*International Journal of Applied Mathematical Research*, 3 (1) (2014) 58-61
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www.sciencepubco.com/index.php/IJAMR
doi: 10.14419/ijamr.v3i1.1701
Research Paper

# On the integer solutions of the Pell equation $x^2 = 13y^2 - 3^t$

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**Abstract**

The binary quadratic Diophantine equation represented by $x^2 = 13y^2 - 3^t$ , $t > 0$ is considered and analyzed for its non-zero distinct integer solutions for the choices of $t$ given by (i) $t = 1$ (ii) $t = 3$ (iii) $t = 5$ (iv) $t = 2k$ and (v) $t = 2k + 5$. A few interesting relations among the solutions are presented. Further, recurrence relations on the solutions are obtained.

***Keywords***: *Pell equation, integer solutions of Pell equation, binary quadratic Diophantine equation.*

## 1. Introduction

It is well known that the Pell equation $x^2 - Dy^2 = 1$ (D > 0 and square free) has always positive integer solutions. When$N \neq 1$, the Pell equation $x^2 - Dy^2 = N$ may not have any positive integer solutions.For example, the equations $x^2 = 3y^2 - 1$ and $x^2 = 7y^2 - 4$ have no integer solutions. When k is a positive integer and$D \in (k^2 \pm 4, k^2 \pm 1)$, positive integer solutions of the equations $x^2 - Dy^2 = \pm 4$ and $x^2 - Dy^2 = \pm 1$ have been investigated by Jones in [9].In [3], [6], [10], [15], some specific Pell equation and their integer solutions are considered. In [1], the integer solutions of the Pell equation $x^2 - (k^2 + k)y^2 = 2^t$ has been considered. In [2], the Pell equation $x^2 - (k^2 - k)y^2 = 2^t$ is analyzed for the integer solutions. In [7], the Pell equation $x^2 - 18y^2 = 4^k$ is considered. In [8], the Pell equation $x^2 - 3y^2 = (k^2 + 4k + 1)^t$ is analyzed for its positive integer solutions.
This communication concerns with the Pell equation $x^2 = 13y^2 - 3^t$, where $t > 0$ and infinitely many positive integer solutions are obtained for the choices of $t$ given by (i) $t = 1$ (ii) $t = 3$ (iii) $t = 5$ (iv) $t = 2k$ and (v) $t = 2k + 5$.A few interesting relations among the solutions are presented. Further, recurrence relations on the solutions are derived.

## 2. Notation

$t_{4,n}$ = Square number of rank $n$.

## 3. Method of analysis

### 3.1. Choice 1:$t = 1$

The Pell equation is
$x^2 = 13y^2 - 3$ (1)
Let $(X_0, Y_0)$ be the initial solution of (1) given by
$X_0 = 7$ ; $Y_0 = 2$
To find the other solutions of (1), consider the Pellian equation
$x^2 = 13y^2 + 1$
whose initial solution $(\tilde{x}_n, \tilde{y}_n)$ is given by

$\tilde{x}_n = \frac{1}{2} f_n$

$\tilde{y}_n = \frac{1}{2\sqrt{13}} g_n$

Where $f_n = \left(649 + 180\sqrt{13}\right)^{n+1} + \left(649 - 180\sqrt{13}\right)^{n+1}$

$g_n = \left(649 + 180\sqrt{13}\right)^{n+1} - \left(649 - 180\sqrt{13}\right)^{n+1}$ , $n = 0,1,2, \ldots$

Applying Brahmagupta lemma between $(X_0, Y_0)$ and $(\tilde{x}_n, \tilde{y}_n)$ , the sequence of non-zero distinct integer solutions to (1) are obtained as

$$X_{n+1} = \frac{1}{2}[7f_n + 2\sqrt{13}g_n] \quad (2)$$

$$Y_{n+1} = \frac{1}{2\sqrt{13}}[2\sqrt{13}f_n + 7g_n] \quad (3)$$

The recurrence relations satisfied by the solutions of (1) are given by

$X_{n+3} - 1298X_{n+2} + X_{n+1} = 0$ ; $X_1 = 9223$ , $X_2 = 11971447$

$Y_{n+3} - 1298Y_{n+2} + Y_{n+1} = 0$ ; $Y_1 = 2558, Y_2 = 3320282$

From (2) and (3), the values of $f_n$ and $g_n$ are found to be

$$f_n = \frac{1}{3}(52Y_{n+1} - 14X_{n+1}) \quad ; \quad g_n = \frac{1}{3}\left(4\sqrt{13}X_{n+1} - 14\sqrt{13}Y_{n+1}\right) \quad (4)$$

