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Research paper

Solving 2^{nd} Order Nonlinear Differential Equations Using Piecewise Analytic Method (Pendulum Equations)

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Abstract

In this paper, piecewise analytic method (PAM) is used for solving highly nonlinear 2^{nd} order differential equation (pendulum equations) which is a big problem for engineers and scientists. PAM is used for showing the nonlinear dynamics of the solution with and without linearizion. The error and accuracy of the solution are controlled easily according to our needs.

Keywords: *Nonlinear Differential Equation; Padé Approximants; Piecewise Analytic Method; Runge-Kutta Method; Pendulum Equations.*

1. Introduction

Nonlinear dynamics is the study of time-evolving systems governed by nonlinear equations where superposition fails. General solutions of nonlinear dynamics are rarely obtainable, numerous analytical and numerical techniques have been developed to analyze such dynamical systems ([1]-[3]). The piecewise analytic method (PAM) is a new technique which helps in showing the nonlinear dynamics and solutions of highly nonlinear differential equation.

In real practice, scientists and engineers search for method that can give an excellent accurate approximate solution with a prescribed precision. In section 4 it is shown how PAM can control the error and give solution with a prescribed precision. PAM is used for solving 1^{st} order nonlinear differential equations ([4]-[8]).

In this paper, The pendulum equation is used as an example of 2^{nd} order nonlinear differential equation which is solved using PAM. The pendulum systems are classical models of nonlinear dynamics, which permit to select different important nonlinear effects. The nonlinearity makes the pendulum equations very difficult to solve analytically. In spite of starting the research on pendulum from over 100 years ago, principal analytic result were not obtained and are still used to describe variety of physical and engineering applications([9]-[15]).

2. Piecwise Analytic Method

The main steps of PAM is shown in Fig 1. For solving the general 2nd order initial value problem

$$u'' = \phi(t,u,u'), \qquad u(t_0) = f_0, \qquad u'(t_0) = f_1, \qquad t \geq t_0. \tag{1}$$

A general approximate analytical solution for each subinterval which is named by $U_m(t)$ where $m = 0,1,2,...,n,...$ is obtained by rewriting equation (1) in the form

$$\frac{d^2U_m(t)}{dt^2} = \phi(t,U_m,U_m'), \qquad U_m(t_m) = f_{m,0},$$
$$\frac{dU_m}{dt}(t_m) = f_{m,1}, \qquad t \in [t_m, t_{m+1}], \qquad m = 0,1,2,...,(n-1). \tag{2}$$

where

$$U_m(t) = \sum_{n=0}^{s} K_{m,n}(t-t_m)^n, \tag{3}$$

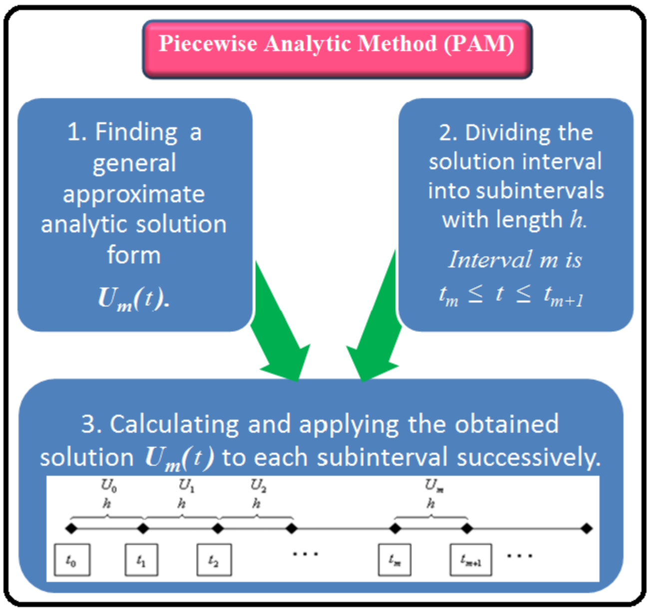


Figure 1: The piecewise analytic method (PAM).

