. (4), $u_2(x) = v(x)u_1(x)$, is a solution of (4), where as $v(x)$ an unknown function (see [20]). So

$$\mathrm{T}_\alpha\big(u_2(x)\big) = v(x).\mathrm{T}_\alpha\big(u_1(x)\big) + u_1(x).\mathrm{T}_\alpha\big(v(x)\big)$$

$$\mathrm{T}_\alpha\,\mathrm{T}_\alpha\big(u_2(x)\big) = v(x).\mathrm{T}_\alpha\mathrm{T}_\alpha\big(u_1(x)\big) + 2\mathrm{T}_\alpha(u_1(x)).\mathrm{T}_\alpha\big(v(x)\big) + u_1(x).\mathrm{T}_\alpha\mathrm{T}_\alpha\big(u_1(x)\big),$$

By substituting the above results into Eq. (4), we get

$$v(x)\Big(\mathrm{T}_\alpha\mathrm{T}_\alpha\big(u_1(x)\big) + P(x)\mathrm{T}_\alpha\big(u_1(x)\big) + Q(x)u_1(x)\Big) + u_1(x)\mathrm{T}_\alpha\mathrm{T}_\alpha\big(v(x)\big) + \mathrm{T}_\alpha\big(v(x)\big)\big(P(x)u_1(x) + 2\mathrm{T}_\alpha\big(u_1(x)\big)\big) = 0.$$

Since $\mathrm{u}_1(\mathrm{x})$ is a solution of Eq. (4), It reduces

$$u_1(x)\mathrm{T}_\alpha\mathrm{T}_\alpha\big(v(x)\big) + \mathrm{T}_\alpha\big(v(x)\big)\big(P(x)u_1(x) + 2\mathrm{T}_\alpha\big(u_1(x)\big)\big) = 0$$

or

$$\frac{\mathrm{T}_\alpha\mathrm{T}_\alpha\big(v(x)\big)}{\mathrm{T}_\alpha\big(v(x)\big)} = -2\frac{\mathrm{T}_\alpha\big(u_1(x)\big)}{u_1(x)} - P(x).$$

A fractional integration now gives

$$\ln(\mathrm{T}_\alpha(v(x))) = -2\ln(u_1(x)) - \mathrm{I}_\alpha(\mathrm{P}(\mathrm{x})),$$

so

$$\mathrm{T}_\alpha\big(v(x)\big) = \frac{1}{u_1^2(x)}e^{-\mathrm{I}_\alpha(P(x))}$$

and

$$v(x) = \mathrm{I}_\alpha\left(\frac{1}{u_1^2(x)}e^{-\mathrm{I}_\alpha(P(x))}\right).$$

Consequently, the general solution of homogeneous fractional equation of (4) is as follows (see [23]),

$$u_h(x) = C_1u_1(x) + C_2(\mathrm{I}_\alpha\left(\frac{1}{u_1^2(x)}e^{-\mathrm{I}_\alpha(P(x))}\right)u_1(x).$$

Example 4.1.1. We know that $\mathrm{u}_1(\mathrm{x}) = 3\sqrt[3]{\mathrm{x}}$ is a solution of the following homogeneous equation

$$9\sqrt[3]{x^2}\,\mathrm{T}_{\frac{2}{3}}\mathrm{T}_{\frac{2}{3}}\big(u(x)\big) - 6\sqrt[3]{x}\,\mathrm{T}_{\frac{2}{3}}\big(u(x)\big) + 2\,u(x) = 0.$$

According to (5), we have

$$v(x) = 3\sqrt[3]{x}\ .$$

Therefore

$$u_2(x) = 9\sqrt[3]{x^2}.$$

So the general solution is as follows,

$$u_h(x) = C_1\sqrt[3]{x} + C_2\sqrt[3]{x^2}.$$

Example 4.1.2. We know that $\mathrm{u}_1(\mathrm{x}) = \mathrm{x}$ is a solution of $2xT_{\frac{1}{2}}\mathrm{T}_{\frac{1}{2}}\big(u(x)\big) + \sqrt{x}\mathrm{T}_{\frac{1}{2}}\big(u(x)\big) - 2u(x) = 0.$