**Properties**

1. $936Y_{2n+2} - 252X_{2n+2} + 108$ is a nasty number.
2. $468Y_{3n+3} - 126X_{3n+3} + 1404Y_{n+1} - 378X_{n+1}$ is a cubic integer.
3. $1404Y_{4n+4} - 378X_{4n+4} + 324t_{4,f_n} - 162$ is a bi-quadratic integer.

## 3.2. Choice 2: $t = 3$.

The Pell equation is

$$x^2 = 13y^2 - 27 \quad (5)$$

Let $(X_0, Y_0)$ be the initial solution of (5) given by

$X_0 = 5$ ; $Y_0 = 2$

Applying Brahmagupta lemma between $(X_0, Y_0)$ and $(\tilde{x}_n, \tilde{y}_n)$ , the sequence of non-zero distinct integer solutions to (5) are obtained as

$$X_{n+1} = \frac{1}{2}[5f_n + 2\sqrt{13}g_n] \quad (6)$$

$$Y_{n+1} = \frac{1}{2\sqrt{13}}[2\sqrt{13}f_n + 5g_n] \quad (7)$$

The recurrence relations satisfied by the solutions of (5) are given by

$X_{n+3} - 1298X_{n+2} + X_{n+1} = 0$ ; $X_1 = 7925$ , $X_2 = 10286645$

$Y_{n+3} - 1298Y_{n+2} + Y_{n+1} = 0$ ; $Y_1 = 2198, Y_2 = 2853002$

From (6) and (7), the values of $f_n$ and $g_n$ are found to be

$$f_n = \frac{1}{27}(52Y_{n+1} - 10X_{n+1}) \quad ; \quad g_n = \frac{1}{27}\left(4\sqrt{13}X_{n+1} - 10\sqrt{13}Y_{n+1}\right) \quad (8)$$

**Properties**

1. $6(468Y_{2n+2} - 90X_{2n+2} + 1458)$ is a nasty number.
2. $468Y_{3n+3} - 90X_{3n+3} + 468Y_{n+1} - 90X_{n+1}$ is a cubic integer.
3. $52Y_{4n+4} - 10X_{4n+4} + 324t_{4,f_n} - 162$ is a bi-quadratic integer.

## 3.3. Choice 3: $t = 5$

The Pell equation is

$$x^2 = 13y^2 - 243 \quad (9)$$

Let $(X_0, Y_0)$ be the initial solution of (9) given by

$X_0 = 15$ ; $Y_0 = 6$

Applying Brahmagupta lemma between $(X_0, Y_0)$ and $(\tilde{x}_n, \tilde{y}_n)$ , the sequence of non-zero distinct integer solutions to (9) are obtained as

$$X_{n+1} = \frac{1}{2}[15f_n + 6\sqrt{13}g_n] \quad (10)$$

$$Y_{n+1} = \frac{1}{2\sqrt{13}}[6\sqrt{13}f_n + 15g_n] \quad (11)$$

The recurrence relations satisfied by the solutions of (9) are given by

$X_{n+3} - 1298X_{n+2} + X_{n+1} = 0$ ; $X_1 = 23775$ , $X_2 = 30859935$

$Y_{n+3} - 1298Y_{n+2} + Y_{n+1} = 0$ ; $Y_1 = 6594, Y_2 = 8559006$

From (10) and (11), the values of $f_n$ and $g_n$ are found to be

$f_n = \frac{1}{243}(156Y_{n+1} - 30X_{n+1})$ ; $g_n = \frac{1}{243}\left(12\sqrt{13}X_{n+1} - 30\sqrt{13}Y_{n+1}\right)$ (12)

**Properties**

1. $104Y_{2n+2} - 20X_{2n+2} + 108)$ is a nasty number.
2. $52Y_{3n+3} - 10X_{3n+3} + 156Y_{n+1} - 30X_{n+1}$ is a cubic integer.
3. $156Y_{4n+4} - 30X_{4n+4} + 324t_{4,f_n} - 162$ is a bi-quadratic integer