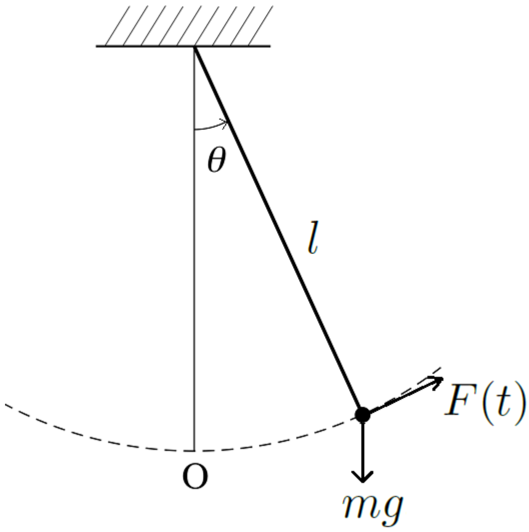


Figure 2: Simple Pendulum.

and s is the order of PAM. Substituting by(3) into (2) and equating the coefficients of each power of $(t-t_m)^n$ to zero to get a successive relations that express the coefficient $K_{m,i}$ in terms of the coefficients $K_{m,a}$ where $a < i$. At the end of this step the solution $U_m(t)$ (equation 3) can be rewritten in the function form

$$U_m(t) = g(t, t_m, K_{m,0}, K_{m,1}), \qquad t \in [t_m, t_{m+1}] \tag{4}$$

Now, the general approximate analytical solution $U_m(t)$ for each subinterval is obtained and we are ready to the final step which is the numerical step. In this step, $U_m(t)$ (equation 4) is calculated successively for each subinterval based on $K_{m,0} = f_{m,0} = U_{m-1}(t_m)$ where $U_{-1}(t_0) = f_0$ and $K_{m,1} = f_{m,1} = \frac{dU_{m-1}}{dt}(t_m)$ where $\frac{dU_{-1}}{dt}(t_0) = f_1$.

Note, There is another PAM solution which can be used instead of the series solution (3). For more details see ([4]-[6]).

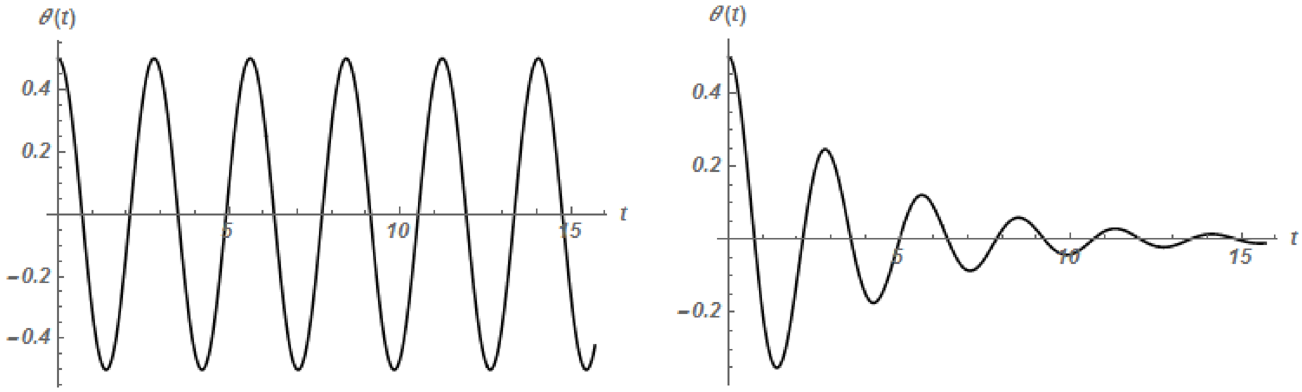
3. Case Studies

The pendulum systems permit to select different important nonlinear effects. For example, a simple pendulum bob of mass m at the end of a weightless rod that has a fixed length l at an angle θ to the vertical, figure 2. Newton's second law gives

mass . acceleration = sum of forces acting on the bob

which leads to the following driving equation of motion of simple pendulum [9]

$$ml\theta'' = -mgsin\theta - bl\theta' + F(t). \tag{5}$$



(a) The undambed case ($b = 0$).

(b) The dambed case ($\frac{b}{m} = 0.5$).

Figure 3: The PAM solution of the linear case (equation 6) using $h = 0.01$, $s = 8$, $\frac{g}{l} = 5$, $f_0 = 0.5$ and $f_1 = 0$.

where g is the gravity acceleration, $F(t)$ is a periodic external force pushes on the bob and $bl\theta'$ is a friction force resistance. Equation (5) is the general second order nonlinear differential equation of simple pendulum. Studies deal with equation (5) in different forms. It is formed as linear by invoking the small angle approximation $sin(\theta) \simeq \theta$ for small $\theta \leq 1$, nonlinear, damped ($b \neq 0$), undamped ($b = 0$) , forced ($F(t) \neq 0$) or unforced ($F(t) = 0$) which will be studied in the following subsections.

3.1. Linear Case

Rewriting equation (5) in dimensionless linear unforced form

$$\theta'' + \frac{b}{m}\theta' + \frac{g}{l}\theta = 0, \qquad \theta(t_0) = f_0, \quad \frac{d\theta}{dt}(t_0) = f_1. \tag{6}$$

The replacement ($sin(\theta) = \theta$) leads to solving equation 5 easily, but also, leads to throwing out some of the physics, like motions where the pendulum whirls, we can see the effect of linearity in the following subsection. This linear case is the unique case which has an exact solution.