According to D'Alambert approach $v(x)$ and second solution $u_2(x)$ are obtained as follows,

$$v(x) = -\frac{x^{-2}}{2}\quad,\quad u_2(x) = -\frac{1}{2x}.$$

Therefor a general solution of above equation has the following form,

$$u_h(x) = C_1x + C_2x^{-1}.$$

4.2. The homogeneous fractional equation with constant coefficients

We are now in a position to give a complete discussion of the homogeneous equation of Eq. (4) for the special case in which p and q are constants (see [23]).

$$\mathrm{T}_\alpha\mathrm{T}_\alpha\big(u(x)\big) + p\,\mathrm{T}_\alpha\big(u(x)\big) + q\,u(x) = 0. \tag{6}$$

Our starting point is the fact that the exponential function $\mathrm{e}^{\mathrm{m}(\frac{1}{\alpha}\mathrm{x}^\alpha)}$, has the property that its conformable fractional derivative are all constant multiples of the function itself. It leads us to consider (see [19])

$$u(x) = e^{m(\frac{1}{\alpha}x^\alpha)} \tag{7}$$

as a possible solution for Eq. (6), we have

$$\mathrm{T}_\alpha\big(u(x)\big) = me^{m(\frac{1}{\alpha}x^\alpha)}, \tag{8}$$

and

$$\mathrm{T}_\alpha\mathrm{T}_\alpha\,\big(u(x)\big) = m^2e^{m(\frac{1}{\alpha}x^\alpha)}. \tag{9}$$

Substituting Eqs. (7), (8), and (9) into (6) yields to

$$(m^2 + p\,m + q)e^{m(\frac{1}{\alpha}x^\alpha)} = 0 \tag{10}$$

and since $\mathrm{e}^{\mathrm{m}(\frac{1}{\alpha}\mathrm{x}^\alpha)}$ is never zero, (7) holds if and only if m satisfies the following auxiliary equation (see [23]),

$$m^2 + p\,m + q = 0. \tag{11}$$

It is clear that the roots m_1 and m_2 of Eq. (11) are distinct real numbers if and only if $p^2 - 4q > 0$.

In this case, we get the two solutions

$u_1(x) = e^{m_1(\frac{1}{\alpha}x^\alpha)}$ and $u_2(x) = e^{m_2(\frac{1}{\alpha}x^\alpha)}$.

Since the ratio $\frac{\mathrm{e}^{\mathrm{m}_1(\frac{1}{\alpha}\mathrm{x}^\alpha)}}{\mathrm{e}^{\mathrm{m}_2(\frac{1}{\alpha}\mathrm{x}^\alpha)}} = \mathrm{e}^{(\mathrm{m}_1\text{-}\mathrm{m}_2)(\frac{1}{\alpha}\mathrm{x}^\alpha)}$ is not constant,

these solutions are linearly independent and

$$u_h(x) = C_1e^{m_1(\frac{1}{\alpha}x^\alpha)} + C_2e^{m_2(\frac{1}{\alpha}x^\alpha)},$$

is the general solution of Eq. (6).

If $m_1 = m_2$, then we obtain only one solution $u_1(x) = e^{m_1(\frac{1}{\alpha}x^\alpha)}$. Therefore, we can easily find a second linearly independent solution by the D'Alambert method as the following form

$$u_2(x) = (\frac{1}{\alpha}x^\alpha)e^{m_1(\frac{1}{\alpha}x^\alpha)}$$

and the general solution of Eq. (6) is

$$u_h(x) = \left(C_1 + C_2\left(\frac{1}{\alpha}x^\alpha\right)\right)e^{m_1\left(\frac{1}{\alpha}x^\alpha\right)}.$$

