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Research Paper

A new formulation for the linearized Navier-Stokes equation

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Abstract

This paper is devoted to study the Navier-Stokes equations by applying the curl and using a current function, we obtain a non-linear biharmonic problem where the pressure disappears and instead of the velocity, we are working with a scalar function. After a linearization, we obtain a sequence of linear problems. We study the existence and uniqueness of its solutions. Finally we show the convergence of the sequence of the linearized problems obtained to the non-linear one.

Keywords: *Bi-Laplacian, Existence and uniqueness, Navier-Stokes equations.*

1. Introduction

We consider the Navier-Stokes problem:

$$(P)\begin{cases} -\nu\Delta u + (u\nabla)u + \nabla P = f & \text{in} \quad \Omega, \\ div\ u = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \Gamma, \end{cases}$$

where Ω is a bounded and connected domain in $\mathbb{R}^2$ with lipschitz boundary $\Gamma = \partial\Omega$,
$\boldsymbol{u}$ the velocity and $\boldsymbol{p}$ the pressure.
ν is a positive parameter called kinematic viscosity and the corresponding function $\boldsymbol{f}$ forces applied to the fluid is given.
After the application of the curl and using a current function, we obtain a non-linear biharmonic problem.
The variational formulation of the Navier-Stokes equations in the classic form is well studied in [2], [4], and [6]. A discretisation by Finite Element Methods of the problem is proposed by [4].

The standard discretization of the Stokes and Navier-Stokes equations in vorticity and stream function formulation by affine finite elements is known for its bad convergence. Amara.M and Bernardi.C in [1] present a modified discretization, they prove that the convergence is improved and they establish a priori error estimates.

The outline of the paper is as follows:

In the second Section, we are concerned with the bi-harmonic equation by applying the curl and using a current function.

In Section 3, we study the sequence of linearized problems. We show the existence and uniqueness of their solutions .

In Section 4, we show the convergence of the sequence of solutions of the linearized problems obtained to the non-linear one.

In section 5, we demonstrated the linear convergence.

2. Application of rotational

We have $div\ u = 0$, then it can be written in the form $u = curl\ \phi$ where ϕ is a scalar function called fairly regular stream function.

$$\begin{cases} u_1 = \dfrac{\partial \phi}{\partial y}, \\ u_2 = -\dfrac{\partial \phi}{\partial x}. \end{cases}$$

Then by applying the curl to our problem (P), we will have:

$curl\,\Delta u = -\Delta^2 \phi,$

$curl\,(\nabla P) = 0,$

$curl((u.\nabla)u) = \dfrac{\partial}{\partial x}(u_1 \dfrac{\partial u_2}{\partial x} + u_2 \dfrac{\partial u_2}{\partial y}) - \dfrac{\partial}{\partial y}(u_1 \dfrac{\partial u_1}{\partial x} + u_2 \dfrac{\partial u_1}{\partial y}),$

since

$div\, u = 0 \qquad$ this implies that $\qquad \dfrac{\partial u_1}{\partial x} = -\dfrac{\partial u_2}{\partial y},$

and then, we obtain:

$curl((u.\nabla)u) = -\dfrac{\partial \phi}{\partial y}\dfrac{\partial \Delta\phi}{\partial x} + \dfrac{\partial \phi}{\partial x}\dfrac{\partial \Delta\phi}{\partial y}.$

The equation becomes

$\nu\Delta^2\phi - \dfrac{\partial \phi}{\partial y}\dfrac{\partial \Delta\phi}{\partial x} + \dfrac{\partial \phi}{\partial x}\dfrac{\partial \Delta\phi}{\partial y} = curl\, f,$

and we have the following problem

$$(Q)\begin{cases} \nu\Delta^2\phi - \dfrac{\partial \phi}{\partial y}\dfrac{\partial \Delta\phi}{\partial x} + \dfrac{\partial \phi}{\partial x}\dfrac{\partial \Delta\phi}{\partial y} = curl\, f & \text{in} \quad \Omega, \\ \phi = 0 & \text{on} \quad \Gamma, \\ \dfrac{\partial \phi}{\partial x} = \dfrac{\partial \phi}{\partial y} = 0 & \text{on} \quad \Gamma. \end{cases}$$

A linearization gives:

