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Computational Model for Cardinality Bounded Multiset Space

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Abstract

In [1], manual generation of the elements of multiset space and mathematical models for computing the corresponding frequency numbers were presented. These approaches are quite tedious and amenable to mistakes. In this paper, we develop and implement an efficient algorithm to generate the elements of a cardinality bounded multiset space $X^n(m)$ for X and the frequency tables for various values of n and m. Similarly, graphical representations of the respective output tables are plotted to depict the behavioural patterns of $X^n(m)$ for some finite values of n and $m > 3$.

Keywords: *Multiset, multiset space, frequency number, cardinality, algorithm.*

1 The Concept of a Multiset

A multiset (mset for short) is a collection of objects in which those objects have multiple occurrences. A finite mset over a set X is an mset M formed with finitely many elements from X such that each element has a finite multiplicity of occurrence in M. Also see [2], [3], and [4] for more details.

A multiset can be represented in several ways. The use of square brackets to represent a multiset is quasi-general. Thus, a multiset containing one occurrence of a, two occurrences of b, and three occurrences of c is notationally written as $[[a, b, b, c, c, c]]$ or $[a, b, b, c, c, c]$ or $[\mathrm{a, b, c}]_{1,2,3}$ or $[a, 2b, 3c]$ or $[a.1, b.2, c.3]$ or $[1/a, 2/b, 3/c]$ or $[a^1, b^2, c^3]$ or $[a^1b^2c^3]$. For convenience, the curly brackets are used in place of the square brackets. In fact, the last form of representation as a string, even without using any brackets, turns out to be the most compact one, especially in computational parlance. The following schematic representation of a multiset as a numeric valued or count function abounds, particularly in the foundational development of multiset theory and its application [5][6][7].

This paper is organized as follows. In section 2 we collect preliminaries and basic definitions based on msets space and some related notions. In sections 3, a mathematical model for computing the frequency number of multiset space is presented. In section 4, we extend mset theoretic results obtained in [1] for generating $X^n(m)$ values for $m \leq 3$ to $m > 3$, by means of computational model.

2 Multiset Space and Some Related Notions

Let X be a finite set with cardinality m, usually called an m-mset. Let $X^n(m)$ denote the set of all multisets each having m objects occurring with multiplicities at most n times, including 0. We call $X^n(m)$, a cardinality bounded multiset space for X [1].

Let $X = \{ a_1, a_2, a_3, \ldots, a_m\}$ be an ordered m-set and let $X^n(m)$ be a cardinality bounded space. Let an arbitrary element of $X^n(m)$ be denoted by $X_{<p_i>}$ where $< p_i >$ is an ordered m-tuple, p_i is the multiplicity of i^{th} object in $X, 1 \leq i \leq m$ and $0 \leq pi \leq n$. For convenience $X_{<pi>}$ will be denoted by X_{pi}. Following the aforementioned notation, the term X_p of $X^n(m)$ would mean that all its objects have the same multiplicity p, that is X_p is a regular mset. Also, X_0 will refer to the empty multiset or the origin of $X^n(m)$. It is easy to see that $X^n(m)$ will have $(n+1)^m$ elements where n and m are positive integers.

2.1 Construction of $X^n(m)$ and their Patterns

Let $X = \{ a, b \}$, that is, $m = 2$. Then,

$X^0(2) = \{[a,b]_{0\,0}\}$

$X^1(2) = \{[a,b]_{0\,0}, [a,b]_{0\,1}, [a,b]_{1\,0}, [a,b]_{1\,1} \}$

$X^2(2) = \{[a,b]_{0\,0}, [a,b]_{0\,1}, [a,b]_{1\,0}, [a,b]_{1\,1}, [a,b]_{2\,0}, [a,b]_{0\,2}, [a,b]_{2\,1}, [a,b]_{1\,2}, [a,b]_{2\,2} \}$

$X^3(2) = \{[a,b]_{0\,0}, [a,b]_{0\,1}, [a,b]_{1\,0}, [a,b]_{1\,1}, [a,b]_{1\,2}, [a,b]_{2\,1}, [a,b]_{0\,2}, [a,b]_{2\,0}, [a,b]_{2\,2}, [a,b]_{3\,0}, [a,b]_{0\,3}, [a,b]_{1\,3}, [a,b]_{3\,1}, [a,b]_{3\,2}, [a,b]_{2\,3}, [a,b]_{3\,3} \}$

$X^4(2) = \{[a,b]_{0\,0}, [a,b]_{0\,1}, [a,b]_{1\,0}, [a,b]_{1\,1}, [a,b]_{1\,2}, [a,b]_{2\,1}, [a,b]_{0\,2}, [a,b]_{2\,0}, [a,b]_{2\,2}, [a,b]_{3\,0}, [a,b]_{0\,3}, [a,b]_{1\,3}, [a,b]_{3\,1}, [a,b]_{3\,2}, [a,b]_{2\,3}, [a,b]_{3\,3}, [a,b]_{4\,0}, [a,b]_{0\,4}, [a,b]_{1\,4}, [a,b]_{2\,4}, [a,b]_{4\,2}, [a,b]_{4\,3}, [a,b]_{3\,4}, [a,b]_{4\,4}, [a,b]_{3\,3}\}$

