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A Note on b-chromatic Number of the Transformation Graph G^{++-} and Corona Product of Graphs

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Abstract

In this paper, we find the b-chromatic number of Transformation graph G^{++-} for Cycle, Path and Star graph. Also we determine the b-chromatic number of Corona product of Path graph with Cycle and Path graph with Completegraph along with its structural properties.

Keywords: *b-chromatic number, b-colouring, chromatic number, Corona product, Transformation graph.*

1 Introduction

All graphs in this paper are finite, undirected graphs, loopless graph without multiple edges. A k-colouring of a graph G[1] is a labeling $f: V(G) \rightarrow T$, where $|T| = k$ and it is proper if adjacent vertices have different labels. A graph is k colourable if it has a proper colouring. The chromatic number $\chi(G)$ is the least number k such that G is k-colourable. The b-chromatic number $\varphi(G)$ [2] of a graph G is the largest integer k such that G admits a proper k-colouring in which every colour class has a representative adjacent to at least one vertex in each of the other colour classes. Such a colouring is called a b-colouring. The concept of b-chromatic number was introduced in 1999 by Irwing and Manlove[3].

For a graph G, let $V(G)$ and $E(G)$ [7]denote the point set, line set of graph G respectively. The Transformation graph G^{++-}[4,8]of G is the graph with point set $V(G)\cup E(G)$ in which the points X and$Y$ are joined by a line if one of the following conditions hold.

- $x,y\in$ V(G)and$x,y$are adjacent in G.
- $x,y\in E(G)$ and $x,y$are adjacent in G.
- one of x and y is in $V(G)$ and the other is in $E(G)$ and they are not incident in G.

Corona product [9] or simply corona of graph G_1 and G_2 is a graph which is the disjoint union of one copy of G_1 and $|v_1|$ copies of G_2 ($|v_1|$ is number of vertices of G_1) in which each vertex copy of G_1 is connected to all vertices of separate copy of G_2 .

2 b-Chromatic Number of G^{++-} of Path Graph

Theorem 2.1: *The b-Chromatic number of G^{++-} of Path graph P_n has n colours.*

Proof

Consider a Path graph of length n-1 with vertex set $V=\{v_1,v_2,v_3..v_n\}$ and edge set $E=\{e_1,e_2,e_3..e_{n-1}\}$. In Path graph P_n, each vertex v_i is adjacent with the vertices v_{i-1} and v_{i+1} for $i=2,3,....n-1$, the vertex v_1 is adjacent with v_2 and v_n is adjacent with v_{n-1} and the lines e_1 and e_n are non-adjacent with n-3 lines and remaining e_i for $i=2,3,...n-1$ are non-adjacent with n-4 lines.

By the definition of Transformation graph G^{++-}, the vertex set of $G^{++-}(P_n)$ corresponds to both vertex set and edge set of Path graph. The vertex set of $G^{++-}(P_n)$ is defined as follows:

i.e.$[G^{++-}(P_n)]=\{v_i: 1 \le i \le n\} \cup \{e_i: 1 \le i \le n-1\}$

Consider the colour class $C=\{c_1,c_2,c_3..c_n\}$. Assign the colour c_i *to* v_i for $i=1,2,3..n$ and assign the colour c_{n+i} to e_i for $i=1,2,3..n-1$. Due to the above mentioned non-adjacency condition the above colouring does not produce a b-chromatic colouring. Thus, to make the above colouring as b-chromatic one, assign the colour c_i to v_i for $1\le i\le n$ and assign the colour c_1 to e_1 and c_{i+1} to e_i for $i=2,3..n-1$. Now the vertices v_i for $i=1,2,3$ and the vertices e_i for $3\le i\le n-1$ realizes its own colour, which produces a b-chromatic colouring.

Thus the given colouring is b-chromatic. And by the very construction, it is the maximal colour class.

Hence the proof.

Example

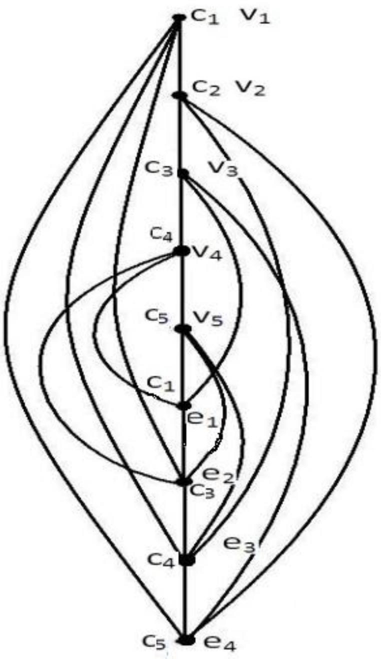

Fig. 1: $G^{++-}(P_5)$

2.1 Structural Properties of $G^{++-}(P_n)$

The number of vertices in $G^{++-}(P_n)$ i.e. $p[G^{++-}(P_n)] = 2n-1$, number of edges in the $G^{++-}(P_n)$ i.e. $q[G^{++-}(P_n)] = n^2-n-1$. The Maximum and Minimum degree of $G^{++-}(P_n)$ is denoted as $\Delta = n$ and $\delta = n-1$ respectively.