$\nu\Delta^2\phi_{n+1} - \dfrac{\partial \phi_n}{\partial y}\dfrac{\partial \Delta\phi_{n+1}}{\partial x} + \dfrac{\partial \phi_n}{\partial x}\dfrac{\partial \Delta\phi_{n+1}}{\partial y} = curl\, f.$

We set

$$c_n = \frac{\partial \phi_n}{\partial y}\ ,\ d_n = \frac{\partial \phi_n}{\partial x},$$

we have:

$$\nu\Delta^2\phi_{n+1} - c_n\frac{\partial\Delta\phi_{n+1}}{\partial x} + d_n\frac{\partial\Delta\phi_{n+1}}{\partial y} = curl\, f.$$

Therefore, our problem is:

$$(Q_{n+1})\begin{cases}\nu\Delta^2\phi_{n+1} - c_n\frac{\partial}{\partial x}\Delta\phi_{n+1} + d_n\frac{\partial}{\partial y}\Delta\phi_{n+1} = curl\, f & \text{in} \quad \Omega,\\ \phi_{n+1} = 0 & \text{on} \quad \Gamma,\\ \frac{\partial\phi_{n+1}}{\partial x} = \frac{\partial\phi_{n+1}}{\partial y} = 0 & \text{on} \quad \Gamma,\end{cases}$$

3. Variational formulation

We multiply both sides of the first equation of (Q_{n+1}) by a test function $v \in V = H_0^2(\Omega)$ and integrating over Ω, we have the following variational problem:

$$(QV)_{n+1}\begin{cases}\text{Find } \phi_{n+1} \in V \text{ such as}\\ a(\phi_{n+1}, v) = L(v), \quad \forall\, v \in V,\end{cases}$$

where $a(.,.)$ is a bilinear form on $V \times V$ given by:

$$a(u, v) = a_0(u, v) + a_n(u, v), \tag{1}$$

where

$$a_0(u, v) = \nu\int_\Omega \Delta u\Delta v dX, \tag{2}$$

$$a_n(u, v) = \int_\Omega \Delta u\left(\frac{\partial\phi_n}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial\phi_n}{\partial x}\frac{\partial v}{\partial y}\right)dX, \tag{3}$$

and $L(.)$ is a linear form on V defined by the following expression:

$$L(v) = \int_\Omega curl\, f\ v\ dX. \tag{4}$$

For the existence and uniqueness of the solution, we need this lemma:

Lemma 3.1. *[5], $\forall\, u \in H^2(\Omega) \cap H_0^1(\Omega)$, there exist $c'(\Omega) > 0$ such that*

$$\|u\|_{H^2}^2 \le c'(\Omega)\|\Delta u\|_2^2 \le c^*\|\Delta u\|_2^2. \tag{5}$$

where c^* will be chosen later.

Theorem 3.2. *For $f \in H^1(\Omega)$ and $curl\, f$ small enough: $\|curl\, f\|_2 < c(\nu)$,the problem $(QV)_{n+1}$ has a unique solution $\phi_{n+1} \in V$.*

Proof. 1. **Continuity of a:**

We show the continuity and coercivity of the bilinear form $a(.,.)$.
We put $u = \phi_{n+1}$ and we deal with each term separately, we begin by:

$$\begin{aligned}|a_0(u, v)| &\le \nu\|\Delta u\|_2\|\Delta v\|_2,\\ &\le \nu\|u\|_{H^2}\|v\|_{H^2}.\end{aligned} \tag{6}$$

And using the continuous injection of $H^1(\Omega)$ in $L^4(\Omega)$, $a_n(.,.)$ will be bounded as bellow

$$\begin{aligned}|a_n(u, v)| &= \left|\int_\Omega \Delta u(\nabla v \wedge \nabla\phi_n)dX\right|,\\ &\le \|\Delta u\|_2\|\nabla v\|_{L^4}\|\nabla\phi_n\|_{L^4},\\ &\le c\|\Delta u\|_2\|\nabla v\|_{H^1}\|\nabla\phi_n\|_{H^1},\\ &\le c\|\phi_n\|_{H^2}\|u\|_{H^2}\|v\|_{H^2}.\end{aligned} \tag{7}$$

Therefore,

$$|a(u,v)| \leq \mathcal{C}_n \|u\|_{H^2} \|v\|_{H^2}, \tag{8}$$

where $\mathcal{C}_n$ is a constant which depends on n given by:

$$\mathcal{C}_n = \nu + c\|\phi_n\|_{H^2}.$$

This implies that, for each fixed n, $a(.,.)$ is continuous on V.