If the roots m_1 and m_2 are distinct complex numbers, then they can be written in the form $a \pm ib$ and our two real solutions of Eq. (6) are as follows

$$u_1(x) = e^{a\left(\frac{1}{\alpha}x^\alpha\right)}(\cos b(\frac{1}{\alpha}x^\alpha))$$

$$u_2(x) = e^{a\left(\frac{1}{\alpha}x^\alpha\right)}(\sin b(\frac{1}{\alpha}x^\alpha)).$$

So the solution of Eq. (6) will be obtained as the following

$$u_h(x) = e^{a\left(\frac{1}{\alpha}x^\alpha\right)}\Big(C_1\cos b(\frac{1}{\alpha}x^\alpha) + C_2\sin b(\frac{1}{\alpha}x^\alpha)\Big).$$

Example.4.2.1. Consider the following homogenous fractional differential equation

$$\mathrm{T}_\alpha\mathrm{T}_\alpha\big(u(x)\big) - 3\mathrm{T}_\alpha\big(u(x)\big) + u(x) = 0.$$

The general solution of the above equation is as follow

$$u_h(x) = C_1e^{(\frac{1}{\alpha}x^\alpha)} + C_2e^{2(\frac{1}{\alpha}x^\alpha)}.$$

Example.4.2.2. The general solution of homogeneous equation of

$$\mathrm{T}_\alpha\mathrm{T}_\alpha\big(u(x)\big) + 4\mathrm{T}_\alpha\big(u(x)\big) + 4u(x) = 0.$$

Will be obtained as follows

$$u_h(x) = \left(C_1 + C_2\left(\frac{1}{\alpha}x^\alpha\right)\right)e^{-2\left(\frac{1}{\alpha}x^\alpha\right)}.$$

Example.4.2.3. Let's consider following homogeneous fractional differential equation

$T_\alpha T_\alpha(u(x)) - 2T_\alpha(u(x)) + 3u(x) = 0.$

Using signature approach result in

$u_h(x) = e^{\left(\frac{1}{\alpha}x^\alpha\right)}\left(C_1 \cos\sqrt{2}(\frac{1}{\alpha}x^\alpha) + C_2 \sin\sqrt{2}(\frac{1}{\alpha}x^\alpha)\right).$

4.3. Euler's equidimensional fractional equation

The homogeneous fractional differential equation,

$$(\frac{1}{\alpha}x^\alpha)^2\, T_\alpha T_\alpha(u(x)) + p\left(\frac{1}{\alpha}x^\alpha\right)T_\alpha(u(x)) + qu(x) = 0, \quad x > 0 \tag{12}$$

where p, q are constant, is called Euler's fractional equation (see [23]). By using the change independent variable (see [19])

$$z = \ln(\frac{1}{\alpha}x^\alpha), \tag{13}$$

we have

$$T_\alpha(u(z)) = (\frac{1}{\alpha}x^\alpha)^{-1}\frac{du}{dz},$$

$$T_\alpha T_\alpha(u(z)) = -\left(\frac{1}{\alpha}x^\alpha\right)^{-2}\frac{du}{dz} + \left(\frac{1}{\alpha}x^\alpha\right)^{-2}\frac{d^2u}{dz^2}. \tag{14}$$

Substituting Eqs. (13) and (14) into Eq. (12), leads to

$$\frac{d^2u}{dz^2} + (p-1)\frac{du}{dz} + qu = 0. \tag{15}$$

That equation (15) is an ordinary differential equation with constant coefficient, and according to this approach the auxiliary equation has the following form

$$m^2 + (p-1)m + q = 0 \tag{16}$$

Suppose m_1 and m_2 are roots of Eq.'s (16) (see [23]). If they are distinct real numbers, then the following solution 0f (12) can be obtained,

$$u_h(x) = C_1(\frac{1}{\alpha}x^\alpha)^{m_1} + C_2(\frac{1}{\alpha}x^\alpha)^{m_2}.$$

If $m_1 = m_2$, we derive

$$u_h(x) = \left(C_1 + C_2\ln\left(\frac{1}{\alpha}x^\alpha\right)\right)(\frac{1}{\alpha}x^\alpha)^{m_1}.$$