$X^5(2) = \{[a,b]_{0\,0}, [a,b]_{0\,1}, [a,b]_{0\,2}, [a,b]_{0\,3}, [a,b]_{0\,4}, [a,b]_{1\,0}, [a,b]_{1\,1}, [a,b]_{1\,2}, [a,b]_{1\,3}, [a,b]_{1\,4}, [a,b]_{2\,0}, [a,b]_{2\,1}, [a,b]_{2\,2}, [a,b]_{2\,3}, [a,b]_{2\,4}, [a,b]_{3\,0}, [a,b]_{3\,1}, [a,b]_{3\,2}, [a,b]_{3\,3}, [a,b]_{3\,4}, [a,b]_{4\,0}, [a,b]_{4\,1}, [a,b]_{4\,2}, [a,b]_{4\,3}, [a,b]_{4\,4}, [a,b]_{5\,0}, [a,b]_{0\,5}, [a,b]_{5\,1}, [a,b]_{1\,5}, [a,b]_{2\,5}, [a,b]_{5\,2}, [a,b]_{3\,5}, [a,b]_{5\,3}, [a,b]_{5\,4}, [a,b]_{4\,5}, [a,b]_{5\,5},\}$

Similarly, $X^n(2)$ are obtained for $n = 6, 7, 8, 9, \ldots$
We take into account the cardinality of the elements of $X^n(m)$ to generate a pattern with reference to the frequency of their occurrences. Table I, and Table II provide a schematic representation of the said patterns for $X^n(2)$ and $X^n(3)$ for some finite values of n as in [1]. Similarly, we extended this idea up to $m = 4, 5, 6$, and 7, that is, covering a cardinality bounded multiset space of $X^n(4)$, $X^n(5)$, $X^n(6)$, and $X^n(7)$ respectively, for some finite values of n.

Table I: Frequency value corresponding to the cardinality of the elements of $X^n(2)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$X^0(2)$	1																		
$X^1(2)$	1	2	1																
$X^2(2)$	1	2	3	2	1														

$X^3(2)$	1	2	3	4	3	2	1												
$X^4(2)$	1	2	3	4	5	4	3	2	1										
$X^5(2)$	1	2	3	4	5	6	5	4	3	2	1								
$X^6(2)$	1	2	3	4	5	6	7	6	5	4	3	2	1						
$X^7(2)$	1	2	3	4	5	6	7	8	7	6	5	4	3	2	1				
$X^8(2)$	1	2	3	4	5	6	7	8	9	8	7	6	5	4	3	2	1		
$X^9(2)$	1	2	3	4	5	6	7	8	9	10	9	8	7	6	5	4	3	2	1

In table I, the first row $0, 1, 2, 3, \ldots, 18$ represents the cardinality of the elements in $X^n(2)$ while the remaining rows represent the corresponding frequency of their occurrence. The set $X^0(2)$ contains only one element with cardinality zero, the set $X^1(2)$ contains three elements; one element with cardinality zero, two elements with cardinality one and one element with cardinality two, and so on for $n = 2, 3, \ldots, 9$.

It can easily be seen from the above that the number of elements in $X^n(m)$ is the same as the sum of the frequencies of the objects. The highest cardinality of an element $X_{pi} \in X^n(m)$ is mn which is unique while the smallest is zero. The highest frequency of an element in $X^n(m)$ is $(n + 1)$ and the corresponding cardinality is n.

Likewise, by adopting the same rigorous method of manually computing the values of $X^n(m)$ for cardinality $m = 2$, we can also generate $X^n(m)$, for cardinality $m = 3$ as shown below:

Let $m = 3$, then we compute $X^n(m)$ for $n = 0, 1, 2, \ldots, 9$.

Let $X = \{a, b, c\}$, that is, $m = 3$. Then,

$X^0(3) = \{[a, b, c]_{0\,0\,0}\}$

$X^1(3) = \{[a, b, c]_{0\,0\,0}, [a, b, c]_{0\,0\,1}, [a, b, c]_{0\,1\,0}, [a, b, c]_{0\,1\,1}\ \{[a, b, c]_{1\,0\,0}, [a, b, c]_{1\,0\,1}, [a, b, c]_{1\,1\,0}, [a, b, c]_{1\,1\,1}\}$

$X^2(3) = \{[a, b, c]_{0\,0\,0}, [a, b, c]_{0\,0\,1}, [a, b, c]_{1\,0}, [a, b, c]_{0\,1\,0}, [a, b, c]_{0\,1\,1}, [a, b, c]_{01\,1\,1}, [a, b, c]_{1\,0\,1}, [a, b, c]_{1\,1\,0}, [a, b, c]_{1\,1\,1}, [a, b, c]_{0\,0\,2}, [a, b, c]_{0\,1\,2}, [a, b]_{0\,2\,0}, [a, b]_{0\,2\,1}, [a, b, c]_{0\,2\,2}, [a, b, c]_{1\,2\,0}, [a, b, c]_{2\,0\,0}, [a, b]_{2\,1\,0}, [a, b]_{2\,0\,1}, [a, b, c]_{1\,0\,2}, [a, b, c]_{1\,2\,1}, [a, b, c]_{1\,1\,2}, [a, b, c]_{1\,2\,2}, [a, b, c]_{2\,0\,0}, [a, b]_{2\,1\,0}, [a, b]_{2\,0\,1}, [a, b, c]_{2\,1\,1}, [a, b, c]_{2\,0\,2}, [a, b, c]_{2\,2\,0}, [a, b, c]_{2\,1\,2}, [a, b, c]_{2\,2\,1}, [a, b]_{2\,1\,0}, [a, b]_{2\,2\,2},\}$