Following the procedures of PAM in section 2, a system of equations in $K_{m,n}$, where $n \leq s$, are obtained. Solving these equations recursively leads to the following relations in terms of $K_{m,0}$ and $K_{m,1}$:

$$\begin{aligned}
K_{m,2} &= -\frac{2K_{m,0}g + bK_{m,1}l}{4l}, \\
K_{m,3} &= -\frac{-2bgK_{m,0} - (b^2 l - 4g)K_{m,1}}{24l}, \\
K_{m,4} &= -\frac{(2b^2 gl - 8g^2)K_{m,0} - (8bgl - b^3 l^2)K_{m,1}}{192l^2}, \\
K_{m,5} &= -\frac{(-2b^3 gl + 16bg^2)K_{m,0} + (12b^2 gl - 16g^2 - b^4 l^2)K_{m,1}}{1920l^2}, \\
K_{m,6} &= -\frac{(2b^4 gl^2 - 24b^2 g^2 l + 32g^3)K_{m,0} + (b^5 l^3 - 16b^3 gl^2 + 48bg^2 l)K_{m,1}}{23040l^3}, \\
K_{m,7} &= -\frac{(-2b^5 gl^2 + 32b^3 g^2 l - 96bg^3)K_{m,0} - (b^6 l^3 + 20b^4 gl^2 - 96b^2 g^2 l + 64g^3)K_{m,1}}{322560l^3}, \\
&\vdots
\end{aligned} \tag{7}$$

using 7, $\theta_m(t) = \sum_{n=0}^{s} K_{m,n}(t - t_m)^n$ is calculated where $K_{m,0} = f_{m,0} = \theta_{m-1}(t_m)$, $\theta_{-1}(t_0) = f_0$, $K_{m,1} = f_{m,1} = \frac{d\theta_{m-1}}{dt}(t_m)$ and $\frac{d\theta_{-1}}{dt}(t_0) = f_1$. Figure 3 shows the PAM solution of the linear case (equation 6).

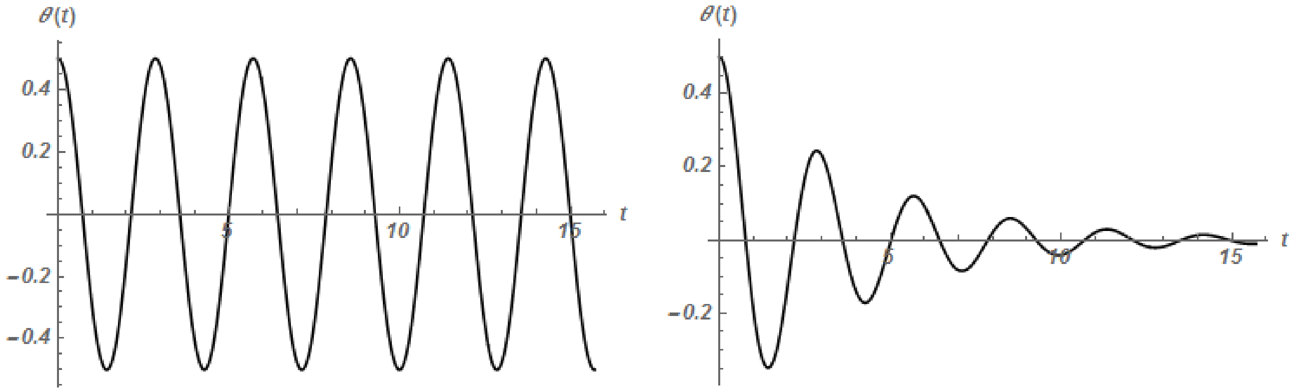
3.2. Nonlinear Case

Considering equation (5) without external force ($F(t) = 0$)

$$\theta'' + \frac{b}{m}\theta' + \frac{g}{l}sin\theta = 0, \qquad \theta(t_0) = f_0, \quad \frac{d\theta}{dt}(t_0) = f_1. \tag{8}$$