And if m_1 and m_2 are distinct complex numbers then the general solution of Eq. (12) will be derive as follows

$$u_h(x) = (\frac{1}{\alpha}x^\alpha)^a[C_1\cos b\ln\left(\frac{1}{\alpha}x^\alpha\right) + C_2\sin b\ln\left(\frac{1}{\alpha}x^\alpha\right)].$$

Example.4.3.1. Let's consider the following homogeneous fractional differential equation

$$(\frac{1}{\alpha}x^\alpha)^2 T_\alpha T_\alpha(u(x)) - 2(\frac{1}{\alpha}x^\alpha)T_\alpha(u(x)) + 2u(x) = 0.$$

The auxiliary equation is as follows

$m^2 - 3m + 2 = 0,$

with the roots are $m_1 = 1$, and $m_2 = 2$.

So the general solution is

$u_h(x) = C_1(\frac{1}{\alpha}x^\alpha) + C_2(\frac{1}{\alpha}x^\alpha)^2.$

Example.4.3.2. consider following homogeneous equation

$(\frac{1}{\alpha}x^\alpha)^2 T_\alpha T_\alpha(u(x)) - 3(\frac{1}{\alpha}x^\alpha)T_\alpha(u(x)) + 4u(x) = 0.$

The root of auxiliary are

$m_1 = m_2 = 2.$

Thus the general solution is as follows,

$$u_h(x) = \left(C_1 + C_2\ln\left(\frac{1}{\alpha}x^\alpha\right)\right)(\frac{1}{\alpha}x^\alpha)^2.$$

Example.4.3.3. Let's consider Euler's fractional equation as follows

$(\frac{1}{\alpha}x^\alpha)^2 T_\alpha T_\alpha(u(x)) + 3(\frac{1}{\alpha}x^\alpha)T_\alpha(u(x)) + 2u(x) = 0.$

The root of auxiliary equation are

$m_1 = -1 + i, m_2 = -1 - i.$

Consequently, we have,

$$u_h(x) = (\frac{1}{\alpha}x^\alpha)^{-1}\left[C_1\cos(\ln\left(\frac{1}{\alpha}x^\alpha\right)) + C_2\sin(\ln\left(\frac{1}{\alpha}x^\alpha\right))\right].$$

5. Determining a particular solution of nonhomogeneous fractional equation

In this section, we have introduced variation of parameters and undetermined coefficients methods for determining a particular solution of nonhomogeneous fractional equations.

5.1. Variation of parameters or Lagrange approach

Assume that $u_1(x), u_2(x)$ are two linearly independent solution homogeneous fractional differential equation of the second order fractional differential equation (3), we suppose that the particular solution $u_p(x)$, is

$$u_p(x) = \vartheta_1(x)u_1(x) + \vartheta_2(x)u_2(x), \tag{19}$$

where $\vartheta_1(x), \vartheta_2(x)$ are two unknown functions (see [23]). By computing the conformable fractional derivative of (19), we derive

$$T_\alpha\left(u_p(x)\right) = (u_1 T_\alpha(\vartheta_1(x)) + u_2 T_\alpha(\vartheta_2(x))) + (\vartheta_1 T_\alpha(u_1(x)) + \vartheta_2 T_\alpha(u_2(x)). \tag{20}$$

To avoid using second conformable fractional derivative, we suppose that

$$u_1 T_\alpha(\vartheta_1(x)) + u_2 T_\alpha(\vartheta_2(x)) = 0. \tag{21}$$

So

$$T_\alpha\left(u_p(x)\right) = \vartheta_1 T_\alpha(u_1(x)) + \vartheta_2 T_\alpha(u_2(x), \tag{22}$$

Therefore,

$$T_\alpha T_\alpha\left(u_p(x)\right) = \vartheta_1 T_\alpha T_\alpha(u_1(x)) + T_\alpha(\vartheta_1(x))T_\alpha(u_1(x)) + T_\alpha(\vartheta_2(x))T_\alpha(u_2(x)) + \vartheta_2 T_\alpha T_\alpha(u_2(x)). \tag{23}$$