### 3.4. Choice 4: $t = 2k$ , $k > 0$.

The Pell equation is

$x^2 = 13y^2 - 3^{2k}, k > 0$ (13)

Let $(X_1, Y_1)$ be the initial solution of (13) given by

$X_1 = 3^k.649$ ; $Y_1 = 3^k.180$

Applying Brahmagupta lemma between $(X_1, Y_1)$ and $(\tilde{x}_n, \tilde{y}_n)$ , the sequence of non-zero distinct integer solutions to (13) are obtained as

$X_{n+1} = 3^k.\frac{1}{2}f_n$ (14)

$Y_{n+1} = 3^k.\frac{1}{2\sqrt{13}}g_n$ , $n = 1,2,3, ...$ (15)

The recurrence relations satisfied by the solutions of (13) are given by

$X_{n+3} - 1298X_{n+2} + X_{n+1} = 0$ ; $X_2 = 3^k.842401$ , $X_3 = 3^k.1093435849$

$Y_{n+3} - 1298Y_{n+2} + Y_{n+1} = 0$ ; $Y_2 = 3^k.233640, Y_3 = 3^k.303264540$

From (14) and (15), the values of $f_n$ and $g_n$ are found to be

$f_n = \frac{1}{3^k}(1298X_{n+2} - 4680Y_{n+2})$ ; $g_n = \frac{1}{3^k}\left(1298\sqrt{13}Y_{n+2} - 360\sqrt{13}X_{n+2}\right)$ (16)

**Properties**

1. When $k \equiv 0 \ (mod\ 2)$, $6(1298X_{2n+3} - 4680Y_{2n+3} + 2.3^{2k})$ is a nasty number.
2. When $k \equiv 0 \ (mod\ 3)$ , $1298X_{3n+4} - 4680Y_{3n+4} + 3(1298X_{n+2} - 4680Y_{n+2})$ is a cubic integer.

### 3.5. Choice 5: $t = 2k + 5$ , $k > 0$

The Pell equation is

$x^2 = 13y^2 - 3^{2k+5}$ (17)

Let $(X_0, Y_0)$ be the initial solution of (17) given by

$X_0 = 3^{k-1}.19$ ; $Y_0 = 3^{k-1}.14$

Applying Brahmagupta lemma between $(X_0, Y_0)$ and $(\tilde{x}_n, \tilde{y}_n)$ , the sequence of non-zero distinct integer solutions to (17) are obtained as

$X_{n+1} = \frac{3^{k-1}}{2}(19f_n + 14\sqrt{13}g_n)$ (18)

$Y_{n+1} = \frac{3^{k-1}}{2\sqrt{13}}(14\sqrt{13}f_n + 19g_n)$ (19)

The recurrence relations satisfied by the solutions of (17) are given by

$X_{n+3} - 1298X_{n+2} + X_{n+1} = 0$ ; $X_1 = 3^{k-1}.45091$ , $X_2 = 3^{k-1}.58528099$

$Y_{n+3} - 1298Y_{n+2} + Y_{n+1} = 0$ ; $Y_1 = 3^{k-1}.12506, Y_2 = 3^{k-1}.16232774$

From (18) and (19), the values of $f_n$ and $g_n$ are found to be

$f_n = \frac{1}{3^{k+6}}(325156Y_{n+2} - 90182X_{n+2})$ ; $g_n = \frac{1}{3^{k+6}}\left(25012\sqrt{13}X_{n+2} - 90182\sqrt{13}Y_{n+2}\right)$ (20)

The integer solutions presented in each of the sections 1 to 5 satisfy the following relations.

1. $X_{n+3} = 649X_{n+2} + 2340Y_{n+2}$.
2. $X_{n+3} = 842401X_{n+1} + 3037230Y_{n+1}$.
3. $Y_{n+3} = 180X_{n+2} + 649Y_{n+2}$.
4. $Y_{n+3} = 233640X_{n+1} + 842401Y_{n+1}$

## 4. Conclusion

To conclude, one may search for other patterns of solutions to the similar equation considered above.

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