Table II: Frequency value corresponding to the cardinality of the elements of $X^n(3)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
$X^0(3)$	1																											
$X^1(3)$	1	3	3	1																								
$X^2(3)$	1	3	6	7	6	3	1																					
$X^3(3)$	1	3	6	10	12	12	10	6	3	1																		
$X^4(3)$	1	3	6	10	15	18	19	18	15	10	6	3	1															
$X^5(3)$	1	3	6	10	15	21	25	27	27	25	21	15	10	6	3	1												

$X^6(3)$	1	3	6	10	15	21	28	33	36	37	36	33	28	21	15	10	6	3	1									
$X^7(3)$	1	3	6	10	15	21	28	36	42	46	48	48	46	42	36	28	21	15	10	6	3	1						
$X^8(3)$	1	3	6	10	15	21	28	36	45	52	57	60	61	60	57	52	45	36	28	21	15	10	6	3	1			
$X^9(3)$	1	3	6	10	15	21	28	36	45	55	63	69	73	75	75	73	69	63	55	45	36	28	21	15	10	6	3	1

In table II, The set $X^0(3)$ contains only one element with cardinality zero, just as in table I, while the set $X^1(3)$ contains four elements; one element with cardinality zero, three elements with cardinality one, three element with cardinality two and one element with cardinality three and so on.
Considering the tediousness encountered by ways of manually computing the various values of $X^n(n)$ with respect to the value of $m = 2$ and 3. A computer program was written based on the implementation of the algorithms presented in section 5. It was observed that the program was able to compute for $X^n(m)$ with respect to the varying values of n and m within an estimated time frame.

2.2. Construction of $X^n(m)$ with Varying Values of m

When $m > 3$ the manual computation of $X^n(m)$ becomes tedious and unfriendly. In this paper, we develop an algorithm with subsequent program to compute $X^n(m)$ and generate the frequency tables. The extended steps taken can be described as follow:
Let $X = \{a, b, c, d\}$, and $m = 4$, then $X^n(4)$ is computed for $n = 0, 1, 2, .., 9$. with cardinality values ranging from $0, 1, 2, \ldots, 36$. Also for $X = \{a, b, c, d, e\}$, and $m = 5$, $X^n(5)$ is likewise computed for $n = 0, 1, 2, .., 9$. with cardinality values ranging from $0, 1, 2, 3, \ldots 40$, and so on for $X = \{a, b, c, d, e, f\}$ and $X = \{a, b, c, d, e, f\}$, The output generated from the computer program is as shown in tables III, IV, V, and VI below.

2.3. Mathematical Models for Computing the Frequency Number of *m* set Space

In [1], equations (1) and (2) were presented as a mathematical model for computing the frequency number of mset space where it was noted that there is no generalization model for computing $fX^n(m)$. However, we adopted the same mathematical concept to design a computational model for obtaining the frequency number for $X^n(m)$, denoted by $fX^n(m)$ using the same recurrence formula.

Let consider some few case of how the recurrence formula can be used to generate the frequency numbers $fX^n(m)$ for when $m = 2$ and $m = 3$. A good instance of this is achieved by considering the frequency number $fX^n(2)$, and $fX^n(3)$, now by using the recurrence formula we have

$$fX^n(2) = \begin{cases} \{k+1\}, \; 0 \le k \le n \\ 2n - k + 1, \; n + 1 \le k \le 2n \end{cases} \tag{1}$$

Similarly, for m=3, the frequency number of $X^n(3)$, which s denoted by $fX^n(3)$, is given by

$$fX^n(3) = \begin{cases} \sum_{i=1}^{k}(i+1), & 0 \le k \le n \\ \sum_{i=1}^{n}(i+1) + \sum_{i=n+1}^{k}(3n-2i+1), & n+1 \le k \le 2n \\ \sum_{i=1}^{n}(i+1) + \sum_{i=n+1}^{2n}(3n-2i) + \sum_{i=2n+1}^{k}(i-3n-2), & 2n+1 \le k \le 3n \end{cases} \quad (2)$$

3 Computational Model for Generating the Frequency Number of a Multiset Space

Algorithm 1 Cardinality computation

```
input: multiplicity n, cardinality m
function generateCardinality(y,z) begin
n: multiplicity of occurrence of an object m with incremental value of +1;
y: incremental value of n;
z: function value; X^n(m);
cardinailty_array = array();          //Array keeping cardinality of values
    for 0 ≤ a < z do
      b ← merge(b, y)   // concatenate the value of b and y
      a ← a +1
    end for
        n ← y + 1
        i ← base_convert(0, n, 10)
        j ← base_convert(b, n, 10)
        k ← 0;
        while i ≤ j do
            res ← base_convert(i, 10, n)
            cardinailty_array() ← res
            i ← i +1
        end while
//Returns an array containing cardinality of values
    return cardinailty_array()
end function generateCardinality
```

