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Research Paper

Positive solutions for one-dimensional p-Laplacian boundary value problems with nonlinear parameter

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Abstract

In this paper, we establish existence of positive solutions of the nonlinear problems of one - dimensional p-Laplacian with nonlinear parameter

$$\varphi_p(u'(t))' + a(t)f(\lambda, u) = 0, \qquad t \in (0,1), \quad u(0) = u(1) = 0.$$

where $a : \Omega \to R$ is continuous and may change sign, $\lambda > 0$ is a parameter, $f(\lambda, 0) > 0$ for all $\lambda > 0$. By applying Leray-Schauder fixed point theorem we obtain the existence of positive solutions.

Keywords: *p-Laplacian, Positive solutions, Leray-Schauder fixed point theorem, nonlinear parameter.*

1. Introduction

The boundary value problem for one- dimensional p-Laplacian

$$\begin{cases} \varphi_p(u'(t))' + \lambda a(t)f(u) = 0, & t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

where $\varphi_p(u(t)) = |u|^{p-2} \quad u, \quad p > 1,$ has been studied extensively. For details, see for example, Refs [1,2,5], in the case p=2 see [6], and for case $\lambda = 1$, see [7,8,9].
In a recent paper [4], Hai considered the boundary value problem

$$\begin{cases} \Delta u + \lambda a(t)f(u) = 0, & t \in \Omega, \\ u = 0, & t \in \partial\Omega, \end{cases} \tag{2}$$

where Ω is a bounded domain in R^N, $a : \Omega \to R$ is continuous and changes its sign, $f(0) > 0$, and $\lambda > 0$ is sufficiently small, under the following assumptions
(A1) $f : [0, \infty) \to R$ is continuous and $f(0) > 0$.
(A2) $a : \bar{\Omega} \to R$ is continuous, $a \not\equiv 0$, and there exists a number $k > 1$ such that

$\int_\Omega G(t,s)a^+(s)ds \geq k\int_\Omega G(t,s)a^-(s)ds, \qquad t\in\Omega,$

where a^+ (resp. a^-) is the positive (resp. negative) part of a, and $G(t,s)$ is the Green's function of $-\Delta$ with Dirichlet boundary conditions.

They obtained the following interesting result:

Theorem A. Let (A1), (A2) hold. Then there exists a positive number $\lambda^\star$, such that (2) has a positive solution for $\lambda < \lambda^\star$. In another recent paper [3], Ma et al investigated the boundary value problem

$$\begin{cases} \Delta u + a(t)f(\lambda,u) = 0, & t\in\Omega,\\ u = 0, & t\in\partial\Omega, \end{cases} \tag{3}$$

By applying Leray-Schauder fixed point theorem they obtained that the problem (3) has a positive solution for $\lambda < \lambda^\star$. Motivated by the results mentioned in [3,4] above, in this paper we study the existence of positive solutions of the nonlinear one- dimensional p-Laplacian

$$\begin{cases} \varphi_p(u'(t))' + a(t)f(\lambda,u) = 0, & t\in(0,1),\\ u(0)=u(1)=0, \end{cases} \tag{4}$$

where $\varphi_p(u(t)) = |u|^p \quad u, p>1$, and hence $\varphi_p(u'(t))'$ is the one- dimensional p-Laplacian, and $a:\Omega\to R$ is continuous and changes its sign, $\lambda>0$ is a parameter, $f(\lambda,0)>0$ for all $\lambda>0$.

The following hypotheses are adopted throughout this paper:

(H1) $f\in C([0,\infty)\times[0,\infty),R)$ and $f(\lambda,0)>0$ for all $\lambda>0$, $f(0,u)=0$ for $u\in[0,\infty)$

(H2) $a:\bar\Omega\to R$ is continuous, $a\not\equiv 0$, and there exists a number $k>1$ such that

$\int_0^t \varphi_p^{-1}\left(h(a^+) + \int_0^s a^+(\tau)d\tau\right)ds \geq k\int_0^t \varphi_p^{-1}\left(h(a^-) + \int_0^s a^-(\tau)d\tau\right)ds,$

where $h: L^1(0,1)\to R$ is continuous function satisfying

$\int_0^1 \varphi_p^{-1}\left(h(a) + \int_0^s a(\tau)d\tau\right)ds = 0.$

The main result of this paper is as follows

Theorem 1.1. Let (H1), (H2) hold. Then there exists a positive number $\lambda^\star$, such that (4) has a positive solution for $\lambda<\lambda^\star$.

The proof of theorem 1.1 is based on the Leray -Schauder theorem see [10], for more details.

Remark 1.1. If we let $f(\lambda,u) := \lambda f(u)$ and $\varphi_p(u) = \Delta u$, in (4), then (4) reduces to (2), (H1) reduces to (A1). Therefore, [4, Theorem 1.1], (see Theorem A above), is the direct consequence of Theorem 1.1.

Clearly, Theorem 1.1 is an extension and improvement of the existence results in [4, Hai], [3, Ma]

The rest of this paper is arranged as follows. In Section 2, we will give some notations and preliminary results, in Section 3, we prove Theorem 1.1 via the Leray - Schauder fixed point theorem.

2. Preliminaries

Throughout the paper, we assume that $f(\lambda,u) = f(\lambda,0)$ for $u\leq 0$ and given $\lambda>0$.

For $u\in C_0^1[0,1]$, define the operator T by

$Tu(t) = \int_0^t \varphi_p^{-1}\left(h(a^+)f(\lambda,u(s)) + \int_0^s a^+(\tau)f(\lambda,u(s))d\tau\right)ds.$

It's not difficult to see that $T: C_0^1[0,1]\to C_0^1[0,1]$ is completely continuous.

Lemma 2.1. Let $0<\delta<1$. Then there exists a positive number $\bar\lambda$ such that, for $0<\lambda<\bar\lambda$, the equation

$\varphi_p(u'(t))' = -a^+(t)f(\lambda,u), \quad 0<t<1 \quad , \qquad u(0)=u(1)=0$

has a positive solution $\tilde u_\lambda$ with $\|\tilde u_\lambda\|_0 \to 0$ as $\lambda\to 0$, and

$\tilde{u}_\lambda(t) \geq \delta f(\lambda, 0) p(t), \qquad t \in \Omega,$

where $p(t) = \int_0^t \varphi_p^{-1}\left(h(a^+) + \int_0^s a^+(\tau) d\tau\right) ds.$

Proof. We shall apply the Leray-Schauder fixed point theorem to prove that T has a fixed point for λ small. Let $\varepsilon > 0$ be such that

$$f(\lambda, u) \geq \delta f(\lambda, 0), \qquad \text{for } 0 \leq u \leq \varepsilon. \tag{5}$$

From $f(0, u) \equiv 0, \forall u \geq 0$, we can suppose that $0 < \lambda < \varepsilon\lambda/2\|p\|_0 \tilde{f}(\lambda, \varepsilon)$, for given $\lambda > 0$, where $\tilde{f}(\lambda, t) = \max_{0 \leq s \leq t} f(\lambda, s)$. Then there exists $A_\lambda \in (0, \varepsilon)$ such that

$$\frac{\tilde{f}(\lambda, A_\lambda)}{\lambda A_\lambda} = \frac{1}{2\lambda \|p\|_0}. \tag{6}$$

Let $u \in C_0^1[0,1]$ and $\theta \in (0,1)$ be such that $u = \theta T u$. Then we have

$\|u\|_0 \leq \lambda \|p\|_0 \frac{\tilde{f}(\lambda, \|u\|_0)}{\lambda}$

or

$\frac{\tilde{f}(\lambda, \|u\|_0)}{\lambda \|u\|_0} \geq \frac{1}{\lambda \|p\|_0},$

which implies that $\|u\|_0 \neq A_\lambda$. Note that $A_\lambda \to 0$ as $\lambda \to 0$. By the Leray-Schauder fixed point theorem, T has a fixed point $\tilde{u}_\lambda$ with $\|\tilde{u}_\lambda\|_0 \leq A_\lambda < \varepsilon$. Consequently, $\tilde{u}_\lambda(t) \geq \lambda\delta \frac{f(\lambda,0)}{\lambda} p(t)$, $t \in [0,1]$, and the proof is completed.

3. Proof of the Theorem 1.1

Let $q(t) = \int_0^t \varphi_p^{-1}\left(h(a^-) + \int_0^s a^-(\tau) d\tau\right) ds$. By (H2), there exist positive numbers $\alpha,\ \gamma \in (0,1)$ such that

$$q(t)|f(\lambda, s)| \leq \gamma p(t) f(\lambda, 0) \tag{7}$$

for $s \in [0, \alpha]$, $t \in [0,1]$. Fix $\delta \in (\gamma, 1)$ and let $\lambda^* > 0$ be such that

$$\|\tilde{u}_\lambda\|_0 + \lambda\delta \frac{f(\lambda, 0)}{\lambda} \|p\|_0 \leq \alpha \tag{8}$$

for $0 < \lambda < \lambda^*$, where $\tilde{u}_\lambda$ is given by Lemma 2.1, and

$$|f(\lambda, x) - f(\lambda, y)| \leq f(\lambda, 0)\left(\frac{\delta - \gamma}{2}\right) \tag{9}$$

for $x, y \in [-\alpha, \alpha]$ with $|x - y| \leq \lambda^* \delta \frac{f(\lambda,0)}{\lambda} \|p\|_0$.

Let $0 < \lambda < \lambda^*$. We look for a solution u_λ of (4) of the form $\tilde{u}_\lambda + v_\lambda$. Thus v_λ satisfies

$$\begin{aligned} v_\lambda(t) = &\int_0^t \varphi_p^{-1}\left(h(af(\lambda, \tilde{u}_\lambda + v_\lambda)) + \int_0^s a(\tau) f(\lambda, \tilde{u}_\lambda + v_\lambda) d\tau\right) ds \\ &- \int_0^t \varphi_p^{-1}\left(h(a^+ f(\lambda, \tilde{u}_\lambda)) + \int_0^s a^+(\tau) f(\lambda, \tilde{u}_\lambda) d\tau\right) ds \quad , \quad 0 < t < 1. \end{aligned}$$

For each $w \in C_0^1[0,1]$, let $v = Tw$ be the solution of

$$\begin{aligned} v(t) = &\int_0^t \varphi_p^{-1}\left(h(af(\lambda, \tilde{u}_\lambda + \omega)) + \int_0^s a(\tau) f(\lambda, \tilde{u}_\lambda + \omega) d\tau\right) ds \\ &- \int_0^t \varphi_p^{-1}\left(h(a^+ f(\lambda, \tilde{u}_\lambda)) + \int_0^s a^+(\tau) f(\lambda, \tilde{u}_\lambda) d\tau\right) ds \quad , \quad 0 < t < 1. \end{aligned}$$

Then $T : C_0^1[0,1] \to C_0^1[0,1]$ is completely continuous. Let $v \in C_0^1[0,1]$ and $\theta \in (0,1)$ be such that $v = \theta T v$. Then we have

$$v = \theta \int_0^t \varphi_p^{-1} \left(h(af(\lambda, \tilde{u}_\lambda + v)) + \int_0^s a(\tau) f(\lambda, \tilde{u}_\lambda + v) d\tau \right) ds$$
$$- \theta \int_0^t \varphi_p^{-1} \left(h(a^+ f(\lambda, \tilde{u}_\lambda)) + \int_0^s a^+(\tau) f(\lambda, \tilde{u}_\lambda) d\tau \right) ds.$$

We claim that $\|v\|_0 \neq \delta f(\lambda, 0)\|p\|_0$. Suppose on the contrary that $\|v\|_0 = \delta f(\lambda, 0)\|p\|_0$. Then, by (8) and (9), we obtain

$$\|\tilde{u}_\lambda + v\|_0 \leq \|\tilde{u}_\lambda\|_0 + \|v\|_0 \leq \alpha$$

and

$$|f(\lambda, \tilde{u}_\lambda + v) - f(\lambda, \tilde{u}_\lambda)| \leq f(\lambda, 0)\tfrac{\delta - \gamma}{2},$$

which together with (7) implies that

$$\begin{aligned} |v(t)| &\leq \lambda \frac{\delta - \gamma}{2} \frac{f(\lambda, 0)}{\lambda} p(t) + \lambda\gamma \frac{f(\lambda, 0)}{\lambda} p(t) \\ &= \lambda \frac{\delta + \gamma}{2} \frac{f(\lambda, 0)}{\lambda} p(t), \end{aligned} \tag{10}$$

In particular

$$\|v\|_0 \leq \lambda \tfrac{\delta+\gamma}{2} \tfrac{f(\lambda,0)}{\lambda} \|p\|_0 < \lambda\delta \tfrac{f(\lambda,0)}{\lambda} \|p\|_0,$$

a contradiction, and the claim is proved. By the Leray -Schauder fixed point theorem, T has a fixed point v_λ with $\|v_\lambda\|_0 \leq \delta f(\lambda, 0)\|p\|_0$. Hence v_λ satisfies (10) and, using Lemma 2.1, we obtain

$$\begin{aligned} u_\lambda(t) &\geq \tilde{u}_\lambda(t) - v_\lambda(t) \\ &\geq \lambda\delta \frac{f(\lambda, 0)}{\lambda} p(t) - \lambda \frac{\delta + \gamma}{2} \frac{f(\lambda, 0)}{\lambda} p(t) \\ &= \lambda \frac{\delta - \gamma}{2} \frac{f(\lambda, 0)}{\lambda} p(t), \end{aligned}$$

i.e., u_λ is a positive solution of (4). This completes the proof of Theorem 1.1.

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