By substituting (19) , (22) and (23) into equation (3), and some manipulation, we get

$$\vartheta_1\left(T_\alpha T_\alpha(u_1(x)) + P(x)T_\alpha(u_1(x)) + Q(x)u_1(x)\right) + \vartheta_2(T_\alpha T_\alpha(u_2(x)) + P(x)T_\alpha(u_2(x)) + Q(x)u_2(x)) + T_\alpha(\vartheta_1(x))T_\alpha(u_1(x)) + T_\alpha(\vartheta_2(x))T_\alpha(u_2(x)) = R(x). \tag{24}$$

Since $u_1(x)$, *and* $u_2(x)$ are solutions of (4), the two expressions in parentheses are equal to zero, and (24) reduces to

$$T_\alpha(\vartheta_1(x))T_\alpha(u_1(x)) + T_\alpha(\vartheta_2(x))T_\alpha(u_2(x)) = R(x). \tag{25}$$

By considering (21) and (25) together, we obtain the following results

$$\vartheta_1(x) = I_\alpha\left(\frac{-u_2(x)R(x)}{W_\alpha(u_1(x), u_2(x))}\right),$$

$$\vartheta_2(x) = I_\alpha\left(\frac{u_1(x)R(x)}{W_\alpha(u_1(x), u_2(x))}\right)$$

so

$$u_p(x) = u_1(x).I_\alpha\left(\frac{-u_2(x)R(x)}{W_\alpha(u_1(x), u_2(x))}\right) + u_2(x).I_\alpha\left(\frac{u_1(x)R(x)}{W_\alpha(u_1(x), u_2(x))}\right).$$

Example.5.1.1. Let's consider the following fractional equation

$$9\sqrt[3]{x^2}T_{\frac{2}{3}}T_{\frac{2}{3}}(u(x)) - 6\sqrt[3]{x}\,T_{\frac{2}{3}}(u(x)) + 2\,u(x) = 9x^2\sqrt[3]{x^2}. \tag{26}$$

By using example.4.1.1. the homogeneous solutions of (26) are as follows,

$u_1(x) = 3\sqrt[3]{x}, u_2(x) = 9\sqrt[3]{x^2},$

and $u_p(x) = -\frac{39}{56}x^3$ a particular solution of (26).

So the equation (26) has a general solution such as

$u(x) = C_1\sqrt[3]{x} + C_2\sqrt[3]{x^2} - \frac{39}{56}x^3.$

Example.5.1.2. we want to find the general solution of

$2x\, T_{\frac{1}{2}}T_{\frac{1}{2}}(u(x)) + \sqrt{x}T_{\frac{1}{2}}(u(x)) - 2u(x) = 4x^3.$

By example.4.1.2. the homogeneous solutions are

$u_1(x) = x,\ and\ u_2(x) = -\frac{1}{2x}$, and $u_p(x) = \frac{1}{4}x^3$, Is a particular solution of it. So the general solution of equation can be presented as the following form
$u(x) = C_1 x + C_2 x^{-1} + 0.25x^3$.

5.2. Undetermined coefficients

Undetermined coefficients is a procedure for finding $u_p(x)$ when (3) has the form

$$\mathrm{T}_\alpha \mathrm{T}_\alpha\big(u(x)\big) + p\ \mathrm{T}_\alpha\big(u(x)\big) + q\ u(x) = R(x), \tag{27}$$

where $p\ , q$ are constant and

$$R(x) = \left(a_0 + a_1\left(\frac{1}{\alpha}x^\alpha\right) + a_2\left(\frac{1}{\alpha}x^\alpha\right)^2 + \cdots + a_n\left(\frac{1}{\alpha}x^\alpha\right)^n\right) e^{\beta\left(\frac{1}{\alpha}x^\alpha\right)} \sin\gamma\left(\frac{1}{\alpha}x^\alpha\right)$$

or

$$\mathrm{R(x)} = \left(a_0 + a_1\left(\frac{1}{\alpha}x^\alpha\right) + a_2\left(\frac{1}{\alpha}x^\alpha\right)^2 + \cdots + a_n\left(\frac{1}{\alpha}x^\alpha\right)^n\right) e^{\beta\left(\frac{1}{\alpha}x^\alpha\right)} \cos\gamma\left(\frac{1}{\alpha}x^\alpha\right).$$