2. **Coercivity of a:**
We have:

$$a_0(u,u) = \nu\|\Delta u\|_2^2, \tag{9}$$

and taking $v = u$ in (7), we obtain:

$$|a_n(u,u)| \leq c\|\phi_n\|_{H^2}\|u\|_{H^2}^2, \tag{10}$$

then using (5)

$$\begin{aligned} a(u,u) &\geq \nu\|\Delta u\|_2^2 - c\|u\|_{H^2}^2\|\phi_n\|_{H^2}, \\ &\geq \frac{\nu}{c'(\Omega)}\|u\|_{H^2}^2 - c\|u\|_{H^2}^2\|\phi_n\|_{H^2}, \\ &\geq (\frac{\nu}{c'(\Omega)} - c\|\phi_n\|_{H^2})\|u\|_{H^2}^2. \end{aligned} \tag{11}$$

To get the coercivity, we should have:

$$\frac{\nu}{c'(\Omega)} - c\|\phi_n\|_{H^2} > 0, \qquad \forall\, n \in \mathbb{N}, \tag{12}$$

which means that

$$\|\phi_n\|_{H^2} < \frac{\nu}{c'(\Omega)c}, \qquad \forall\, n \in \mathbb{N}. \tag{13}$$

Let

$$\alpha^* = \frac{\nu}{2c'(\Omega)c}. \tag{14}$$

We take $\phi_0 \in B_{\alpha^*}$ where $B_\alpha = \{v \in H_0^2(\Omega); \|v\|_{H^2} \leq \alpha\}$ and assume that $\phi_n \in B_{\alpha^*}$, we must show by induction that: $\phi_{n+1} \in B_{\alpha^*}$.
Indeed, we have , if we put $u = \phi_{n+1}$:

$$a(u,u) = L(u) = \int_\Omega curl\, f\; u\; dX,$$

$$a(u,u) \leq \|curl\, f\|_2\|u\|_2, \tag{15}$$

then

$$(\frac{\nu}{c'(\Omega)} - c\|\phi_n\|_{H^2})\|u\|_{H^2}^2 \leq \|curl\, f\|_2\|u\|_{H^2},$$

and

$$(\frac{\nu}{c'(\Omega)} - c\|\phi_n\|_{H^2})\|u\|_{H^2} \leq \|curl\, f\|_{L^2}, \tag{16}$$

$$\|u\|_{H^2} \leq \frac{\|curl\, f\|_2}{\frac{\nu}{c'(\Omega)} - c\|\phi_n\|_{H^2}}.$$

If we assume that:

$$\|curl\, f\|_2 < \frac{\nu^2}{4c'(\Omega)^2 c},$$

then we have :

$$\|u\|_{H^2} \leq \frac{\|curl\, f\|_2}{\frac{\nu}{c'(\Omega)} - c\alpha^*} \leq \alpha^*,$$

which implies that $a(.,.)$ is coercive on V.

3. **Continuity of L:**
In the other hand, the linear form L is continuous:

$$\|L(v)\|_2 \leq \|curl\, f\|_2 \, \|v\|_2. \tag{17}$$

Then, using Lax-Milgram Theorem, the problem $(QV)_{n+1}$ has a unique solution $\phi_{n+1} \in V$.

□

4. Convergence of the sequence

The sequence $(\phi_n)_{n\in\mathbb{N}}$ obtained in the preceding section verifies:

$$\|\phi_n\|_V \leq \alpha^*, \quad \forall\, n \geq 0,$$

which implies that the sequence $(\phi_n)_{n\in\mathbb{N}}$ is bounded in V.
Then there exist a subsequence that converges weakly to w in V.
Since the injection of V in $H_0^1(\Omega)$ is continuous, there exists a subsequence still noted ϕ_n which converges strongly to w in $H_0^1(\Omega)$.
For the convergence, we need this regularity result:

Lemma 4.1. *Assume that Ω is of class $\mathcal{C}^2$ and $\phi_0 \in H_0^2(\Omega) \cap H^3(\Omega)$, we have:*

$$\forall\, n \in \mathbb{N}, \quad \phi_{n+1} \in H_0^2(\Omega) \cap H^3(\Omega).$$