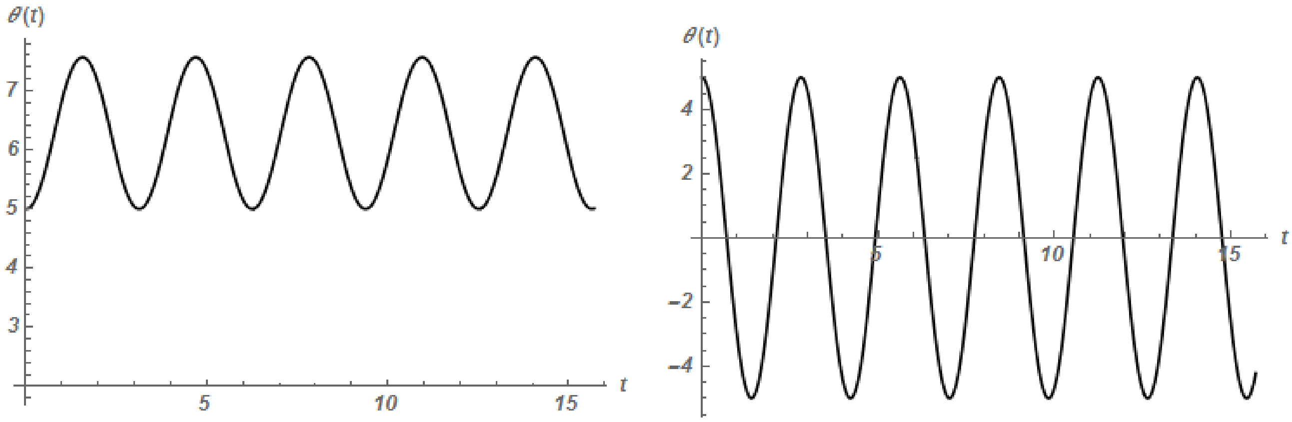
Following the procedures of PAM and solving the system of equations lead to obtain the following successive formula:

$$\begin{aligned}
K_{m,2} &= \frac{1}{4l}(-bK_{m,1}l - 2g\sin(K_{m,0})), \\
K_{m,3} &= \frac{1}{24l}(b^2 K_{m,1}l + 2bg\sin(K_{m,0}) - 4K_{m,1}g\cos(K_{m,0})), \\
K_{m,4} &= \frac{1}{192l^2}(b^3 K_{m,1}\left(-l^2\right) - 2b^2 gl\sin(K_{m,0}) + 8bK_{m,1}gl\cos(K_{m,0}) + 8g^2\sin(K_{m,0})\cos(K_{m,0}) + 8K_{m,1}^2 gl\sin(K_{m,0})), \\
&\vdots
\end{aligned} \tag{9}$$



(a) The undambed case ($b = 0$). **(b)** The dambed case ($\frac{b}{m} = 0.5$).

Figure 4: The PAM solution for the nonlinear case (Equation 8) using $h = 0.01$, $s = 8$, $\frac{g}{l} = 5$, $f_0 = 0.5$ and $f_1 = 0$.



(a) The nonlinear undambed case (equation 8, $b = 0$). **(b)** The linear undambed case (equation 6, $b = 0$).

Figure 5: The PAM solution using $h = 0.01$, $s = 8$, $\frac{g}{l} = 5$, $f_0 = 5$ and $f_1 = 0$.

calculating $\theta_m(t) = \sum_{n=0}^{s} K_{m,n}(t-t_m)^n$ using 9 where $K_{m,0} = f_{m,0} = \theta_{m-1}(t_m)$, $\theta_{-1}(t_0) = f_0$, $K_{m,1} = f_{m,1} = \frac{d\theta_{m-1}}{dt}(t_m)$ and $\frac{d\theta_{-1}}{dt}(t_0) = f_1$. Figure 4 shows the solution for the nonlinear equation 8, damped and undambed, for small θ. It is clear from figure 3 and figure 4 that the results from the linear equation 6 and the nonlinear equation 8 are approximately the same but the results are completely different for large θ. Figures 5,6 and 7 show the effect of linearity on the results which gives results differ from reality of the nonlinear case.

3.3. Forced Case

In this case, equation (5) takes the form

$$\theta'' + \frac{b}{m}\theta' + \frac{g}{l} sin\theta + \frac{1}{ml}F(t) = 0, \qquad \theta(t_0) = f_0, \quad \frac{d\theta}{dt}(t_0) = f_1. \tag{10}$$

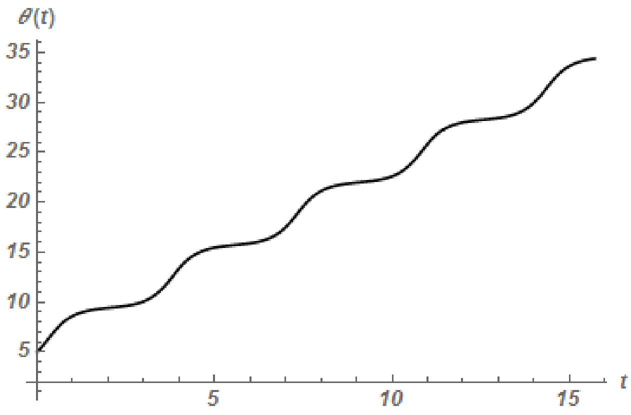
Following the same procedures of PAM and solving the obtained system of equations lead to obtain the following successive formula:

$$\begin{aligned}
K_{m,2} &= \frac{1}{2lm}\left(-bK_{m,1}l - gm\sin\left(K_{m,0}\right) - F\left(-t_m\right)\right),\\
K_{m,3} &= \frac{1}{6lm^2}\left(b^2K_{m,1}l + bgm\sin\left(K_{m,0}\right) + bF\left(-t_m\right) - K_{m,1}gm^2\cos\left(K_{m,0}\right) - mF'\left(-t_m\right)\right),\\
K_{m,4} &= \frac{1}{24l^2m^3}\left(b^3K_{m,1}\left(-l^2\right) - b^2glm\sin\left(K_{m,0}\right) - b^2lF\left(-t_m\right) + 2bK_{m,1}glm^2\cos\left(K_{m,0}\right) + blmF'\left(-t_m\right) + gm^2\cos\left(K_{m,0}\right)F\left(-t_m\right) + \right.\\
&\quad \left. g^2m^3\sin\left(K_{m,0}\right)\cos\left(K_{m,0}\right) + K_{m,1}^2glm^3\sin\left(K_{m,0}\right) - lm^2F''\left(-t_m\right)\right),\\
&\ \vdots
\end{aligned} \tag{11}$$

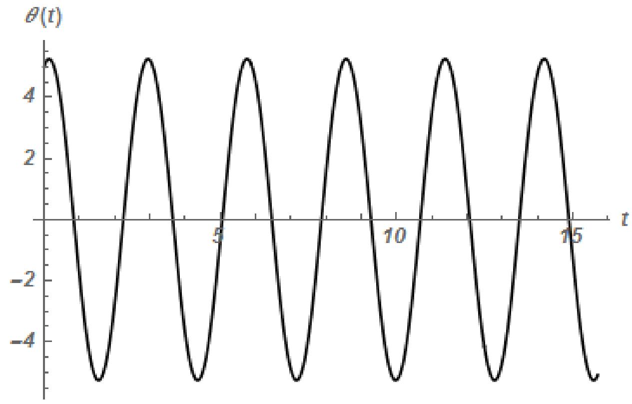
Figure 8 shows the PAM solution $\theta_m(t) = \sum_{n=0}^{s} K_{m,n}(t-t_m)^n$ for the forced nonlinear case (equation 10) where $F(t) = sin(t)$, damped and undamped.

4. Error Estimation

The error in PAM can be controlled by two methods, the first method through changing the solution order s and the second method through changing the subintervals length h. In the limit as h approaches zero and the order s approach infinity, PAM solution approaches the exact

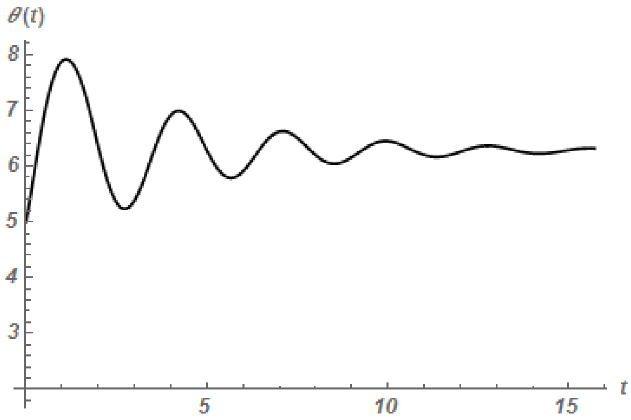


(a) The nonlinear undambed case (equation 8, $b = 0$).

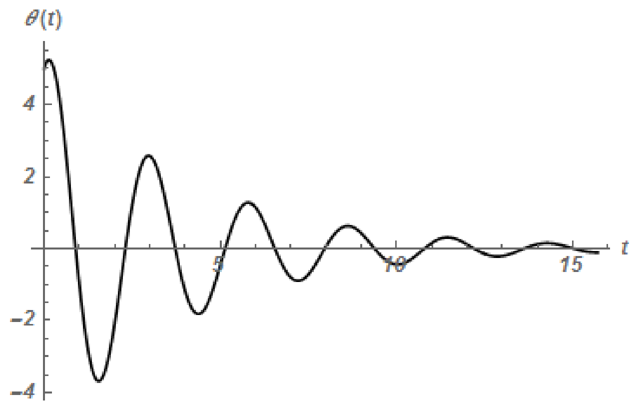


(b) The linear undambed case (equation 6, $b = 0$).

Figure 6: The PAM solution using $h = 0.01$, $s = 8$, $\frac{g}{l} = 5$, $f_0 = 5$ and $f_1 = 3.6$.