Algorithms 2 computation of cardinality summation

```
function getCardinalitySummation(cardinailty_array()) begin
    for each cardinailty_array() as val do
//Tokenize digits into array
    b ← array() //Initialize array
    c ← 0 //Initialize summation

        for 0 ≤ a < strlen(val) do
            b[] ← intval(substr(val, a, 1))
```

```
            a ← a + 1
        end for
//Sums array values

            for each b as new_val do
                c ← c + new_val
            end for

//Pushes sum into new array
            summ_array[] ← c
        end for
    return summ_array()
end function getCardinalitySummation
```

Algorithm3 Frequency number of a multiset space generation

```
m: cardinality value
    for 1 ≤ g ≤ 8 do
        for 0 ≤ i < cell_length do
            occurence_counter ← " "
            getCardinalitySummation_arr ← array()
            getCardinalitySummation_arr ←
                                    getCardinalitySummation(generateCardinality(g, m))
            for each getCardinalitySummation_arr as loop_val do
                if  loop_val == i do
                        occurence_counter ← occurence_counter + 1;
                        else
                            occurence_counter ← " "
                    end if
                            i ← i + 1
        end for
                            g ← g + 1
    end for
```

This algorithm has been implemented into a corresponding applet program named mset.java, its takes two arguments, n and m. As an example of it use, we computed some instances of $X = \{a,b\}$, $X = \{a,b,c\}$, $X = \{a,b,c,d\}$, $X = \{a,b,c,d,e\}$, $X = \{a,b,c,d,e,f\}$, that is $m = 2,3,\ldots,6$ for $n = 0,2,3,\ldots,9$.

4 Conclusion and Further Direction

This study has been carried out as a result of the promising application interests eminent with the introduction of the cardinality bounded multiset space. There are major fields in computer science, for which this area is considered possible areas of application, such as data mining, search optimization techniques and conceptual proving of some program termination. Some other possible areas of application of mset theory and concepts as a whole has been studied in detail and presented in [8][9]. Most importantly, we have been able to use a computer program to generate $X^n(m)$ for varying values of n and m. However, there has been no generalization case of mathematical model covering the computation of cardinality bounded mset space. We therefore present this issue as an open research topic that deserve full attention.

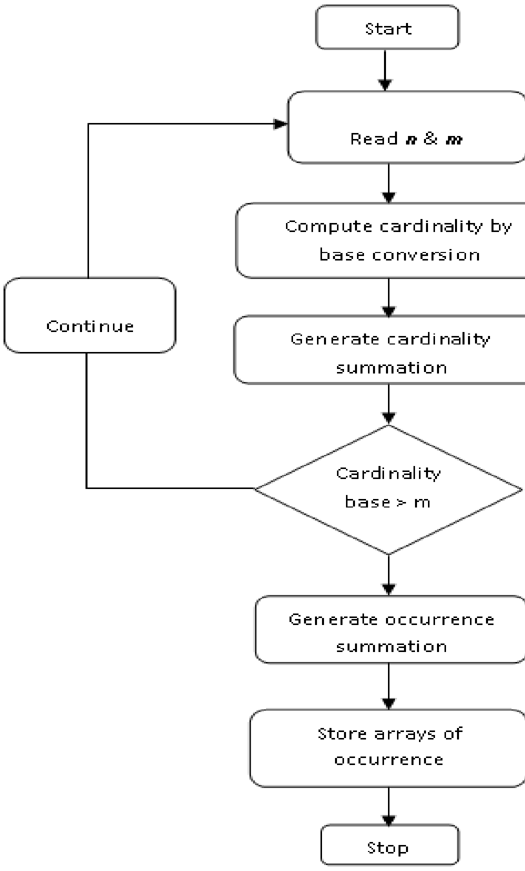

Figure 1: Program flow for the $X^n(m)$ computation

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Table III: Frequency value corresponding to the cardinality of the elements of $X^n(4)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
$X^0(4)$	1																																				
$X^1(4)$	1	4	6	4	1																																
$X^2(4)$	1	4	10	16	19	16	10	4	1																												
$X^3(4)$	1	4	10	20	31	40	44	40	31	20	10	4	1																								
$X^4(4)$	1	4	10	20	35	52	68	80	85	80	68	52	35	20	10	4	1																				
$X^5(4)$	1	4	10	20	35	56	80	104	125	140	146	140	125	104	80	56	35	20	10	4	1																
$X^6(4)$	1	4	10	20	35	56	84	116	149	180	206	224	231	224	206	180	149	116	84	56	35	20	10	4	1												
$X^7(4)$	1	4	10	20	35	56	84	120	161	204	246	284	315	336	344	336	315	284	246	204	161	120	84	56	35	20	10	4	1								
$X^8(4)$	1	4	10	20	35	56	84	120	165	216	270	324	375	420	456	480	489	480	456	420	375	324	270	216	165	120	84	56	35	20	10	4	1				
$X^9(4)$	1	4	10	20	35	56	84	120	165	220	282	348	415	480	540	592	633	660	670	660	633	592	540	480	415	348	282	220	165	120	84	56	35	20	10	4	1