Proof. The problem (Q_{n+1}) can be written as

$$\begin{cases} -\nu\Delta\omega_{n+1} + c_n\dfrac{\partial \omega_{n+1}}{\partial x} - d_n\dfrac{\partial \omega_{n+1}}{\partial y} = -curl\, f \qquad \text{in } \Omega, \qquad (1) \\ \Delta\phi_{n+1} = \omega_{n+1} \qquad \text{in } \Omega, \qquad (2) \\ \phi_{n+1} \in H_0^2(\Omega). \end{cases}$$

The variational formulation of (1) is

$$\begin{cases} \text{Find } \omega_{n+1} \in H_0^1(\Omega) \text{ such as} \\ A_n(\omega_{n+1}, v) = l(v), \ \ \forall\, v \in H_0^1(\Omega), \end{cases}$$

where

$$A_n(\omega_{n+1}, v) = \nu \int_\Omega \nabla\omega_{n+1}\, \nabla v \, dX + \int_\Omega \omega_{n+1}\, (d_n \frac{\partial v}{\partial y} - c_n \frac{\partial v}{\partial x})\, dX,$$

$$l(v) = -\int_\Omega curl\, f \ v \, dX.$$

For the coercivity of A_n, we have:

$$\begin{aligned} A_n(v,v) &= \nu \int_\Omega |\nabla v|^2 \, dX + \int_\Omega v\,(\nabla v \wedge \nabla \omega_n)\, dX \\ &\geq \nu \, \|\nabla v\,\|_2^2 - c\|\nabla v\|_2 \|v\|_{H^1} \|\nabla \omega_n\|_{H^1} \\ &\geq \nu c_1 \, \|v\,\|_{H^1}^2 - c\alpha^* \|v\|_{H^1}^2 . \end{aligned}$$

It suffices to choose in inequality (5): $c^* > \dfrac{1}{2c_1}$.
And by Lax-Milgram Theorem, we have $\omega_{n+1} \in H^1(\Omega)$.

The theory of regularity for weak solutions of the laplace problem applied to the variational formulation of (2) gives $\phi_{n+1} \in H^3(\Omega)$. □

Then, we have this result of convergence :

Lemma 4.2. *1. We have,* $\forall v \in H_0^2(\Omega) \cap H^3(\Omega)$*:*

$$\lim_{n \to +\infty} a_0(\phi_{n+1}, v) = a_0(w, v).$$

2. We have, $\forall v \in H_0^2(\Omega) \cap H^3(\Omega)$*:*

$$\lim_{n \to +\infty} a_n(\phi_{n+1}, v) = a_\infty(w, v) = \int_\Omega \Delta w \,(\nabla v \wedge \nabla w)\, dX.$$

Proof. 1. We have:

$$\begin{aligned} |a_0(\phi_{n+1}, v) - a_0(w, v)| &= |\nu \int_\Omega \Delta(\phi_{n+1} - w)\, \Delta v \; dX| \\ &\leq \nu \|\nabla(\phi_{n+1} - w)\|_2 \|v\|_{H^3}, \end{aligned}$$

then, we obtain the result.

2. On the other hand, we have:

$$|a_n(\phi_{n+1}, v) - a_\infty(w, v)| = T_{1_n} + T_{2_n}$$

where

$$T_{1_n} = |\int_\Omega (\Delta\phi_{n+1} - \Delta w)(\nabla v \wedge \nabla \phi_n) dX|,$$

and

$$T_{2_n} = |\int_\Omega \Delta w \cdot [\nabla v \wedge (\nabla \phi_n - \nabla w)] dX|$$

By Green formula we have :

$$T_{1_n} = |\int_\Omega (\nabla\phi_{n+1} - \nabla w)\; \nabla(\nabla v \wedge \nabla \phi_n) dX|$$

which gives :

$$T_{1_n} \leq \sum_{i,j=1}^{2} |\int_\Omega \frac{\partial(\phi_{n+1} - \omega)}{\partial x_i} \frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial(\phi_n)}{\partial x_j} dX| + \sum_{i,j=1}^{2} |\int_\Omega \frac{\partial(\phi_{n+1} - \omega)}{\partial x_i} \frac{\partial v}{\partial x_j} \frac{\partial^2(\phi_n)}{\partial x_i \partial x_j} dX|.$$

and

$$T_{2_n} \leq \sum_{i,j=1}^{2} |\int_\Omega \frac{\partial(\phi_n - \omega)}{\partial x_i} \frac{\partial^2 \omega}{\partial x_i \partial x_j} \frac{\partial v}{\partial x_j} dX|$$

According to the Sobolev imbedding Theorem the space $H^1(\Omega)$ is continuously imbedded in $L^4(\Omega)$ for $n=2$.