We choose a particular solution in the following form

$$u_p(x) = [\ \left(A_0 + A_1\left(\frac{1}{\alpha}x^\alpha\right) + A_2\left(\frac{1}{\alpha}x^\alpha\right)^2 + \cdots + A_n\left(\frac{1}{\alpha}x^\alpha\right)^n\right) e^{\beta\left(\frac{1}{\alpha}x^\alpha\right)} \sin\gamma\left(\frac{1}{\alpha}x^\alpha\right) + \left(B_0 + B_1\left(\frac{1}{\alpha}x^\alpha\right) + B_2\left(\frac{1}{\alpha}x^\alpha\right)^2 + \cdots + B_n\left(\frac{1}{\alpha}x^\alpha\right)^n\right) e^{\beta\left(\frac{1}{\alpha}x^\alpha\right)} \sin\gamma\left(\frac{1}{\alpha}x^\alpha\right)]\ \left(\frac{1}{\alpha}x^\alpha\right)^m \tag{28}$$

that $A_0, A_1, \dots, A_n, B_0, B_1, \dots, B_n$, unknown coefficients and m is the lowest non-negative integer number, that removes homogeneous solutions, in choosing $u_p(x)$. By substituting (28) into (27) unknown coefficients will be obtained.

Example.5.2.1. let's consider the following nonhomogeneous fractional equation

$$\mathrm{T}_{\frac{2}{3}}\mathrm{T}_{\frac{2}{3}}\big(u(x)\big) - 2\mathrm{T}_{\frac{2}{3}}\big(u(x)\big) = 18\sqrt[3]{x^2} - 10\ . \tag{29}$$

Homogenous solutions of (29) are as follows
$u_1(x) = 1\ , u_2(x) = e^{\frac{4}{3}\sqrt[3]{x^2}}$,
And a particular solution of it has the following form,
$u_p(x) = 3\sqrt[3]{x^2} - \frac{27}{4}\left(\sqrt[3]{x^2}\right)^2$.
So the general solution of (29) is
$u(x) = C_1 + C_2 e^{\frac{4}{3}\sqrt[3]{x^2}} + 3\sqrt[3]{x^2} - \frac{27}{4}\left(\sqrt[3]{x^2}\right)^2$.

Example.5.2.2. The general solution of
$\mathrm{T}_{\frac{1}{2}}\mathrm{T}_{\frac{1}{2}}\big(u(x)\big) - 2\mathrm{T}_{\frac{1}{2}}\big(u(x)\big) + u(x) = 2\sqrt{x}\, e^{2\sqrt{x}}$,
can be presented as the following form
$u(x) = C_1 e^{2\sqrt{x}} + C_2\sqrt{x}\, e^{2\sqrt{x}} + \frac{4}{3}\ x\sqrt{x}\, e^{2\sqrt{x}}$.

Example.5.2.3. Consider nonhomogeneous fractional equation as follows
$\mathrm{T}_{\frac{1}{2}}\mathrm{T}_{\frac{1}{2}}\big(u(x)\big) + 4\ u(x) = 4\cos 4\sqrt{x} + 32x - 8\sqrt{x}$,
This equation has a general solution such as,
$u(x) = C_1\cos 4\sqrt{x} + C_2\sin 4\sqrt{x} + 2\sqrt{x}\sin 4\sqrt{x} + 8x - 2\sqrt{x} - 2$.

6. Conclusion

In this article, some methods such as, the use of a known solution to find another one, homogenous equation with constant coefficients and Euler's equidimensional equation have been introduced for determining a general solution of homogenous fractional equations. The variation of parameters or Lagrange method, undetermined coefficients approach, for specifying a particular solution of second order linear fractional differential equations, are presented. This approach leads to an exact solution, thus there is no need of using any numerical approach.

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