Table IV: Frequency value corresponding to the cardinality of the elements of $X^n(5)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
$X^0(5)$	1																																								
$X^1(5)$	1	5	10	10	5	1																																			
$X^2(5)$	1	5	15	30	45	51	45	30	15	5	1																														
$X^3(5)$	1	5	15	35	65	101	135	155	155	135	101	65	35	15	5	1																									
$X^4(5)$	1	5	15	35	70	121	185	255	320	365	381	365	320	255	185	121	70	35	15	5	1																				
$X^5(5)$	1	5	15	35	70	126	205	305	420	540	651	735	780	780	735	651	540	420	305	205	126	70	35	15	5	1															
$X^6(5)$	1	5	15	35	70	126	210	325	470	640	826	1015	1190	1330	1420	1451	1420	1330	1190	1015	826	640	470	325	210	126	70	35	15	5	1										
$X^7(5)$	1	5	15	35	70	126	210	330	490	690	926	1190	1470	1750	2010	2226	2380	2460	2460	2380	2226	2010	1750	1470	1190	926	690	490	330	210	126	70	35	15	5	1					
$X^8(5)$	1	5	15	35	70	126	210	330	495	710	976	1290	1645	2030	2430	2826	3195	3510	3750	3900	3951	3900	3750	3510	3195	2826	2430	2030	1645	1290	976	710	495	330	210	126	70	35	15	5	1

Table V: Frequency value corresponding to the cardinality of the elements of $X^n(6)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$X^0(6)$	1																					
$X^1(6)$	1	6	15	20	15	6	1															
$X^2(6)$	1	6	21	50	90	126	141	126	90	50	21	6	1									
$X^3(6)$	1	6	21	56	120	216	336	456	546	580	546	456	336	216	120	56	21	6	1			
$X^4(6)$	1	6	21	56	126	246	426	666	951	1246	1506	1686	1751	1686	1506	1246	951	666	426	246	126	56
$X^5(6)$	1	6	21	56	126	252	456	756	1161	1666	2247	2856	3431	3906	4221	4332	4221	3906	3431	2856	2247	1666
$X^6(6)$	1	6	21	56	126	252	462	786	1251	1876	2667	3612	4676	5796	6891	7872	8652	9156	9331	9156	8652	7872
$X^7(6)$	1	6	21	56	126	252	462	792	1281	1966	2877	4032	5432	7056	8856	10752	12642	14412	15946	17136	17892	18152

	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
$X^0(6)$																					
$X^1(6)$																					
$X^2(6)$																					
$X^3(6)$																					
$X^4(6)$	21	6	1																		
$X^5(6)$	1161	756	456	252	126	56	21	6	1												
$X^6(6)$	6891	5796	4676	3612	2667	1876	1251	786	462	252	126	56	21	6	1						
$X^7(6)$	17892	17136	15946	14412	12642	10752	8856	7056	5432	4032	2877	1966	1281	792	462	252	126	56	21	6	1

Table VI: Frequency value corresponding to the cardinality of the elements of $X^n(7)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
$X^0(7)$	1																												
$X^1(7)$	1	7	21	35	35	21	7	1																					
$X^2(7)$	1	7	28	77	161	266	357	393	357	266	161	77	28	7	1														
$X^3(7)$	1	7	28	84	203	413	728	1128	1554	1918	2128	2128	1918	1554	1128	728	413	203	84	28	7	1							
$X^4(7)$	1	7	28	84	210	455	875	1520	2415	3535	4795	6055	7140	7875	8135	7875	7140	6055	4795	3535	2415	1520	875	455	210	84	28	7	1