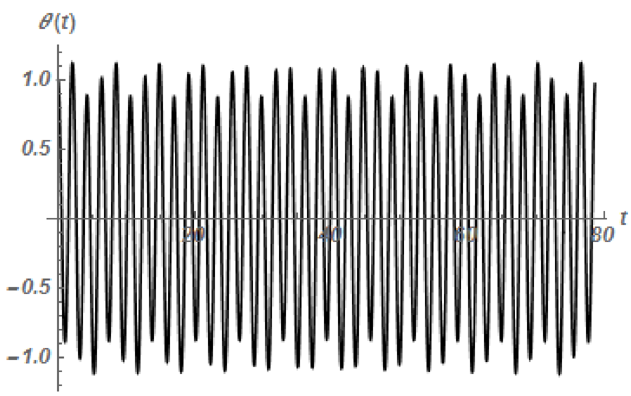


(a) The nonlinear undambed case (equation 8, $\frac{b}{m} = 0.5$).

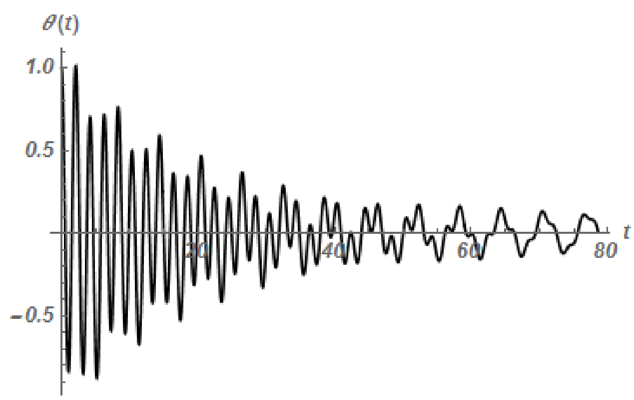


(b) The linear undambed case (equation 6, $\frac{b}{m} = 0.5$).

Figure 7: The PAM solution using $h = 0.01$, $s = 8$, $\frac{g}{l} = 5$, $f_0 = 5$ and $f_1 = 3.6$.

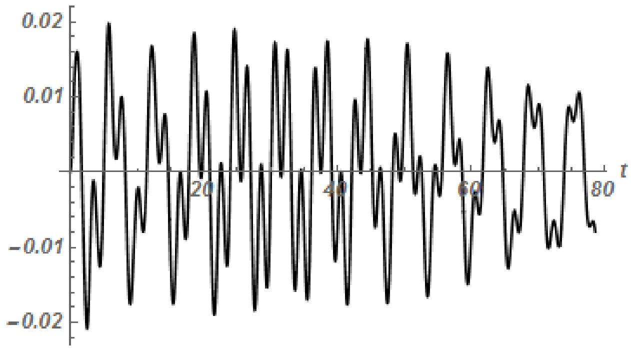


(a) The undambed case ($b = 0$).

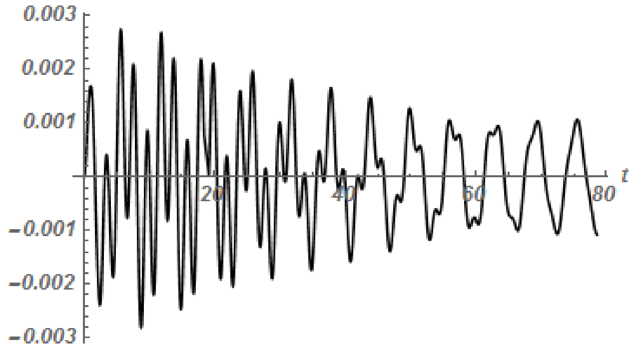


(b) The dambed case ($\frac{b}{m} = 0.1$).

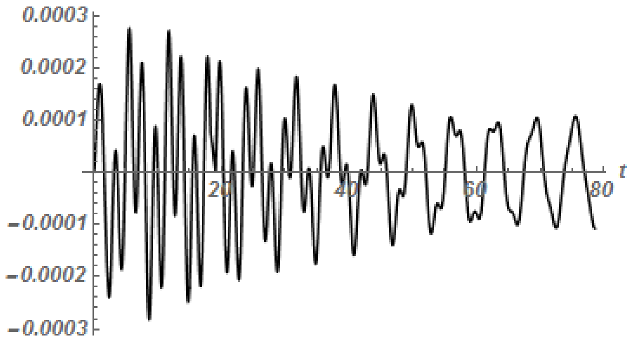
Figure 8: The PAM solution for the nonlinear forced case (equation 10) using $h = 0.01$, $\frac{g}{l} = 5$, $\frac{1}{ml} = 1$, $F(t) = sin(t)$, $s = 8$, $f_0 = 1$ and $f_1 = 0$.