3 b-Chromatic Number of G^{++-} of Cycle

Theorem 3.1: *The b-Chromatic number of G^{++-} of the Cycle C_n is n.*

Proof

Consider a Cycle of length n, whose vertices are denoted as $v_1,v_2,v_3....v_n$ and edges are denoted as $e_1,e_2,e_3....e_n$. We see that every point in Cycle C_n is non-adjacent with n-2 lines. Now consider $G^{++-}(C_n)$, here there is no non-incident lines. By the definition of Transformation graph G^{++-}, the vertex set of $G^{++-}(C_n)$ corresponds to both vertex set and edge set of Cycle.

i.e. $V[G^{++-}(C_n)]=\{v_i: 1 \le i \le n\} \cup \{e_i: 1 \le i \le n\}$

By observation, $G^{++-}(C_n)$ forms an-regular graph. Therefore the b-chromatic number of $G^{++-}(C_n) \geq n$. Now we prove for $\varphi[G^{++-}(C_n)] \leq n$, for this consider a proper colouring of $G^{++-}(C_n)$ as follows.
Consider the colour class C=$\{c_1,c_2,c_3..c_n\}$. Assign the colour c_i for $i=1,2,3..n$ to the inner cycle of C_n. Next if we assign the colour c_{n+1} to any vertices in outer cycle, it does not realize the colour $c_{n+1,}$ So we should assign only the existing colours to the vertices in outer cycle . Hence by the colouring procedure, we cannot assign more than n colours to $G^{++-}(C_n)$ i.e. $\varphi[G^{++-}(C_n)] \leq n$. Therefore $\varphi[G^{++-}(C_n)] = n$. Thus by the colouring procedure the above said colouring is maximal and b-chromatic.
Hence the Proof.

Example

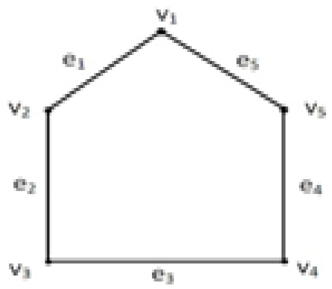

Fig. 2: C_5

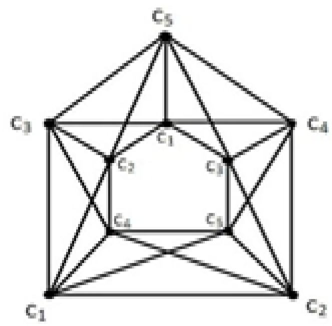

Fig. 3: $\varphi[G^{++-}(C_5)]=5$

3.1 Structural Properties of $G^{++-}(C_n)$

The numberof vertices in $G^{++-}(C_n)$ i.e.$p[G^{++-}(C_n)] = 2n$, number of edges in the $G^{++-}(C_n)$ i.e. $q[G^{++-}(C_n)] = n^2$. The Maximum and Minimum degree of $G^{++-}(C_n)$ are denoted as $\Delta = n$ and $\delta = n$ respectively. Thus $G^{++-}(C_n)$ is an n-regular graph.

4 b-Chromatic Number of $G^{++-}$of Star Graph

Theorem 4.1: *If G is $K_{1,n}$, then clearly $\varphi[G^{++-}(K_{1,n})] = n+1$*

Proof

Consider the graph $K_{1,n}$ with pendant vertices $v_1,v_2,v_3..v_n$ and v where v is the root vertex with degree n. i.e. $V(K_{1,n}) = \{v\} \cup \{v_i : 1 \leq i \leq n\}$and $E(K_{1,n}) = \{e_i : 1 \leq i \leq n\}$ between the vertices vv_i for $i=1,2,3..n$. Here in $K_{1,n}$ we see there is no incident lines.

Consider $G^{++-}(K_{1,n})$. By the definition of the Transformation graph G^{++-}, the vertex set is defined as $V[G^{++-}(K_{1,n})] = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\}$. Here the vertices $\{e_i : 1 \leq i \leq n\}$ forms a clique of order n(say K_n) in $G^{++-}(K_{1,n})$. Therefore we say that the b-chromatic number of $G^{++-}(K_{1,n}) \geq n$. Consider the colour class $C=\{c_1,c_2,c_3...c_{n+1}\}$. Assign a proper colouring to the vertices as follows.