Then by the $H\ddot{o}lder$'s inequality we have for $\omega, v, \phi_n \in H_0^2(\Omega)\cap H^3(\Omega)$:

$$\frac{\partial \omega}{\partial x_i}\frac{\partial^2 v}{\partial x_i \partial x_j}\frac{\partial(\phi_{n+1}-\omega)}{\partial x_j} \in L^1(\Omega), \quad 1 \le i,j \le 2,$$

with for T_{1n}

$$\begin{aligned}|\int_\Omega \frac{\partial \phi_n}{\partial x_i}\frac{\partial^2 v}{\partial x_i \partial x_j}\frac{\partial(\phi_{n+1}-\omega)}{\partial x_j}dX| &\le \|\frac{\partial^2 v}{\partial x_i \partial x_j}\|_4\|\frac{\partial \phi_n}{\partial x_i}\|_4\|\frac{\partial(\phi_{n+1}-\omega)}{\partial x_j}\|_2\\ &\le \mathcal{C}\|\frac{\partial^2 v}{\partial x_i \partial x_j}\|_{H^1}\|\frac{\partial \phi_n}{\partial x_i}\|_{H^1}\|\frac{\partial(\phi_{n+1}-\omega)}{\partial x_j}\|_2\\ &\le \mathcal{C}\|v\|_{H^2}\|\phi_n\|_{H^2}\|\frac{\partial(\phi_n-\omega)}{\partial x_j}\|_2,\end{aligned}$$

and

$$\begin{aligned}|\int_\Omega \frac{\partial^2 \phi_n}{\partial x_i \partial x_j}\frac{\partial v}{\partial x_j}\frac{\partial(\phi_{n+1}-\omega)}{\partial x_i}dX| &\le \|\frac{\partial^2 \phi_n}{\partial x_i \partial x_j}\|_4\|\frac{\partial v}{\partial x_j}\|_4\|\frac{\partial(\phi_{n+1}-\omega)}{\partial x_i}\|_2\\ &\le \mathcal{C}\|\frac{\partial^2 \phi_n}{\partial x_i \partial x_j}\|_{H^1}\|\frac{\partial v}{\partial x_j}\|_{H^1}\|\frac{\partial(\phi_{n+1}-\omega)}{\partial x_i}\|_2\\ &\le \mathcal{C}\|v\|_{H^2}\|\phi_n\|_{H^2}\|\frac{\partial(\phi_{n+1}-\omega)}{\partial x_i}\|_2,\end{aligned}$$

and for T_{2n} we have

$$\begin{aligned}|\int_\Omega \frac{\partial^2 \omega}{\partial x_i \partial x_j}\frac{\partial v}{\partial x_j}\frac{\partial(\phi_n-\omega)}{\partial x_i}dX| &\le \|\frac{\partial^2 \omega}{\partial x_i \partial x_j}\|_4\|\frac{\partial v}{\partial x_j}\|_4\|\frac{\partial(\phi_n-\omega)}{\partial x_i}\|_2\\ &\le \mathcal{C}\|\frac{\partial^2 \omega}{\partial x_i \partial x_j}\|_{H^1}\|\frac{\partial v}{\partial x_j}\|_{H^1}\|\frac{\partial(\phi_n-\omega)}{\partial x_i}\|_2\\ &\le \mathcal{C}\|v\|_{H^2}\|\omega\|_{H^2}\|\frac{\partial(\phi_n-\omega)}{\partial x_i}\|_2,\end{aligned}$$

Then, by the strongly convergence in $H_0^1(\Omega)$, we will have

$$\lim_{n\to+\infty} a_n(\phi_{n+1},v) = a_\infty(w,v).$$

□

Proposition 1. *w is a solution of Q.*

Proof. It follows from Lemma (4.1) that:

$$\lim_{n\to+\infty} a_0(\phi_{n+1},v) + a_n(\phi_{n+1},v) = a_0(w,v) + a_\infty(w,v) = L(v). \tag{18}$$

