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Research Paper

Determination of two-time dependent coefficients in a parabolic partial differential equation by homotopy analysis method

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Abstract

In this paper the solution procedure in obtaining two times - dependent coefficients in a one dimensional partial differential equation and the temperature distribution is discussed and solved. We use the homotopy analysis method to obtain the solution of both the unknown coefficients and the temperature distribution. The solutions to the unknown coefficients are obtained by reducing our problem to a system of equations at every time step. There are advantages to using HAM, firstly it is independent of small/large physical parameters, there is flexibility on the choice of base function and initial guess of solution and lastly there is great generality. The results obtained from this method shows high accuracy, computational efficiency and a strong rate of convergence.

Keywords: *Heat Equation, Homotopy Analysis Method, Inverse Problem, Time - Dependent Diffusion Coefficients.*

1. Introduction

The study of parabolic inverse problems has gained much attention in recent years. The literature, analysis, research and implementation of numerical and approximate analytic methods for the solution of parabolic inverse problems, i.e. the determination of some unknown functions and coefficients in the parabolic partial differential equations have increased. There have been many contributions to the study of inverse parabolic differential equations [1], [3], [5]. In this paper we utilize the homotopy analysis method (HAM) [2],proposed by Liao[6-8], to help us obtain approximate and exact solutions to the following parameter identification problem of finding coefficients p(t) and q(t) in the diffusion equation:

$$u_t = u_{xx} + q(t)u_x + p(t)u + f(x,t), \quad 0 < x < l,\ 0 < t \le \tau, \tag{1.1}$$

With the initial condition

$$u(x,0) = k(x),\ \ 0 < x < l, \tag{1.2}$$

And boundary condition

$$u(0,t) = g_0(t),\ \ 0 < t \le \tau, \tag{1.3}$$

$$u(l,t) = g_1(t),\ \ 0 < t \le \tau, \tag{1.4}$$

And subject to the additional specifications

$$u(x^*,t) = E_1(t),\ \ 0 < x^* < l,\ 0 < t \le \tau, \tag{1.5}$$

And

$$u(x^{**},t) = E_2(t),\ \ 0 < x^* < l,\ 0 < t \le \tau, \tag{1.6}$$

Where f(x, t), k(x), $g_0(t), g_1(t)$, $E_1(t) \ne 0$ and $E_2(t) \ne 0$ are known functions, while u(x,t) , p(t) and q(t) are unknown.

The points x^* and x^{**} are given, and are found in the spatial domain of the problem.

This problem has been solved by Saadatmandi and Dehghan [3] where they used the tau technique to determine the unknown techniques to determine p (t) and q(t). The technique used by Saadatmandi and Dehghan consisted of reducing the problem to a set of algebraic equations by expanding the approximate solutions of u(x,t), p(t) and q(t) in terms of shifted Legendre polynomials with unknown coefficients. In utilizing operational matrices and given derivatives, they used the tau method to evaluate the unknown coefficients of shifted Legendre polynomials.

Unlike other perturbation methods, HAM avoids discretization, provides us with efficient numerical solution with high accuracy; there is minimal calculation and the avoidance of physically unrealistic assumptions. The convergence region for the series solution obtained by HAM is determined by the convergence-control parameter h.

The paper contains the following sections:

In Section 2, we discuss the methodology of the homotopy analysis method. We look at the solution of the parabolic inverse problem with two unknown time – dependent coefficients in Section 3. In section 4, we look at numerical examples; we discuss and analyze the results.

2. Homotopy analysis method

To illustrate the basic idea of the HAM, we consider the following differential equation:

$$N\left[u(x,t)\right]=s(x,t), \tag{2.1}$$

Where N is a nonlinear operator, x and t denotes independent and dependent variables respectively, u is an unknown function and s(x, t) is the nonhomogeneous term. By means of HAM, we first construct a zeroth-order deformation equation

$$(1-r)L\left[\varphi(x,t;r)-u_0(x,t)\right]=rhN\left[\varphi(x,t;r)-s(x,t)\right], \tag{2.2}$$

Where $r\in[0,1]$ is the embedding parameter, $h\neq 0$ is a convergence-control, L is an auxiliary linear operator, $\varphi(x,t;r)$ is an unknown factor, $u_0(x,t)$ is an initial guess of u(x,t). It is obvious that when the embedding parameter r goes from 0 to 1, the values for $\varphi(x,t;r)$ becomes

$$\varphi(x,t;0)=u_0(x,t),\qquad \varphi(x,t;1)=u(x,t), \tag{2.3}$$

Respectively. Thus as r increases from 0 to 1, the solution $\varphi(x,t;r)$ varies from the initial guess $u_0(x,t)$ to the solution u(x, t). Expanding $\varphi(x,t;r)$ in Taylor series with respect to r, we obtain

$$\varphi(x,t;r)=u_0(x,t)+\sum_{m=1}^{+\infty}u_m(x,t)r^m, \tag{2.4}$$

Where

$$u_m(x,t)=\frac{1}{m!}\frac{\partial^m\varphi(x,t;r)}{\partial r^m}\Big|_{r=0}, \tag{2.5}$$

The convergence of the series (2.4) depends upon the convergence-control parameter h .

With HAM, we have the freedom to choose the initial guess $u_0(x,t)$, the auxiliary linear operator L, and the nonzero convergence-control parameter h . We assume that all of them are properly chosen so that:

1) The solution $\varphi(x,t;r)$ of the zeroth-order deformation equation (2.2) exists for all $r\in[0,1]$.
2) The homotopy analysis derivative $D_m(\varphi(x,t;r))$ exists for m = 1, 2, 3,…, $+\infty$.
3) The power series (2.4) of $\varphi(x,t;r)$ converges at r =1.

Then from Eqs. (2.3) and (2.4), we have under these assumptions the solution series

$$u(x,t)=u_0(x,t)+\sum_{m=1}^{+\infty}u_m(x,t), \tag{2.6}$$

Which must be one of the solutions of the original nonlinear equation, as proven by Liao [8]?

Define the vectors

$$\vec{u}_n=\left\{u_0(x,t),u_1(x,t),...,u_n(x,t)\right\}, \tag{2.7}$$

Differentiating the zeroth-order deformation equation (2.2) m – times with respect to r and then dividing them by m! And finally setting r = 0 (Taking the mth – order homotopy derivative). Firstly, since L is a linear operator independent of r, it holds

$$\begin{aligned}&D_m\left((1-r)L\left[\varphi(x,t;r)-u_0(x,t)\right]\right)\\&=D_m\left(L\left[\varphi(x,t;r)-r\varphi(x,t;r)+u_0(x,t)r-u_0(x,t)\right]\right),\end{aligned} \tag{2.8}$$

And using Homotopy Properties [8]

$$D_m\left(L\left[\varphi(x,t;r)-r\varphi(x,t;r)+u_0(x,t)r-u_0(x,t)\right]\right)$$
$$=L\left[D_m\varphi(x,t;r)-D_m\left(r\varphi(x,t;r)+u_0D_m(r)\right)\right], \tag{2.9}$$

And Homotopy Property [8]

$$L\left[D_m\left(\varphi(x,t;r)\right)-D_m\left(r\varphi(x,t;r)\right)+u_0(x,t)D_m(r)\right]$$
$$=L\left[u_m(x,t)-u_{m-1}(x,t)+u_0(x,t)D_m(r)\right], \tag{2.10}$$

Which equals t the $L[x_m]$ when m=1, and $L[x_m-x_{m-1}]$ when m > 1, respectively. Thus, the m-th – order deformation equation becomes

$$L\left[u_m(x,t)-\chi_m u_{m-1}(x,t)\right]=hR_m\left(\vec{u}_{m-1}\right), \tag{2.11}$$

Where

$$R_m\left(\vec{u}_{m-1}\right)=\frac{1}{(m-1)!}\frac{\partial^{m-1}N\left[\varphi(x,t;r)-s(x,t)\right]}{\partial r^{m-1}}\Big|_{r=0}, \tag{2.12}$$

And

$$\chi_m=\begin{cases}0, & m\le 1\\ 1, & m>1\end{cases}, \tag{2.13}$$

It should be emphasized that $u_m(x,t)$ for $m\ge 1$ is governed by the linear equation (2.11) with linear boundary conditions that come from the original problem. Therefore the solution to the differential equation obtained by HAM is a family of solutions expressed using the convergence-control parameter h .

3. Approximate analytical solution to the inverse problem with HAM

In this paper, we use HAM to obtain the approximate analytical solution to the inverse problem by obtaining $u(x,t)$, p (t) and q (t).
Using the solution procedure by HAM, we define a linear operator in the form.
Solution procedure by HAM, we define a linear operator in the form

$$L\left[\phi(x,t;r)\right]=\frac{\partial\phi(x,t;r)}{\partial t}, \tag{3.1}$$

With the property

$$L\left[c_1(x)\right]=0, \tag{3.2}$$

Where $c_1(x)$ is the integration constant? The nonlinear operator is taken as

$$N\left[\phi(x,t;r)\right]=-u_t(x,t;r)+u_{xx}(x,t;r)+q(t)u_x(x,t)+p(t)u(x,t)+f(x,t), \tag{3.3}$$

So we can define R_m as

$$R_m\left(\vec{u}_{m-1}\right)=-\frac{\partial u_{m-1}(x,t)}{\partial t}+\frac{\partial^2 u_{m-1}(x,t)}{\partial x^2}+q_{m-1}(t)\frac{\partial u_{m-1}(x,t)}{\partial x}+p_{m-1}(t)u_{m-1}(x,t)+f(x,t) \tag{3.4}$$

Using (2.11), (2.13) and (3.3), we can get

$$u_m(x,t)=\chi_m u_{m-1}(x,t)+h\int R_m\left(\vec{u}_{m-1}\right)dt+c_1(x), \tag{3.5}$$

The parameters used during the application of HAM are as follows:
The initial guess is $u_0(x,t)$, which is found by analyzing (1.2)-(1.6)

$$u(x,t)=\sum_{m=0}^{+\infty}u_m(x,t)=u_0+u_1+u_2+..., \tag{3.7}$$

The solutions for p(t) and q(t) are found by solving the system of the following equations:

$$\begin{cases}\left.\left(\frac{\partial u_{m-1}(x,t)}{\partial t}-\frac{\partial^2 u_{m-1}(x,t)}{\partial x^2}-f(x,t)\right)\right|_{x=x^*}=\left.\left(q_{m-1}(t)\frac{\partial u_{m-1}(x,t)}{\partial x}+p_{m-1}(t)u_{m-1}(x,t)\right)\right|_{x=x^*}\\ \left.\left(\frac{\partial u_{m-1}(x,t)}{\partial t}-\frac{\partial^2 u_{m-1}(x,t)}{\partial x^2}-f(x,t)\right)\right|_{x=x^{**}}=\left.\left(q_{m-1}(t)\frac{\partial u_{m-1}(x,t)}{\partial x}+p_{m-1}(t)u_{m-1}(x,t)\right)\right|_{x=x^{**}}\end{cases} \tag{3.8}$$

Note that

$$p(t)=\sum_{m=0}^{+\infty}p_m(t)=p_0(t)+p_1(t)+p_2(t)+\dots \tag{3.9}$$

$$q(t)=\sum_{m=0}^{+\infty}q_m(t)=q_0(t)+q_1(t)+q_2(t)+\dots \tag{3.10}$$

The convergence-control parameter h is calculated by setting $\left(u(l,t)\right)_{HAM}=\left(u(l,t)\right)_{exact}$, and then solving for h.

4. Numerical examples

Example 1: Consider (1.1)-(1.6) with $l=1,\tau=3$ and

$g_0(t)=1,$

$g_1(t)=t\sin(1)+1,$

$k(x)=1,$

$x^*=0.2\ ,x^{**}=0.6$

$E_1(t)=t\sin(0.2)+1,$

$E_2(t)=t\sin(0.6)+1,$

$f(x,t)=(1+t)\sin(x)-0.1t^3\cos(x)-(t^3+t)(t\sin(x)+1)\exp(-t^2)$

Solutions obtained with HAM:

In analyzing (1.2)-(1.6) the initial guess is $u_0(x,t)=t\sin(x)+1$

$u(x,t)=t\sin(x)+1$

$p(t)=(t^2+1)e^{-t^2}t$

$q(t)=0.1t^2$

Exact Solutions:

$u(x,t)=t\sin(x)+1$

$p(t)=(t^3+t)e^{-t^2}$

$q(t)=0.1t^2$

Example 2: Consider (1.1)-(1.6) with $l=1,\tau=1$ and

$g_0(t)=1+t^3,$

$g_1(t)=\exp(t)+\cos(1)+t^3,$

$k(x)=x^2+\cos(x),$

$x^*=0.15\ ,x^{**}=0.75$

$E_1(t)=0.0225\exp(t)+\cos(0.15)+t^3,$

$E_2(t)=0.4125\exp(t)+\cos(0.75)+t^3,$

$$f(x,t)=\left(1-t^2-\sin(t)\right)x^2\exp(t)+\left(t\sin x-2\right)\exp(t)-2xt\exp(2t)+\left(1-t^2-\sin(t)\right)\cos(x)$$
$$+\left(3-t^3-t\sin(t)\right)t^2,$$

In analyzing (1.2)-(1.6) our initial guess is $u_0(x,t)=x^2\exp(t)+\cos(x)+t^3$

$(u(x,t))_{HAM}=(u(x,t))_{EXACT}=x^2\exp(t)+\cos(x)+t^3$

The results obtained for p (t) and q(t) using HAM when h=0 are listed below

Table 1: Results for $p(t)$

t	Exact p(t)	HAM	Absolute Error
0.0	0	0	0
0.05	0.05247916927	0.05247916927	0
0.10	0.1098334166	0.1098334166	0
0.15	0.1719381325	0.1719381323	2×10^{-10}
0.20	0.2386693308	0.2386693311	3×10^{-10}
0.25	0.3099039593	0.3099039593	0
0.30	0.3855202067	0.3855202067	0
0.35	0.4653978075	0.4653978073	2×10^{-10}
0.40	0.5494183423	0.5494183423	0
0.45	0.6374655341	0.6374655340	1×10^{-10}
0.50	0.7294255386	0.7294255390	4×10^{-10}
0.55	0.8251872289	0.8251872290	1×10^{-10}
0.60	0.9246424734	0.9246424734	0
0.65	1.027686406	1.027686405	1×10^{-9}
0.70	1.134217687	1.134217686	1×10^{-9}
0.75	1.244138760	1.244138759	1×10^{-9}
0.80	1.357356091	1.357356089	2×10^{-9}
0.85	1.473780405	1.473780404	1×10^{-9}
0.90	1.593326910	1.593326911	1×10^{-9}
0.95	1.715915505	1.715915503	2×10^{-9}
1.0	1.841470985	1.841470985	0

Table 2: Results for q (t)

t	Exact q(t)	HAM	Absolute Error
0.0	0	0	0
0.05	0.05256355480	0.05256355482	2×10^{-11}
0.10	0.1105170918	0.1105170918	0
0.15	0.1742751364	0.1742751363	1×10^{-10}
0.20	0.2442805516	0.2442805516	0
0.25	0.3210063542	0.3210063540	2×10^{-10}
0.30	0.4049576424	0.4049576420	4×10^{-10}
0.35	0.4966736422	0.4966736412	1.01×10^{-7}
0.40	0.5967298792	0.5967298784	8×10^{-10}
0.45	0.7057404832	0.7057404834	2×10^{-10}
0.50	0.8243606355	0.8243606348	7×10^{-10}
0.55	0.9532891599	0.9532891592	7×10^{-10}
0.60	1.093271280	1.093271281	1×10^{-9}
0.65	1.245101539	1.245101538	1×10^{-9}
0.70	1.409626895	1.409626894	1×10^{-9}
0.75	1.587750013	1.587750009	4×10^{-9}
0.80	1.780432742	1.780432740	2×10^{-9}
0.85	1.988699824	1.988699822	2×10^{-9}
0.90	2.213642800	2.213642800	0
0.95	2.456424176	2.456424174	2×10^{-9}
1.0	2.718281828	2.718281828	0

Example 3: Consider (1.1)-(1.6) with $l = 1, \tau = 1$ and

$g_0(t) = (1+t)^3,$

$g_1(t) = \cos(t) + t^2 + (t+1)^3,$

$k(x) = 1 + x^3,$

$x^* = 0.25 \ , x^{**} = 0.85$

$E_1(t) = 0.015625\cos(t) + 0.0625t^2 + (t+1)^3,$

$E_2(t) = 0.0614125\cos(t) + 0.7225t^2 + (t+1)^3,$

$f(x,t) = x^2(2t - x\sin(t)) - t^2(x^2 + t + 2) - x(x^2 + 3x + 6 + 15xt)\cos(t) - t(1+t)^3\cos(t) - 2xt^2(1 + 5t + t^3)$

$- x^2t(4t^2 + x\cos(t))\cos(t) + 3t + 2,$

In analyzing (1.2)-(1.6) our initial guess is $u_0(x,t) = x^3\cos(t) + x^2t^2 + (t+1)^3$

$(u(x,t))_{HAM} = (u(x,t))_{EXACT} = x^3\cos(t) + x^2t^2 + (t+1)^3$

The results obtained for p (t) and q(t) using HAM when h=0 are listed below

Table 3: Results for p (t)

t	Exact p(t)	HAM	Absolute Error
0.0	1	1	0
0.05	1.049937513	1.049937513	0
0.10	1.099500417	1.099500416	1×10^{-9}
0.15	1.148315662	1.148315661	1×10^{-9}
0.20	1.196013316	1.196013316	0
0.25	1.242228105	1.242228105	0
0.30	1.286600947	1.286600947	0
0.35	1.328780450	1.328780449	1×10^{-9}
0.40	1.368424398	1.368424397	1×10^{-9}
0.45	1.405201196	1.405201194	2×10^{-9}
0.50	1.438791281	1.438791281	0
0.55	1.468888487	1.468888487	0
0.60	1.495201369	1.495201368	1×10^{-9}
0.65	1.517454469	1.517454470	1×10^{-9}
0.70	1.535389531	1.535389530	1×10^{-9}
0.75	1.548766652	1.548766652	0
0.80	1.557365367	1.557365367	0
0.85	1.560985674	1.560985675	1×10^{-9}
0.90	1.559448972	1.559448970	2×10^{-9}
0.95	1.552598935	1.552598934	1×10^{-9}
1.0	1.540302306	1.540302306	0

Table 4: Results for q (t)

t	Exact q(t)	HAM	Absolute Error
0.0	1	1	0
0.05	1.250125	1.250124999	1×10^{-9}
0.10	1.501000	1.501000	0
0.15	1.753375	1.753374998	2×10^{-9}
0.20	2.008000	2.008000001	1×10^{-9}
0.25	2.265625	2.265625	0
0.30	2.527000	2.527000	0
0.35	2.792875	2.792875	0
0.40	3.064000	3.064000	0
0.45	3.341125	3.341124998	2×10^{-9}
0.50	3.625000	3.624999998	2×10^{-9}
0.55	3.916375	3.916374997	3×10^{-9}
0.60	4.216000	4.216000	0
0.65	4.524625	4.524625	0
0.70	4.843000	4.842999992	8×10^{-9}

t	Exact q(t)	HAM	Absolute Error
0.75	5.171875	5.171875004	4×10^{-9}
0.80	5.512000	5.511999996	4×10^{-9}
0.85	5.864125	5.864125004	4×10^{-9}
0.90	6.229000	6.228999996	4×10^{-9}
0.95	6.607375	6.607375	0
1.0	7	7	0

5. Conclusion

We have shown that HAM can be used to accurately predict the results u(x, t), p (t) and q (t). The inverse parabolic partial differential equation has been discussed theoretically and analyzed numerically. The freedom in choosing h enables us to adjust and control the convergence of the solution series and this differentiates the homotopy analysis method from other existing methods such as the homotopy perturbation method, adomain decomposition method and variational iteration method.
MAPLE was used for the computation presented in this paper.

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