(a) $[U_m^{(h=0.01)}(t) - U_m^{(h=0.1)}(t)]$



(b) $[U_m^{(h=0.001)}(t) - U_m^{(h=0.01)}(t)]$



(c) $[U_m^{(h=0.0001)}(t) - U_m^{(h=0.001)}(t)]$

Figure 9: The effect of changing h on the solution precision (equation 10) using $F(t) = sin(t)$, $s = 5$, $\frac{b}{m} = 0.1$, $\frac{g}{l} = 10$, $f_0 = 1$ and $f_1 = 0$.

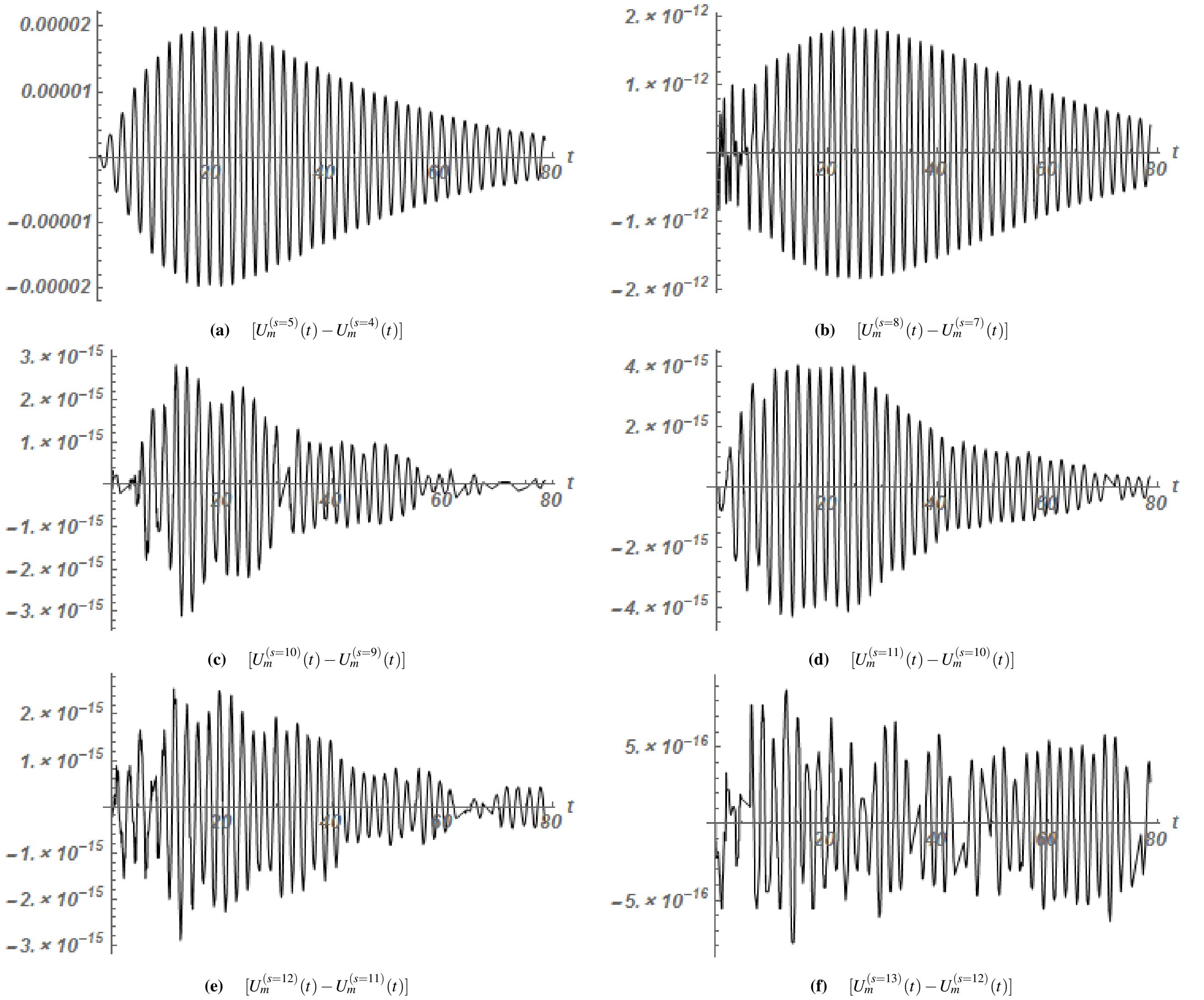
solution. Of course, it does not make sense to apply a zero interval size and infinity order to PAM solutions. Practically, the error in PAM is bounded, controlled and small by selecting the appropriate subinterval size h and the appropriate order of accuracy s.
The error in PAM is often estimated in a posteriori manner as follows. One calculates the solution for subinterval length h and another smaller subinterval length h and takes those figures which are in agreement for the two calculations. For example, if we take the case study equation 10 (forced case), $\theta = 0.1244563$ for $h = 0.1$, $\theta = 0.1244784$ for $h = 0.01$ and $\theta = 0.1244789$ for $h = 0.001$ one can be sure that θ is accurate to 4 decimal points if $h = 0.01$ and θ is accurate to 6 decimal points if $h = 0.001$. Figure 9 shows the difference between two PAM solutions for two different values of h, fixing PAM order at $(s = 4)$, which indicates that the precision is increased as the step size h is reduced. In the same manner, the error can be estimated through changing the order s. Figure 10 shows the difference between two PAM solutions for two different order, fixing the step size $(h = 0.01)$, which indicates that the accuracy is increased as the order is increased.

5. Conclusion

One can see, through the obtained results, that PAM is a promising approximate method. It is not a traditional approximate method. It is a combination between numerical and analytic method. It can be used for solving highly nonlinear differential equation without linearizion. It can show and analyze the nonlinear dynamics easily. Scientists and engineers can use it for knowing the effect of changing the parameters, coefficients and initial conditions on the solution of differential equations in addition to control the accuracy and the error as needed very easily.

References

[1] S. H. Strogatz. Nonlinear Dynamics and Chaos. Addison-Wesley, 1994.
[2] D. Jordan and P. Smith, Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, Fourth Edition, Oxford University Press, 2007.
[3] Greenspan, D., Numerical Solution of Ordinary Differential Equations for Classical, Relativistic and Nano Systems. WILEY-VCH Verlag GmbH Co. KGaA, Weinheim, 2008.
[4] T. A. Abassy, Piecewise analytic method, International Journal of Applied Mathematical Research 1 (2012), no. 1, 77-107, DOI: 10.14419/ijamr.v1i1.22.
[5] T. A. Abassy, Introduction to piecewise analytic method, Journal of Fractional Calculus and Applications 3(S) (2012), 1-19.
[6] T. A. Abassy, Piecewise Analytic Method (Solving Any Nonlinear Ordinary Differential Equation of 1st Order with Any Initial Condition), International Journal of Applied Mathematical Research 2 (2013), no.1, 16-39.
[7] T. A. Abassy, Piecewise Analytic Method VS Runge-Kutta Method (Comparative Study), nternational Journal of Applied Mathematical Research, 9 (2) (2020) 41-49, DOI: 10.14419/ijamr.v9i2.31118.
[8] T. A. Abassy, Piecewise Analytic Method (PAM) is a New Step in the Evolution of Solving Nonlinear Differential Equations, International Journal of Applied Mathematical Research, 8 (1) (2019) 12-19, DOI: 10.14419/ijamr.v8i1. 24984.
[9] C.ODE.E, ODE ARCHITECT Companion.JOHN WILEY and SONS, INC, New York (1966).
[10] A. A. Klimenko, Y. V. Mikhlin , J. Awrejcewicz, Nonlinear normal modes in pendulum systems, Nonlinear Dynamics, October 2012, Volume 70, Issue 1, pp 797-813.
[11] I. Malkin, Certain Problems of the Theory of Nonlinear Vibrations. Geotechteorizdat, Moscow (1956) (in Russian)
[12] A.Blaquiere, Nonlinear System Analysis. Academic Press, New York (1966).
[13] A.H. Nayfeh, Mook, D.T.: Nonlinear Oscillations. Wiley, New York (1979).
[14] J. Mawhin, Global results for the forced pendulum equation. Handbook of differential equations, 533-589, Elsevier/North-Holland, Amsterdam,2004.



(a) $[U_m^{(s=5)}(t) - U_m^{(s=4)}(t)]$

(b) $[U_m^{(s=8)}(t) - U_m^{(s=7)}(t)]$

(c) $[U_m^{(s=10)}(t) - U_m^{(s=9)}(t)]$

(d) $[U_m^{(s=11)}(t) - U_m^{(s=10)}(t)]$

(e) $[U_m^{(s=12)}(t) - U_m^{(s=11)}(t)]$

(f) $[U_m^{(s=13)}(t) - U_m^{(s=12)}(t)]$

Figure 10: The effect of changing the order s on the solution precision (equation 10) using $F(t) = sin(t)$, $h = 0.01$, $\frac{b}{m} = 0.1$, $\frac{g}{l} = 10$, $f_0 = 1$ and $f_1 = 0$.

[15] L.P. Pook,Understanding Pendulums: A Brief Introduction,Springer,2011.