Case 1

First assign the proper colouring to the vertex e_i. Assign the colour c_i to the vertex e_i for $i=1,2,3..n$, and assign the colour c_{n+1} to v_i for $i=1,2,3..n$ and assign any colour to root vertex other than the colour c_{n+1}.Now the vertices e_i realizes its own colour. Thus, by the colouring procedure the above said colouring produces a maximal and b-chromatic colouring.

Example

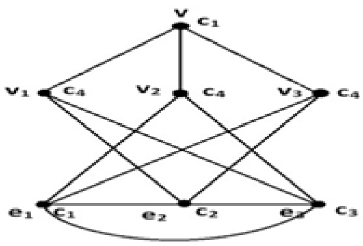

Fig 4: $\varphi[G^{++-}(K_{1,3})]=4$

Case 2

Next assign proper colouring to the vertex v and vi for $i=1,2,3..n$.
Assign the colour c_1 to the root vertex v and c_{i+1} to v_i for $i=1,2,3..n$ and assign the same set of colour to e_i which is already assigned for v_i because v_i is not adjacent with e_i for $i=1,2,3..n$, which produces a b-chromatic colouring. Thus by colouring procedure the above said colouring is maximal and b-chromatic. Hence the proof.

Example

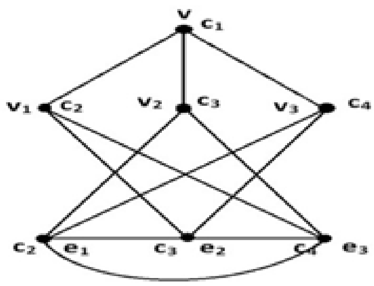

Fig. 5: $\varphi[G^{++-}(K_{1,3})]=4$

4.1 Structural Properties of $G^{++-}(K_{1,n})$

The Number of vertices in $G^{++-}(K_{1,n})$ i.e. $p[G^{++-}(K_{1,n})]= 2n+1$, number of edges in the $G^{++-}(K_{1,n})$ i.e. $q[G^{++-}(K_{1,n})] =\left[\frac{n(3n-1)}{2}\right]$. The Maximum and Minimum degree of $G^{++-}(K_{1,n})$is denoted as $\Delta = n+1$ and $\delta = n-1$ respectively. The number of vertices having maximum and minimum degree in $G^{++-}(K_{1,n})$ is denoted by $n(p_{\Delta})=n$ and $n(p_{\delta})=n+1$.

Theorem 4.2: *For any Star graph $K_{1,n}$, the number of edges in $G^{++-}(K_{1,n})$ is*

$\left[\frac{n(3n-1)}{2}\right]$.

Proof

$q[G^{++-}(K_{1,n})]$ = Number of edges in $K_{1,n}$ + Number of edges in K_n + Number of edges in crown graph S_n

$$= \binom{n}{1}+\binom{n}{2}+n(n-1)$$

$$= n+\left[\frac{n(n-1)}{2}\right]+n(n-1)$$

$$= n+n(n-1)\left[\frac{2+1}{2}\right]$$

$$= \frac{2n+3n^2-3n}{2}$$

$$=\frac{3n^2-n}{2}$$

$$=\left[\frac{n(3n-1)}{2}\right]$$

Therefore $q[G^{++-}(K_{1,n})]=\left[\frac{n(3n-1)}{2}\right]$

5 b-Chromatic Number of Corona Product of Path Graph with Cycle

Theorem 5.1: *For any integer $n>3$, $\varphi\ (P_n o C_n) = n$*

Proof:

Let $G_1= P_n$ be a Path graph of length $n-1$ with vertices $v_1,v_2,v_3....v_n$ and edges $e_1,e_2,e_3....e_{n-1}$. Consider $G_2=C_n$ be a Cycle of length n whose vertices are denoted as $v_1,v_2,v_3....v_n$ and edges are denoted by $e_1,e_2,e_3....e_n$.

Consider the Corona product of $G_1$and G_2 i.e. $G= P_n$oC_n is obtained by taking unique copy of P_n with n vertices and n copies of C_n and joining the i^{th} vertex of P_n to every vertex in i^{th} copy ofC_n..

i.e.$V(G) =V(P_n) \cup V(C^1n) \cup V(C^2n) \cup V(C^3n) \cup \ldots\ldots.. V(C^nn)$

where $V(P_n) = \{v_1,v_2,v_3....v_n\}$ and $V(C^i_n) = \{u_i^j : 1 \le i \le n, 1 \le j \le n\}$

