International Journal of Applied Mathematical Research, 3 (2) (2014) 292-300
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www.sciencepubco.com/index.php/IJAMR
doi: 10.14419/ijamr.v3i3.3122
Research Paper

A numerical method for singular and singularly perturbed Dirichlet-type boundary-value problems

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Abstract

This note is concerned with a new numerical method for the solution of singular and singularly perturbed boundary value problems. The method uses Fourier sine series and is suitable for Dirichlet-type boundary value problems. It removes the singularity of the problem in a natural way. A number of numerical examples are used to study the applicability of the method.

Keywords: *Dirichlet-type boundary valueproblems, Singular boundary value problems, Singularly perturbed baoundary value problems.*

1. Introduction

In this note we consider a numerical method for singular linear Dirichlet-type boundary value problems. Such problems arise very naturally in various applications including gas dynamics, chemical reactions, and structural mechanics. Existence and uniqueness of the solution for such problems have been reported in [1]. Traditional numerical methods fail to produce good approximate solutions for such equations. As a result, a number of investigators have considered various non-classical methods, including series solutions, Chebyshev polynomials, B-splines, and cubic splines [2,3,4,5]. Additional methods include fitted mesh [6], Green's functions and decomposition [7] and the Green's matrix [8]. Recent results also include methods based on reproducing kernel space [9,10], Sinc collocation method [11], Sinc-Galerkin method [12], and an iterative predictor-corrector type method based on finite difference approximation [13]. A recent review of the existing methods based on spline functions can also be found in [14].

The purpose of this note is to develop a numerical method for singular and singularly perturbed Dirichlet-type boundary value problems. The extension of the method to other boundary conditions will be considered in future works. Section 2 discusses some preliminaries. Section 3 introduces the method in details for a special finite element space. Section 4 studies the method using Fourier sine functions and section 5 uses a number of examples to investigate the applicability of the method, and compares the results to exact solutions.

2. Preliminary notes

Consider a 1-D bounded function $f(x)$, $x \in [0,1]$ with Dirichlet condition given by $f(0) = f(1) = 0$. In a standard finite-element method one seeks to approximate the function by a projection given by

$$f(x) = \sum_{j=1}^{N} a_j h_j(x) \tag{1}$$

where $h_i(x), i = 1, 2, .., N$ is the finite-element space. Using linear hat functions, Fig. 1 shows the approximating space. If the domain is divided into n_e equal elements then, for $f(0) = f(1) = 0$, there are $N = n_e - 1$ linear independent hat functions that span the space. As a result, there are a_j, $j = 1, 2, ..., N$ unkown costants.

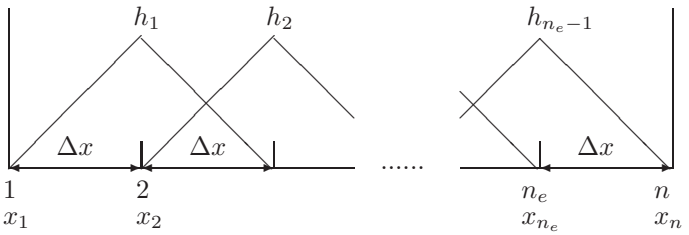

Fig. 1: A uniform mesh with equal intervals

In order to obtain these constants one multiplies the equation by $h_i(x)$ and integrate over the domain. The coefficients can be obtained after inverting the symmetric matrix $[< h_i, h_j >]$. This is one way to obtain a linear system for the unknown coefficients. The functions $h_i(x)$ form a linearly independent set and, as a result, the coefficient matrix is symmetric and positive definite. It is also possible to obtain a linear set of equations for the unknown a_j according to the following. Using equal elements Δx, the finite element mesh leads to the $n = n_e + 1$ nodes with $x_1 = 0$, $x_i = (i-1)\Delta x$, for $i = 2, 3, ..., n$, and $x_n = 1$. It is possible to integrate Eqn. (1) from $x = 0$ to $x = x_2$ according to

$$\int_0^{x_\ell} f(x)dx = \sum_{j=1}^{N} a_j \int_0^{x_\ell} h_j(x)dx., \quad \ell = 2. \tag{2}$$

Additional linearly independent equations can be obtained by extending the domain of integration according to $\ell = 2, 3, ..., N$. It is then possible to obtain a linear system of equations given by

$$\begin{bmatrix} \frac{1}{2}\sigma & 0 & \dots & 0 \\ \sigma & \frac{1}{2}\sigma & \dots & 0 \\ \dots & \dots & \dots & 0 \\ \sigma & \dots & \sigma & \frac{1}{2}\sigma \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} \int_0^{x_2} f(x)dx \\ \int_0^{x_3} f(x)dx \\ \vdots \\ \int_0^{x_{n_e}} f(x)dx \end{bmatrix}, \quad \text{where,} \quad \sigma = \int_0^1 h_2(x)dx. \tag{3}$$

The coefficent matrix is lower-triangular and it is clear that the eigenvalues of the coefficient matrix are all nonzero, and equal to $\frac{1}{2}\sigma$. As a result, the linear system can be uniquely solved. For Dirichlet type problems, it is also possible to project the function in the space of Fourier sine functions according to

$$f(x) = \sum_{j=1}^{\infty} a_j sin(j\pi x) \approx \sum_{j=1}^{K} a_j sin(j\pi x), \tag{4}$$

where the first K term in the infinite series is taken into account. By integrating the above equation over similar intervals, it is possible to obtain at least K linearly independent equations. It is also possible to construct more than K equations. This leads to an over-specified system. However, the equations are consistent and the least-square solution is identical to the solution where K independent equations are used. Fig. 2 compares the exact function $f(x) = x - x^2$ with its representation using the present method when $K = 30$. Using $n_e = 30$ leads to a square matrix. For $n_e = 40$ and $n_e = 60$, it leads to an over-specified system. However, the equations are consistent, and the least-square solution is identical to the square case.

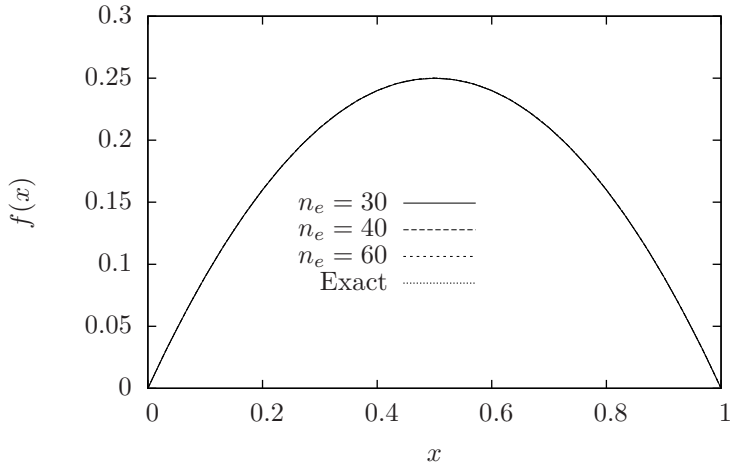

Fig. 2: Comparison of the function $f(x) = x - x^2$ and its representation for different values of n_e.

3. Numerical method based on linear hat functions

Consider a second-order singular differential equation given by

$$P(x)y'' + Q(x)y' + R(x)y = F(x), \quad 0 < x \leq 1 \tag{5}$$

with Dirichlet-type boundary condition

$$y(0) = y(1) = 0, \tag{6}$$

where, the functions $P(x), Q(x)$ and $F(x)$ are analytic in $x \in (0, 1)$. Nonzero Dirichlet conditions can be transformed into zero Dirichlet condition by altering the function $F(x)$. The function $P(x)$ can vanish at the boundary points. Consider the same discretization with equal intervals given in Fig. 1.

The proposed method starts by integrating Eqn. (5) from $x = 0$ to $x = z_i$ according to

$$\int_0^{z_i} P(x)y''dx + \int_0^{z_i} Q(x)y'dx + \int_0^{z_i} R(x)ydx = \int_0^{z_i} F(x)dx, \tag{7}$$

where $x_i - z_i = \epsilon$, with $0 < \epsilon << 1$. With hat functions, the derivative is discontinuous at the nodal points. The domain of intergations are arbitrary and it is possible to choose $z_i \approx x_i^-$. The first term can be integrated by parts which leads to

$$P(x)y'|_{x=0}^{x=z_i} + \int_0^{z_i} (Q(x) - P(x)')y'dx + \int_0^{z_i} R(x)ydx = \int_0^{z_i} F(x)dx. \tag{8}$$

Expanding the unknown function in the space of linear hat functions according to Eqn. (1), and writting in the form of a linear system leads to the coefficient matrix that is again lower triangular with the diagonal elements given by

$$\begin{bmatrix} \Gamma_2 & 0 & \dots & \dots & 0 \\ : & \Gamma_3 & 0 & \dots & 0 \\ : & : & \Gamma_\ell & \dots & 0 \\ : & : & : & \dots & 0 \\ : & : & : & : & \Gamma_{N+1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} \int_0^{x_2} F(x)dx \\ \int_0^{x_3} F(x)dx \\ \vdots \\ \int_0^{x_{n_e}} F(x)dx \end{bmatrix}, \tag{9}$$

where

$$\Gamma_\ell = p(z_\ell) + \int_{x_{\ell-1}}^{z_\ell} (Q(x) - P'(x))dx + \int_{x_{\ell-1}}^{z_\ell} (x - x_{\ell-1})R(x)dx, \quad \ell = 2, 3, , , N + 1 \tag{10}$$

It is now possible to make the following statement on the solution of the given boundary value problem.

Remark: Assume that the functions $P(x), Q(x)$ and $R(x)$ are such that all of the diagonal elements i.e., Γ_ℓ, in the coefficient matrix are nonzero. Then, the unique solution to the given boundary value problem can be obtained using the expansion in the space of linear hat functions.

In the next section, we proceed to apply this procedure using Fourier sine functions. Foourier sine functions are global and provide a bit more flexibility

4. A numerical method based on Fourier sine functions

Consider the same boundary value problem given in Eqns. (5) and (6). Consider the same discretization with equal intervals given in Fig. 1. The proposed method starts by integrating Eqn. (5) from $x = 0$ to $x = x_i$ according to

$$\int_0^{x_i} P(x)y''dx + \int_0^{x_i} Q(x)y'dx + \int_0^{x_i} R(x)ydx = \int_0^{x_i} F(x)dx. \tag{11}$$

The first term can be integrated by parts which leads to

$$P(x)y'|_{x=0}^{x=x_i} + \int_0^{x_i} (Q(x) - P(x)')y'dx + \int_0^{x_i} R(x)ydx = \int_0^{x_i} F(x)dx. \tag{12}$$

With the zero boundary conditions, it is possible to expand the unknown function in the Fourier sine series according to

$$y(x) = \sum_{j=1}^{\infty} A_j sin(j\pi x), \quad \text{with} \quad y'(x) = \sum_{j=1}^{\infty} (j\pi)A_j cos(j\pi x), \tag{13}$$

where now, the coefficients A_j, $j = 1, 2, .., \infty$ are unknowns. Substituting the above expression in Eqn. (12) leads to

$$\begin{aligned} P(x_i)\sum_{j=1}^{\infty}(j\pi)A_j cos(j\pi x_i) + \sum_{j=1}^{\infty}(j\pi)A_j \int_{x=0}^{x_i}(Q(x) - P(x)')cos(j\pi x)dx + \\ \sum_{j=1}^{\infty} A_j \int_{x=0}^{x_i} R(x)sin(j\pi x)dx = \int_0^{x_i} F(x)dx. \end{aligned} \tag{14}$$

Keeping the first K terms in the above summations leads to

$$\begin{aligned} P(x_i)\sum_{j=1}^{K}(j\pi)A_j cos(j\pi x_i) + \sum_{j=1}^{K}(j\pi)A_j \int_{x=0}^{x_i}(Q(x) - P(x)')cos(j\pi x)dx + \\ \sum_{j=1}^{K} A_j \int_{x=0}^{x_i} R(x)sin(j\pi x)dx = \int_0^{x_i} F(x)dx. \end{aligned} \tag{15}$$

Here, we have one equation for K unknowns. Repeating the procedure for K different domains of integrations leads to K linearly independent equations according to

$$\begin{aligned} &\sum_{j=1}^{K} A_j \left[j\pi P(x_2)cos(j\pi x_2) + j\pi \int_0^{x_2}(Q - P')cos(j\pi x)dx + \int_0^{x_2} Rsin(j\pi x)dx \right] = \int_0^{x_2} Fdx. \\ &\sum_{j=1}^{K} A_j \left[j\pi P(x_3)cos(j\pi x_3) + j\pi \int_0^{x_3}(Q - P')cos(j\pi x)dx + \int_0^{x_3} Rsin(j\pi x)dx \right] = \int_0^{x_3} Fdx. \\ &\vdots \\ &\sum_{j=1}^{K} A_j \left[j\pi P(x_n)cos(j\pi x_n) + j\pi \int_0^{x_n}(Q - P')cos(j\pi x)dx + \int_0^{x_n} Rsin(j\pi x)dx \right] = \int_0^{x_n} Fdx. \end{aligned} \tag{16}$$

Using a value of n_e larger than K, leads to an over-specified, but consistent system. In the next section, we use a number of examples to examine these features, and study the applicability of the method

5. Numerical examples

Example 1: Consider a singular boundary value problem [9] given by

$$y'' + \frac{1}{x}y' + y(x) = f(x) = 4 - 9x + x^2 - x^3, \quad 0 < x \leq 1, \quad y(0) = y(1) = 0 \tag{17}$$

The exact solution is given by $y(x) = x^2 - x^3$. After multiplying the above equation by x and integrating from $x = 0$ to $x = x_i$ one arrives at

$$x_i y'(x_i) + \int_0^{x_i} xy(x)dx = \int_0^{x_i} x(4 - 9x + x^2 - x^3)dx. \tag{18}$$

Assuming a Fourier Sine expansion for the unknown function leads to

$$\sum_{j=1}^{K} A_j \left[x_i(j\pi)cos(j\pi x_i) + \int_0^{x_i} xsin(j\pi x)dx \right] = 2x_i^2 - 3x_i^3 + \frac{1}{4}x_i^4 - \frac{1}{5}x_i^5. \tag{19}$$

Using n_e equal intervals and having $n_e = K$ makes it possible to solve for the unknown coefficients A_j, for $j = 1, 2, ..K$. Table 1 presents the numerical results and compares their values to the exact solution. The error is computed according to

$$E_r = |\frac{y_{num} - y_{exact}}{y_{exact}}|. \tag{20}$$

Table 1: Computed value and the relative error at different values of x for the example 1 using Fourier sine functions.

	$x = 0.2$	$x = 0.2$	$x = 0.4$	$x = 0.4$	$x = 0.6$	$x = 0.6$	$x = 0.8$	$x = 0.8$
n_e	y_{num}	E_r	y_{num}	E_r	y_{num}	E_r	y_{num}	E_r
20	0.033051	0.32E-1	0.097011	0.10E-1	0.14494	0.65E-2	0.12880	0.62E-2
40	0.032266	0.83E-2	0.096257	0.26E-2	0.14424	0.16E-2	0.12822	0.16E-2
80	0.032067	0.20E-2	0.096065	0.67E-3	0.14406	0.42E-3	0.12806	0.43E-3
160	0.032017	0.52E-3	0.096016	0.16E-3	0.14402	0.10E-3	0.12801	0.11E-3
320	0.032004	0.13E-3	0.096004	0.42E-4	0.14400	0.26E-4	0.12800	0.27E-4

Fig. 3 presents the numerical solution for the case where 100 sine functions ($K = 100$) are used to approximate the unknown function, but the number of equal intervals n_e are chosen as 120, 140, and 160. In these cases, the matrices are over-specified. However, the equations are consistent and the least-square solutions are identical to the numerical solution for a square matrix case, i.e. ($n_e = 100$). Fig. 3 presents these solutions and compares them to the exact solution.

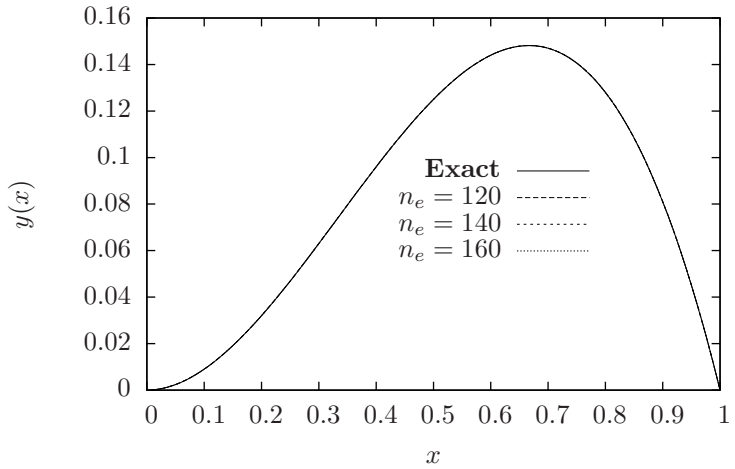

Fig. 3: Comparison of the numerical solution for the example 1 with the exact solution with over-specified systems. The systems are over-specified but consist with $K = 100$ and $n_e = 120, 140, 160$.

It is also instructive to use hat functions. After applying the steps in section 3 to this example one arrives at the working equation given by

$$z_i\left[\sum_{j=1}^{N} a_j h_j'(z_i)\right] + \sum_{j=1}^{N} a_j \int_0^{z_i} x h_j(x)dx = 2z_i^2 - 3z_i^3 + \frac{1}{4}z_i^4 - \frac{1}{5}z_i^5, \quad i = 1, N$$

where $x_i - z_i = 0.005$. Table 2 presents the numerical results at the same locations, and compares them to the exact solution. Numerical results showed little changes with respect to the choice of z_i. Comparison of the results in Table 1 and Table 2 indicates that, for this example, Fourier sine series can obtain a better numerical results with the same mesh density. This could be due to the fact that Fourier sine functions are global and, unlike linear hat functions, are in fact infinitely differentiable.

Table 2: Computed value and the relative error at different values of x for the example 1 using linear hat functions.

	$x = 0.2$	$x = 0.2$	$x = 0.4$	$x = 0.4$	$x = 0.6$	$x = 0.6$	$x = 0.8$	$x = 0.8$
n_e	y_{num}	E_r	y_{num}	E_r	y_{num}	E_r	y_{num}	E_r
100	0.033109	0.34E-1	0.097232	0.12E-1	0.144372	0.25E-2	0.126561	0.11E-1
200	0.032556	0.70E+0	0.096619	0.64E-2	0.144190	0.13E-2	0.127285	0.55E-2
400	0.032278	0.86E-2	0.096310	0.32E-2	0.144096	0.66E-3	0.127643	0.27E-2
800	0.032139	0.43E-2	0.096155	0.16E-2	0.144048	0.33E-3	0.127822	0.13E-2
1600	0.032070	0.21E-2	0.096078	0.80E-3	0.144024	0.16E-3	0.127911	0.69E-3

Example 2: We next Consider a singular boundary value problem [15] given by

$$y'' - \frac{1}{x}y' + \frac{1}{x(x+1)}y(x) = f(x) = -x^3, \quad 0 < x \leq 1, \quad y(0) = y(1) = 0 \tag{21}$$

Using the equal interval mesh given in Fig. 1, multiplying the equation by $(x^2 + x)$, and following the steps in the algorithm, the working equation leads to

$$(x_i + x_i^2)\sum_{j=1}^{K}(j\pi)A_j cos(j\pi x_i) - (2 + 3x_i)\sum_{j=1}^{K} A_j sin(j\pi x_i) - 4\sum_{j=1}^{K}\frac{A_j}{j\pi}(cos(j\pi x_i) - 1) = -\frac{1}{6}x_i^6 - \frac{1}{5}x_i^5. \tag{22}$$

Fig. 4 presents the solution for three different mesh sizes, i.e., $n_e = K = 20, 40, 80$, and compares their values to the exact solution.

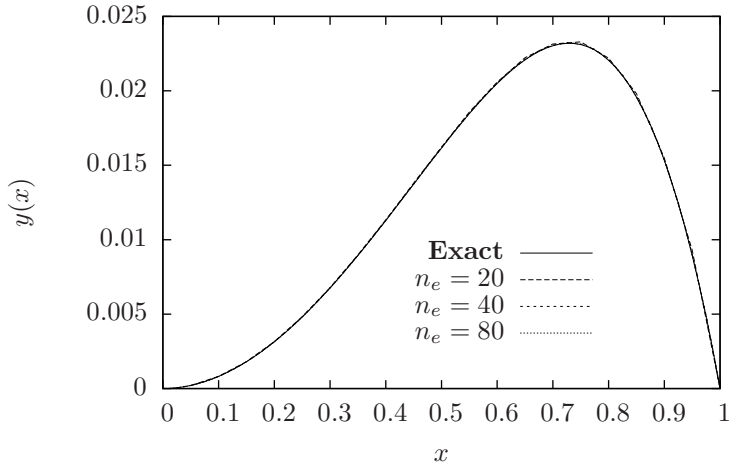

Fig. 4: Comparison of the numerical solution with the exact solution for the example 2. The numerical solution converges with the increasing number of mesh.

Fig. 5 presents the absolute value of the error $|y_{\text{numerical}} - y_{\text{exact}}|$ for the increasing mesh size, and shows the convergence of the numerical solution.

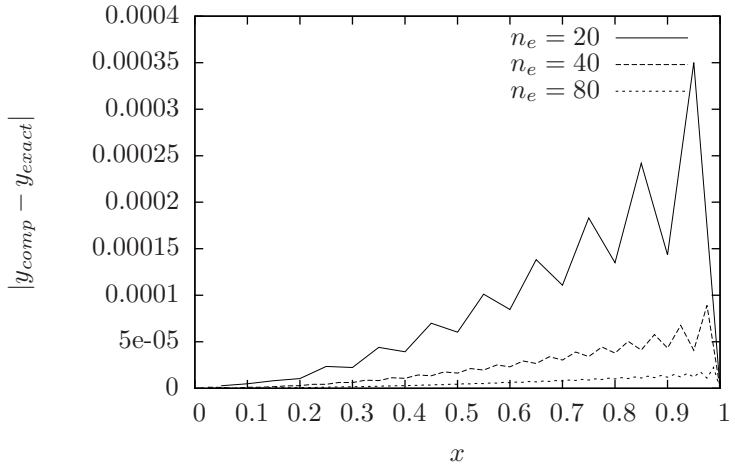

Fig. 5: Absolute value of the error for the example 2 for three different mesh sizes.

Example 3: We next consider a singularly perturbed reaction-diffusion boundary value problem with interior layer and a discontinuous source term given by [6]

$$\epsilon y'' + \mu(1+x)^2 y' - y(x) = f(x), \quad y(0) = y(1) = 0, \quad \text{where} \tag{23}$$

$$f(x) = \begin{cases} 2x+1 & \text{for} \quad 0 \le x \le 0.5 \\ -(3x+4) & \text{for} \quad 0.5 < x \le 1. \end{cases} \tag{24}$$

Using the equal interval mesh given in Fig. 1, and applying the method leads to

$$\epsilon y'|_{x=0}^{x=x_i} + \mu \int_0^{x_i} (1+x)^2 y' dx - \int_0^{x_i} y(x) dx = \int_0^{x_i} f(x) dx. \tag{25}$$

Now, expanding the unknown function in terms of Fourier sine series leads to

$$\epsilon \sum_{j=1}^{K} (j\pi) A_j \left[cos(j\pi x_i) - 1\right] + \mu \sum_{j=1}^{K} (j\pi) A_j \int_{x=0}^{x=x_i} (1+x)^2 cos(j\pi x) dx + \sum_{j=1}^{K} \frac{A_j}{j\pi} \left[cos(j\pi x_i) - 1\right] = \int_0^{x_i} f(x) dx. \tag{26}$$

Fig. 6 shows the computed value for three mesh sizes.

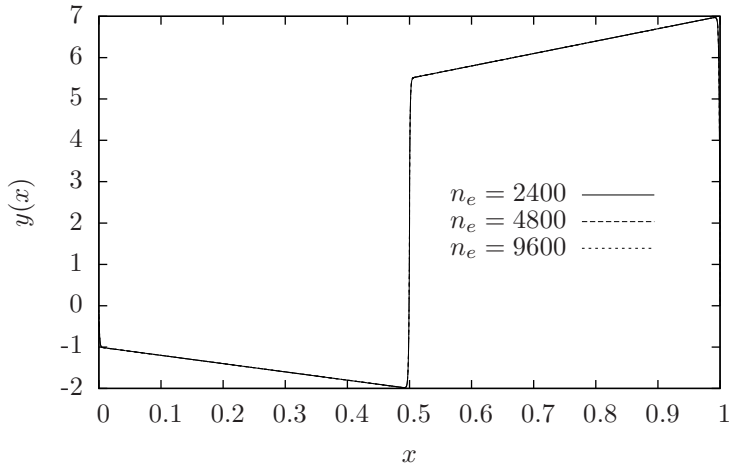

Fig. 6: Numerical solution for the example 3 for three different mesh sizes. The result shows the boundary layers at the end points and the interior layer.

Fig. 7 and 8 present the absolute value of the difference between the two mesh sizes.

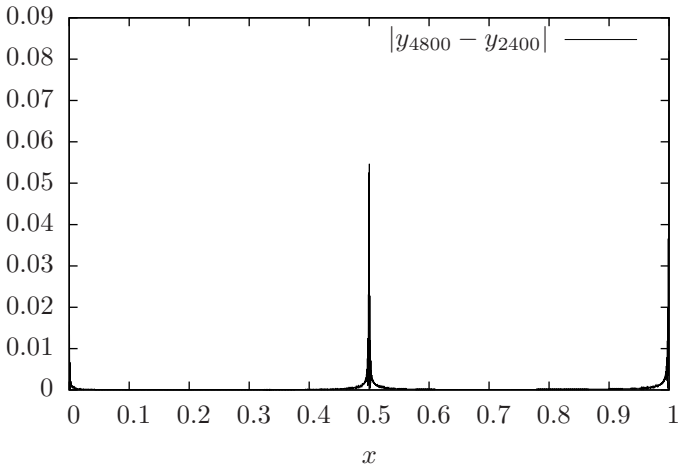

Fig. 7: Absolute value of the difference between the numerical results for two mesh sizes for the example 3.

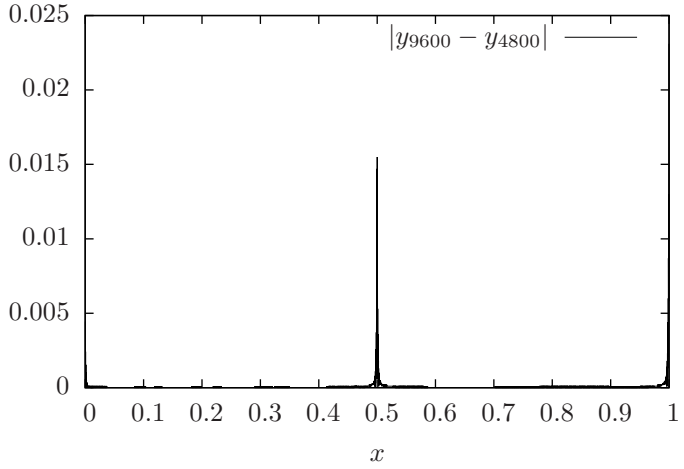

Fig. 8: Absolute value of the difference between the numerical results for two mesh sizes for the example 3.

Fig. 7 and 8 show that the solution is converging for increasing mesh sizes. Larger values of the difference are located at the boundary layers including the interior layer. Here, we are using a mesh with equal intervals. The computational effort can be reduced by using a finer mesh over the boundary layers.

6. Conclusion

In this note we presented a numerical method for obtaining the solution of linear singular boundary value problems. It uses Fourier sine series to approximate the unknown function and treats Dirichlet-type boundary conditions. It can also be applied to singularly perturbed boundary value problems. Three numerical examples were used to study the applicability of the method.

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