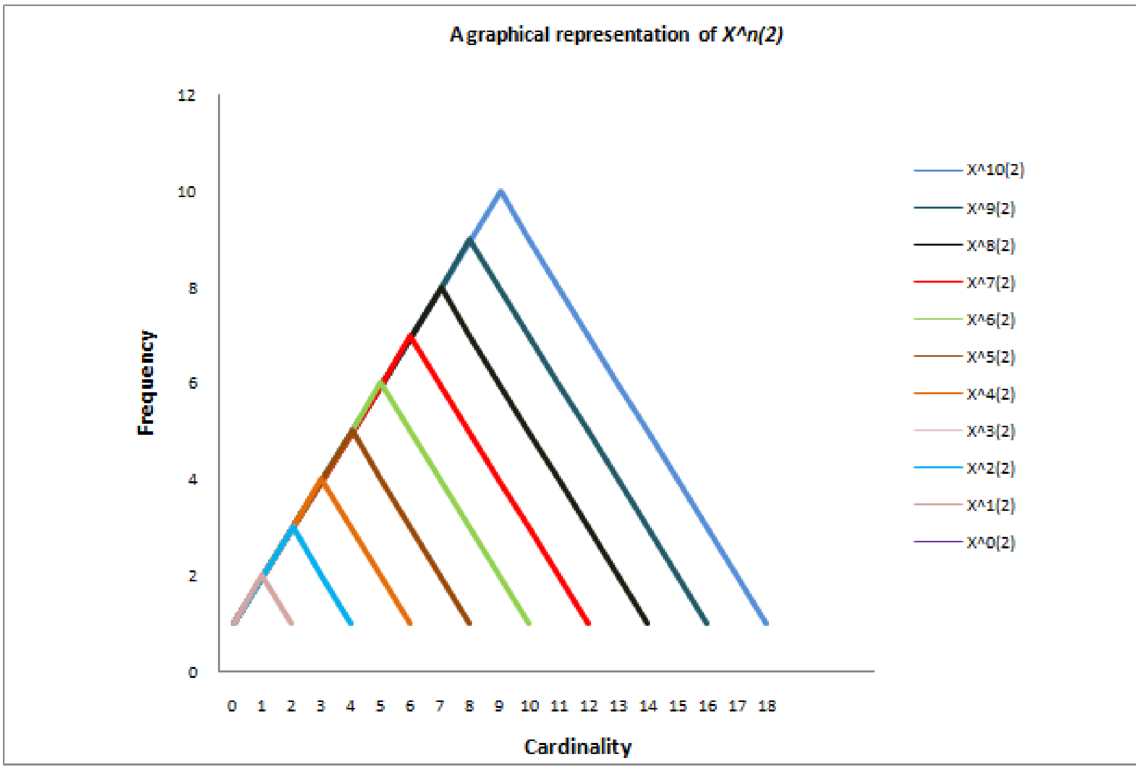

Figure 1:

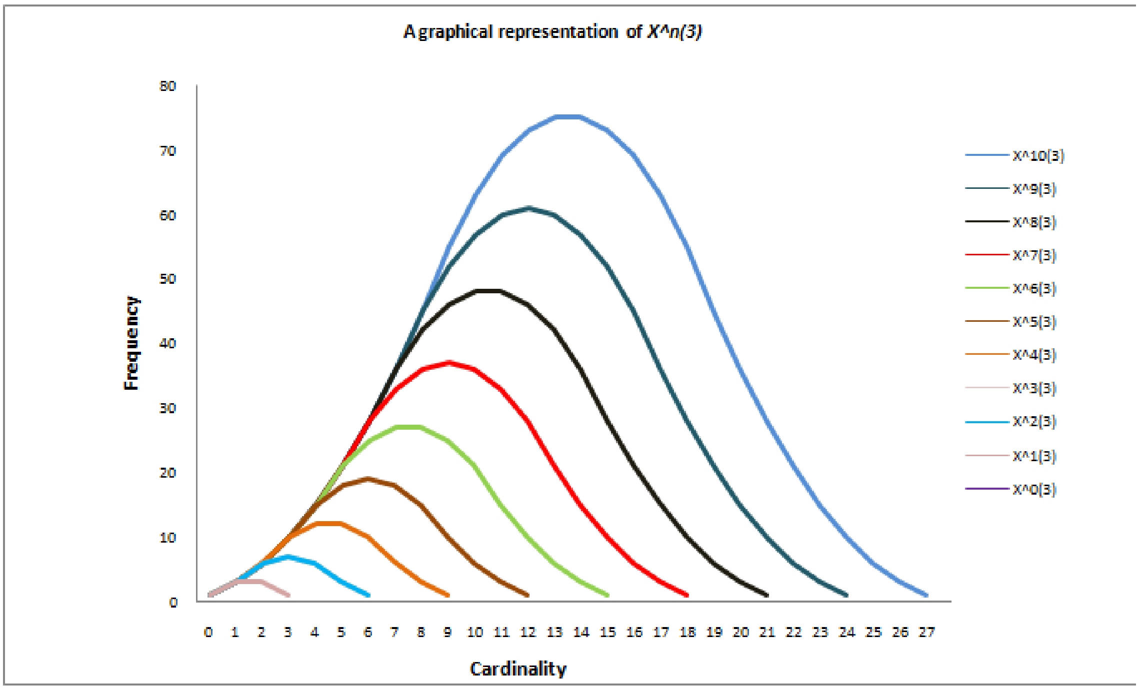

Figure 2:

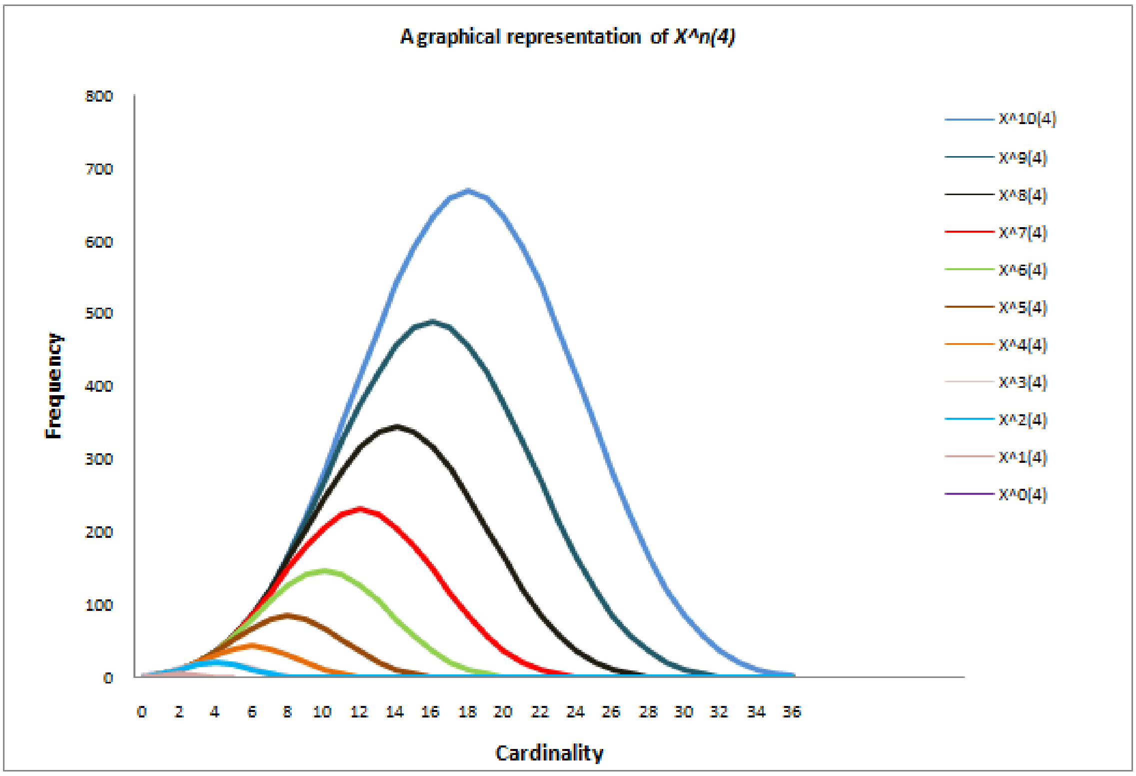

Figure3:

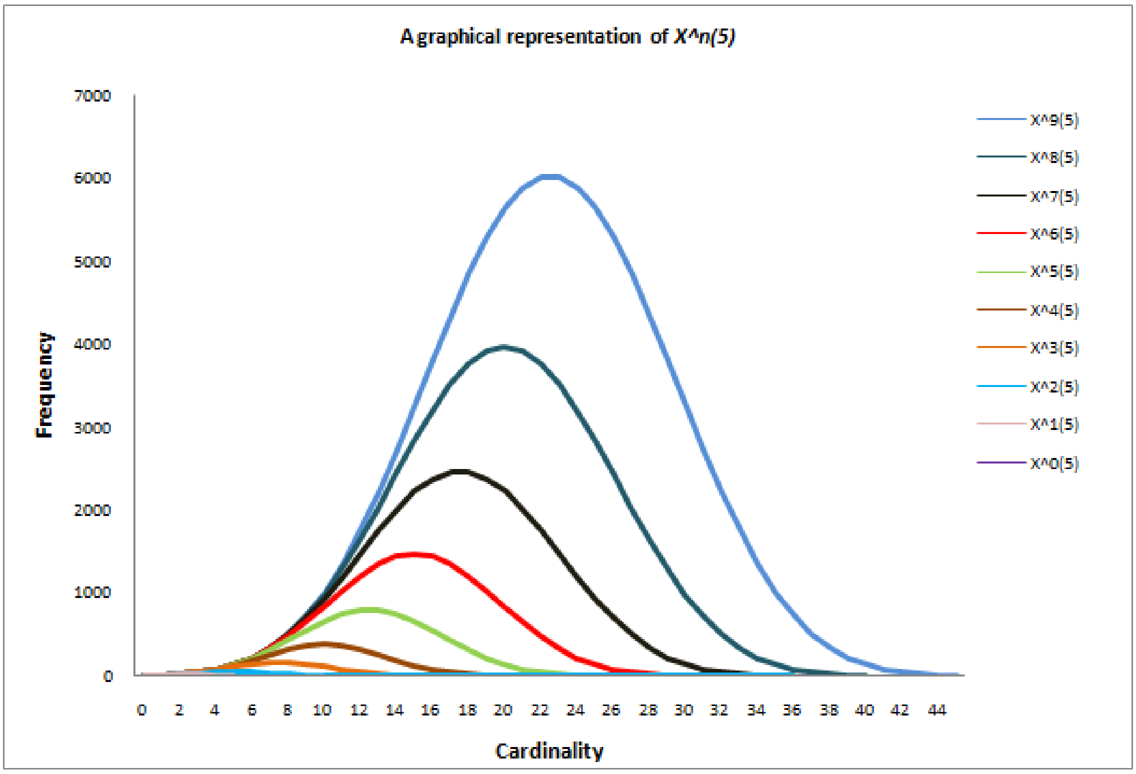

Figure 4:

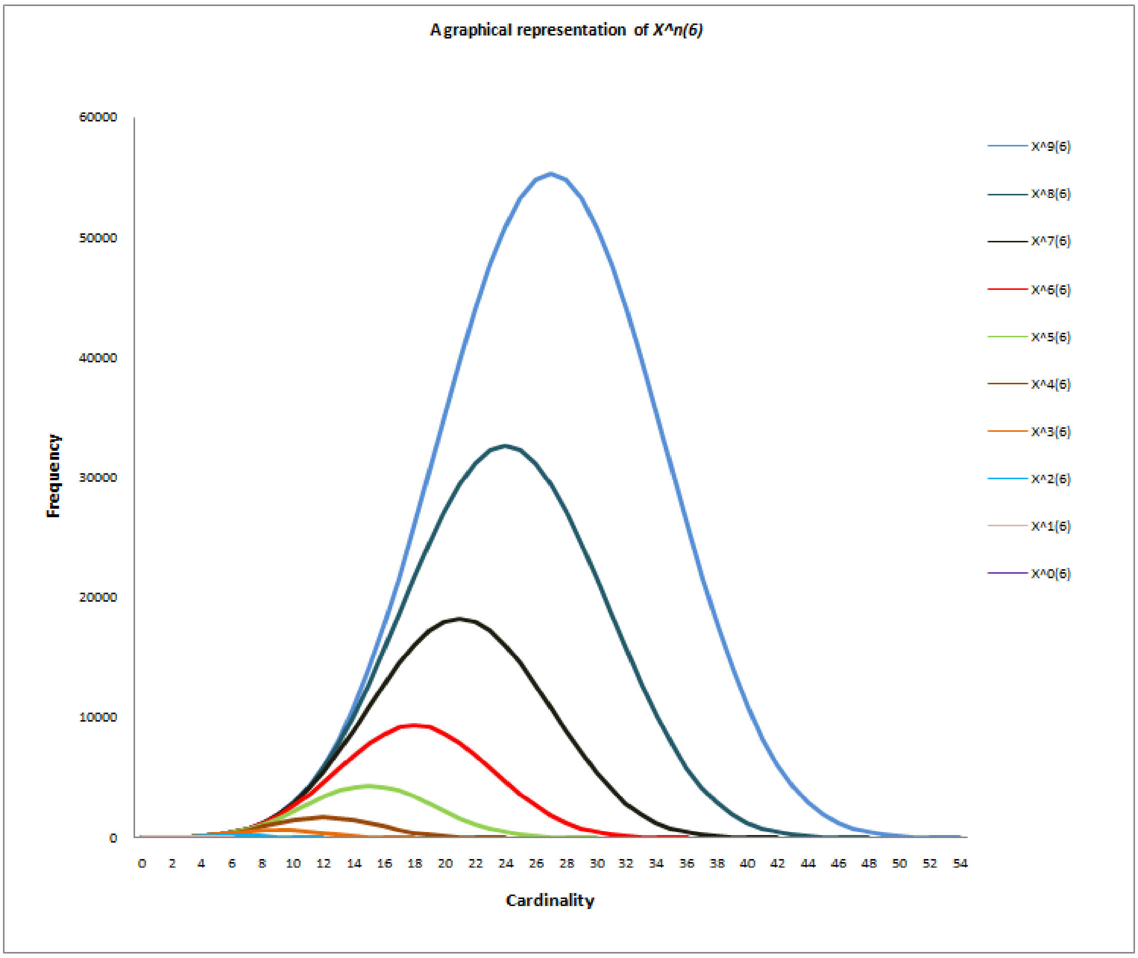

Figure 5:

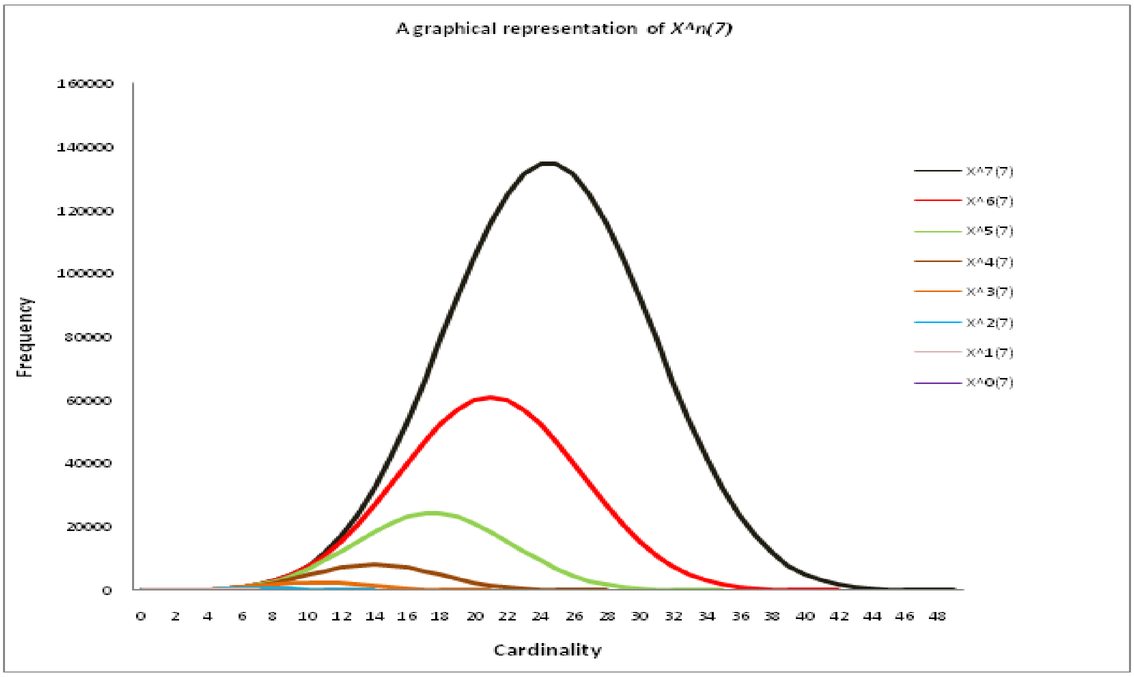

Figure 6: