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Research Paper

Dual series method for solving Helmholtz equation with mixed boundary conditions of the third kind

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Abstract

The paper involves with the application of the dual series equation to the problem of Helmholtz equation of cylindrical coordinates subject to inhomogeneous mixed boundary conditions of the third kind located on the surface of the cylinder of bounded radius and infinite high. By choosing the Hankel integral transform, the dual series equations were reduced to a Fredholm integral equation of the second kind which is solved conveniently by using numerical techniques.

Keywords: *Helmholtz Equation, Dual Series Equations, Mixed Boundary Conditions.*

1. Introduction

It is well known that heat and wave equations with different coordinate systems and various initial and boundary conditions can be reduced to Helmholtz equation, for example temperature function $T(t,P)$ is transformed to $T(t,P)=e^{-t\gamma}u(P)$, such transformation simplify the solution of the required partial differential equation. In this paper we presented a dual series equations related to Helmholtz equation in cylindrical coordinates subject to inhomogeneous mixed boundary conditions of the third kind located on a level surface of the cylinder of the bounded radius and infinite high. The radius of the surface cylinder R is divided into two intervals such that, inside the disk $0<r<R_1$ located a third kind boundary conditions which is different with the boundary conditions of the third kind located outside the disk $R_1<r<R_2$. No boundary conditions on the line of discontinuity $r=R_1$. Application of separation of variables to the Helmholtz partial differential equation, we obtain
a general solution with unknown coefficients, next step the use of the boundary conditions leads to dual series equations with Bessel function as a kernel for determination unknown coefficients.
Problems of the heat and Helmholtz equations with mixed boundary conditions of the first, the second and of the third kind in axial cylindrical coordinates have previously been treated and founded in the scientific or technical monographs on this subject [1-6], [8]. Much attention was received to investigate mixed problems with partly infinite boundaries can often be reduced to study dual integral equations[1-6],while for the finite regions the problems can often reduce to study dual series equations. Dual series equations related to Laplace equation in cylindrical, spherical and other coordinates involving potential theory, diffraction theory, elasticity theory and other applications can be found in [10-12] and other monographs. In this paper, we treated the dual series equations by defining Hankel integral transform of a function related to the unknown function satisfying the dual series equations, and then the inverse transformation produces ultimately the function sought for. In [9-11] Hankel integral transform method is used for investigation solutions for dual integral equations involving a Bessel function of the first kind of order zero as a kernel. The analysis of the dual series equations led to a Fredholm integral equation of the second kind which is solved numerically.

2. Formulation and solution of the problem

Find the solution of Helmholtz mixed boundary value problem in a axial symmetrical solid cylinder

$$\nabla^2 v+k^2 v=0, \qquad (1)$$

$$a_1 v_z - b_1 v = -f_1(r), \quad r \in D \tag{2}$$

$$a_2 u_z - b_2 u = -f_2(r), \quad r \in \overline{D} \tag{3}$$

Where $v = v(r,z)$, $\nabla^2 = (\partial^2/\partial r^2 + \frac{1}{r}\partial/\partial r + \partial^2/\partial z^2)$, $0 < r < R_2$, $0 < z < \infty$ Are the corresponding cylindrical coordinates, k is a constant, $k \neq 0$ $D = \{0 < r < R_1, z = 0\}$, $\overline{D} = \{R_1 < r < R_2, z = 0\}$ R_2 and is the radius of the cylinder $a_i, b_i, i = 1,2$ Are constants, $f_i(r), i = 1,2$ known continuous functions, further we assume that as $\sqrt{r^2 + z^2} \to 0$, $v \to 0$.

Separation variables in (1), under the assumption that the conditions bounded at infinity and zero, we obtain the general solution of the problem

$$v(r,z) = \sum_{n=1}^{\infty} C_n \exp(-z\sqrt{\lambda_n^2 + k^2})J_0(\lambda_n r) \tag{4}$$

C_n Unknown coefficients, λ_n is the root of Bessel function of the first kind order zero $J_0(\lambda_n R_2) = 0$.

Use mixed conditions (2) and (3) for (4), we obtain the following dual series equations to determine the unknown coefficients C_n

$$\sum_{n=1}^{\infty} C_n \left\{a_1\sqrt{\lambda_n^2 + k^2} + b_1\right\} J_0(\lambda_n \rho) = f_1(r), \quad r \in D \tag{5}$$

$$\sum_{n=1}^{\infty} C_n \left\{a_2\sqrt{\lambda_n^2 + k^2} + b_2\right\} J_0(\lambda_n \rho) = f_2(r), \quad r \in \overline{D} \tag{6}$$

If $k = 0$, and some of the coefficients a_i, b_i is zero, the dual equations (5) and (6) reduced to known solutions [10], [11]. Now to solve (5) and (6), let us to introduce the substitution

$$A_n = C_n(a_2\sqrt{\lambda_n^2 + k^2} + b_2), \quad G_n = \frac{a_1\sqrt{\lambda_n^2 + k^2} + b_1}{a_2\sqrt{\lambda_n^2 + k^2} + b_2}$$

Dual equations (5) and (6) accept the form

$$\sum_{n=1}^{\infty} A_n J_0(\lambda_n r) G_n(\lambda_n) = f_1(r), \quad r \in D \tag{7}$$

$$\sum_{n=1}^{\infty} A_n J_0(\lambda_n r) = f_2(r), \quad r \in \overline{D} \tag{8}$$

To solve (7), (8), let us to write down the weight function $G_n(\lambda_n)$ as

$G_n(\lambda_n) = G_n(\lambda_n) - q + q = g_n(\lambda_n) + q$

Where q is constant, g_n is continuous function

$$g_n = \frac{\sqrt{\lambda_n^2 + k^2}(a_1 - qa_2) + (b_1 - qb_2)}{a_2\sqrt{\lambda_n^2 + k^2} + b_2} \tag{9}$$

Dual series equations (7), (8) by meaning (9) will be

$$\sum_{n=1}^{\infty} A_n J_0(\lambda_n r)(g_n(\lambda_n) + q) = f_1(r), \ r \in D \tag{10}$$

Series (8) is written over the interval $0 < r < R_2$ such that

$$\sum_{n=1}^{\infty} A_n J_0(\lambda_n r) = \begin{cases} y(r), & r \in D, \\ f_2(r), & r \in \overline{D}. \end{cases}$$

$y(r)$ Is continuous unknown function defined over the interval $0 < r < R_1$ appears is the nature of a mixed boundary conditions. Use an inversion formula for determination of A_n in (8), we have

$$A_n = \frac{2}{R_2^2 J_1^2(\lambda_n R_2)}\left\{\int_0^{R_1} y(u)J_0(\lambda_n u)udu + \int_{R_1}^{R_2} f_2(u)J_0(\lambda_n u)udu\right\} \tag{11}$$

Putting (11) into (10), we have

$$q\sum_{n=1}^{\infty} A_n J_0(\lambda_n r) + \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) g_n(\lambda_n) =$$

$$q\sum_{n=1}^{\infty} A_n J_0(\lambda_n r) + \frac{2}{R_2^2 J_1^2(\lambda_n R_2)}\sum_{n=1}^{\infty}\left\{\int_0^{R_1} y(u)J_0(\lambda_n u)udu + \int_{R_1}^{R_2} f_2(u)J_0(\lambda_n u)udu\right\} \\ J_0(\lambda_n r) g_n(\lambda_n) = f_1(r), \quad r \in D \tag{12}$$

First sum in (12), is equal to $y(r)$, simplifying the second sum and integration yields a Fredholm integral equation of the second kind

$$y(r) + \frac{1}{q}\int_0^{R_1} y(u)\, K(r,u)\, du = \frac{1}{q}F(r) \qquad , r \in D \tag{13}$$

With kernel and free term respectively

$$K(r,\mathrm{u})=\frac{2}{R_2{}^2}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n u)J_{-1/2}(\lambda_n \mathrm{r})}{J_1^2(\lambda_n R_2)}g_n$$

$$\mathrm{F}(r)=f_1(r)+\frac{2}{R_2{}^2}\int_{R_1}^{R_2}f_2(\mathrm{u})\left(\sum_{n=1}^{\infty}\frac{J_0(\lambda_n u)J_{-1/2}(\lambda_n \mathrm{r})}{J_1^2(\lambda_n R_2)}g_n\right)du$$

The known function and the kernel satisfy the inequalities [12]

$$\int_0^{R_2}|F(r)|\,dr<\infty\quad,\ L^2=\int_0^{R_1}\int_0^{R_1}K^2(r,\mathrm{u})dr\,du<\infty\,.\ \lambda<1/L,\quad \lambda=1/\mathrm{q}$$

And $\lambda<\dfrac{1}{MR_1}$, $\quad \mathrm{M}=\max K(r,u),\ 0<\mathrm{r,u}<\mathrm{R}_1$

3. Numerical solution for integral equation

It can be seen that the problem are reduced to determine the solution of an integral equation for the unknown function $y(r)$ as presented in (13).This equation can be approximated by a sum discrete values of r and u leading to the system of linear equations[8]

$$\mathrm{y}(r_i)+\sum_{j=1}^{n}B_jK(r_i,\mathrm{u}_j)\mathrm{y}(\mathrm{u}_j)=F(r_j)+E_i \tag{14}$$

B_j Is the weight function depending on the numerical integration? Both r_i,u_j are equally from 0 to R_1, $K(r_i,\mathrm{u}_j)$ is the discretized kernel integral equation, $\mathrm{y}(r_j),F(r_j)$ discretized values for the right-hand side function and auxiliary function respectively, E_i is the error resulted by replacing integral by series. In expression (14) neglecting the error E_i, then replacing the integral by a set of simultaneous algebraic equations, we have

$$y_i+\sum_{j=1}^{n}B_jK_{ij}y_j=F_i,\ i=1,...,\mathrm{n}\,. \tag{15}$$

Where $y_i=y(r_i)$, the values of $F_i=F(r_i)$, $K_{ij}=K(r_i,y_j)\quad 0\le r_i,u_j\le R_1$

Approximation $\tilde{\mathrm{y}}(r)$ of the unknown function $\mathrm{y}(r)$ by using (14) and (15) is

$$\tilde{\mathrm{y}}(r)=F(r)-\sum_{j=1}^{n}A_jK(r,r_j)\mathrm{y}_j \tag{16}$$

Where

$$K(\mathrm{r},\mathrm{r}_j)=\frac{2}{R_2{}^2}\sum_{n=1}^{\infty}\frac{J_{-1/2}(\lambda_n \mathrm{r})J_0(\lambda_n r_j)}{J_1^2(\lambda_n R_2)}g_n$$

$$K(\mathrm{u}_i,\mathrm{r}_j)=\frac{2}{R_2{}^2}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n u_i)J_{-1/2}(\lambda_n \mathrm{r}_j)}{J_1^2(\lambda_n R_2)}g_n$$

To solve equation (15), Simpson rule is used for this purpose to set up the simultaneous equations [10] , the weight function B_j is

$$B_1=B_n=\frac{R_1}{3(n-1)}$$

$$B_2=B_4=...=\frac{4R_1}{3(n-1)}$$

$$B_3=B_5=...=\frac{2R_1}{3(n-1)}$$

In view of equation (15), one expands to the linear n equations in the matrix

$Ky=F\Rightarrow y=K^{-1}F$ (17)

$$K=\begin{bmatrix}1+\lambda B_1K(r_1,u_1) & B_2K(r_1,u_2) & \cdots & B_nK(r_1,u_n)\\ B_1K(r_2,u_1) & 1+\lambda B_2K(r_1,u_2) & \cdots & B_nK(r_2,u_n)\\ \vdots & \vdots & & \vdots\\ B_1K(r_n,u_1) & B_1K(r_n,u_1) & \cdots & 1+\lambda B_nK(r_n,u_n)\end{bmatrix}$$

$$y=\begin{bmatrix}y(r_1)\\ y(r_2)\\ \vdots\\ y(r_n)\end{bmatrix}\quad F=\begin{bmatrix}\mathrm{F}(r_1)\\ \mathrm{F}(r_2)\\ \vdots\\ \mathrm{F}(r_n)\end{bmatrix},\ \lambda=1/q$$

It is noted that for simplification the infinite series $K(r,\mathrm{u})$ and free term $\mathrm{F}(r)$ appeared in (13) can be treated numerically by approximated $K(r,\mathrm{u})$ such that

$$\tilde{K}(r,u) \approx \frac{2}{R_2{}^2} \sum_{n=1}^{N} \frac{J_0(\lambda_n u)J_{-1/2}(\lambda_n \mathrm{r})}{J_1^2(\lambda_n R_2)} g_n$$

Without loss of generality if we consider $f_1(r) = \exp(-\mathrm{r}^2), f_2(r) = 0, \mathrm{R}_1 = 1$ in the equality

(17) Calculation will be more simple moreover, the functional series $K(r,\mathrm{u})$ converges for any values r,u and r_1 .Notice that as $R_2 \to \infty$, the dual series (5), (6) reduced to dual integral equations of the form

$$\int_0^\infty A(\lambda)J_0(\lambda r)(\mathrm{a}_1\sqrt{\lambda^2+k^2}+b_1)d\lambda = f_1(r),\ r \in (0,R_1)\quad \int_0^\infty A(\lambda)J_0(\lambda r)(\mathrm{a}_2\sqrt{\lambda^2+k^2}+b_2)d\lambda = f_2(r),\ r \in (R_1,\infty)$$

The solution of the above dual integral equations was discussed with details in [4].

4. Conclusion

The obtained results the conclusion can be drown that the paper aim an analytical method the mixed boundary value problem with boundary conditions of the third kind will lead to study dual series equations. The use of a Hankel integral transform always reduces to inhomogeneous Fredholm integral equation of the second kind which is treated numerically by using simple numerical method (Simpson's rule of integration). It can be revealed that there is no approximation before the numerical evaluation of the integral equation solution. The present results can be served other investigations mixed problems, in particular for mixed boundary condition of the third kind with various applications and coordinate systems.

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