Which gives:

$$\nu\int_\Omega \Delta w\,\Delta v dX + \int_\Omega \Delta w\,(\nabla v \wedge \nabla w)dX = \int_\Omega curl\,f\;vdX, \tag{19}$$

then

$$\int_\Omega (\nu\Delta^2 w - \frac{\partial w}{\partial y}\frac{\partial \Delta w}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \Delta w}{\partial y} - curl\,f)vdX = 0, \qquad \forall v \in H_0^2(\Omega), \tag{20}$$

then

$$\nu\Delta^2 w - \frac{\partial w}{\partial y}\frac{\partial \Delta w}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \Delta w}{\partial y} = curl\,f, \qquad \forall w \in H_0^2(\Omega). \tag{21}$$

So we can conclude that w is the solution of Q. □

5. Linear convergence

For this part, we set

$$w_{n+1} = \phi_{n+1} - w,$$

we have the following result:

Proposition 2. *For $w_{n+1} \in H^2(\Omega)$, there exist a constant C such that:*

$$\|w_{n+1}\|_{H^2} \leq C\|w_n\|_{H^2}. \tag{22}$$

Proof. Taking the difference between the problem (Q_{n+1}) and the problem (Q), we have:

$$\nu\Delta^2 w_{n+1} - \frac{\partial\phi_n}{\partial y}\frac{\partial(\Delta w_{n+1})}{\partial x} + \frac{\partial\phi_n}{\partial x}\frac{\partial(\Delta w_{n+1})}{\partial y} = F_n, \tag{23}$$

where

$$F_n = \frac{\partial(\Delta w)}{\partial x}\frac{\partial w_n}{\partial y} - \frac{\partial(\Delta w)}{\partial y}\frac{\partial(w_n)}{\partial x}. \tag{24}$$

The variational formulation gives us:

$$\begin{aligned}
\beta\|w_{n+1}\|^2_{H^2} \leq a(w_{n+1}, w_{n+1}) &\leq \int_\Omega |F_n| w_{n+1} dX, \\
&= \int_\Omega |\Delta w(\nabla w_n \wedge \nabla w_{n+1})| dX, \\
&\leq \|\Delta w\|_2 \|\nabla w_n\|_4 \|\nabla w_{n+1}\|_4, \\
&\leq c\|\Delta w\|_2 \|\nabla w_n\|_{H^1} \|\nabla w_{n+1}\|_{H^1}, \\
&\leq c'\|w_n\|_{H^2}\|w_{n+1}\|_{H^2},
\end{aligned}$$

then, we obtain:

$$\|w_{n+1}\|_{H^2} \leq C\|w_n\|_{H^2}, \tag{25}$$

where C is given by:

$$C = \frac{c'}{\beta}, \tag{26}$$

and β the constant of the coercivity.
Which implies the linear convergence. □

6. Conclusion

In this paper, we studied the Navier-Stokes equations by applying the curl and using a current function through the application of rotational, we obtained a non-linear biharmonic problem.
After a linearization, we proved the existence and uniqueness of weak solution of the variational formulation using Lax-Milgram Theorem and which we can compute by finite element method.
And in a second part we showed the convergence of the sequence as well as the linear convergence.

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References

[1] Amara.M, Bernardi.C, "Convergence Of A Finite Element Discretization of The Navier-Stokes Equations In Vorticity And Stream Function Formulation", *M2AN*, Vol.33, No.5, (1999), pp.1033-1056.

[2] Amara.M, Capatina-Papaghiuc.D, Chacon-Vera.E, Trijullo.D, "Vorticity velocity pressure formulation for Navier-Stokes equations", *Comput. Vis. Sci*, No.6, (2004), pp.47-52.

[3] Bernardi.C, Mady.Y, Rapetti.F, "Discretisations variationnelles de problmes aux limites elliptiques", *Springer-Verlag Berlin Heidelberg*, (2004).

[4] Girault.V, Raviart.P-A, "Finite Element Methods for the Navier-Stokes Equations, Theory and Algorithms", *Springer-Verlag*, (1986).

[5] Ndlec.J-C, "Cours d'Analyse Numrique", *Ecole Nationale de Ponts et Chausses.*

[6] Temam.R, "Navier-Stokes Equations, Theory and Numerical Analysis", *North-Holland, Amsterdam*, (1984).