Now assign a proper colouring to these vertices as follows. Consider the colour class $C= \{ c_1,c_2,c_3....c_n\}$. First assign the colour c_i to vertex v_i for $i=1,2,3...n$ and assign the colour to u_i^j as c_{i+j} when $i+j \le n$ and c_{i+j-n} when $i+j > n$ for $1\le i \le n$, $1\le j \le n-1$. Now the only vertex remaining to be coloured is u_i^j for $j=n$. Suppose if we assign any new colour to u_i^j for $i=1,2,3.n$, $j=n$ it will not produce a b-chromatic colouring, because u_i^j $(i=1,2,3..n, j=n)$is adjacent only with u_i^1 and u_i^{n-1}. So we assign the colour to u_i^jother than the colour which we assign for u_i^1 and u_i^{n-1}. Now the vertices$\{v_i : 1 \le i \le n\}$ realize its own colours, which produces a b-chromatic colouring. Thus by the colouring procedure the above said colouring is maximal and b-chromatic. Hence the proof

Example

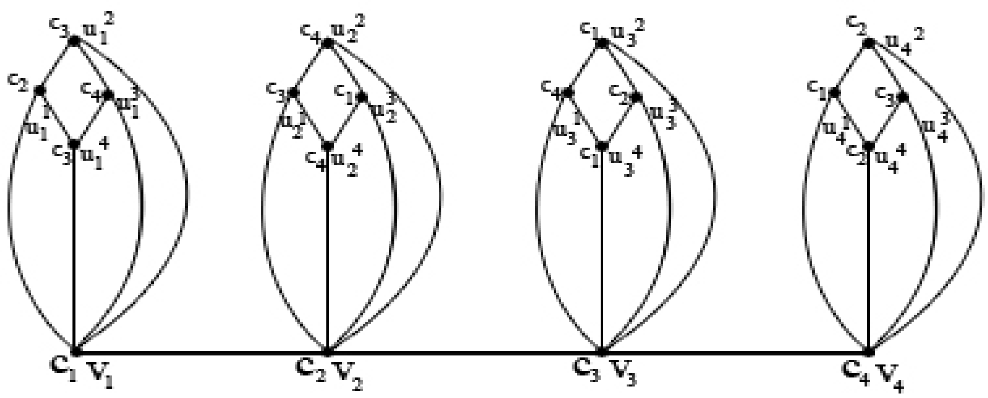

Fig. 6: $\varphi(P_4oC_4) = 4$

5.1 Structural Properties of (P_noC_n)

The Number of vertices in $P_noC_n(n>3)$ i.e. $p(P_noC_n)= n(n+1)$, $n$umber of edges in the P_noC_n i.e. $q(P_noC_n) =2n^2+n-1$. The Maximum and Minimum degree of P_noC_n is denoted as $\Delta = n+2$ and $\delta = n-1$ respectively. The number of vertices having maximum and minimum degree in P_noC_n is denoted by $n(p_\Delta)=n-2$ and $n(p_\delta)=n^2$.

Corollary 5.1: *For any integer n<4,* φ $(P_noC_n) = n+1$

Theorem 5.2: $q(P_no\ c_n)= 2n^2+n-1$

Proof

$q(P_n\ o\ C_n)$ = Number of edges in largest subgraph + Number of edges not in any of the largest subgraph

$$= n\times(2n)+n-1$$

$$= 2n^2+n-1$$

6 b-Chromatic Number of Corona Product of Path with Complete Graph

Theorem 6.1: Let P_n and K_2 be the Path graph and Complete graphs with n vertices respectively.Then

$$\varphi\ (P_n\ oK_2) = \begin{cases} n+1\ for\ n=2 \\ n\ for\ n=3\ and\ 4 \\ n-1\ for\ n=5 \\ 5\ for\ every\ n>6 \end{cases}$$

6.1 Structural Properties of P_no K_2

The Number of vertices in P_noK_2 i.e. $p(P_n\text{o}K_2)= 3n$, number of edges in the P_noK_2 i.e. $q(P_n\text{o}K_2) =3n+(n-1)$. The Maximum and Minimum degree of $P_n\text{o}K_2(n>3)$ is denoted as $\Delta = 4$ and $\delta = 3$ respectively. The number of vertices having maximum and minimum degree in P_noK_2 is denoted by $n(p_\Delta)=n-2$ and $n(p_\delta)=2$.

Theorem 6.2: *For any integer n,* $\varphi(P_noK_n) = n+1$

Proof

Let $G_1= P_n$ be a Path graph of length $n-1$ with n vertices and $G_2 = K_n$ be a Complete graph of n vertices.
Consider the Corona product of $G_1$and G_2 i.e.$G= P_n\text{o}K_n$ is obtained by taking unique copy of P_n with n vertices and n copies of K_n and joining the i^{th} vertex of P_n to every vertex in i^{th} copy of K_n.

i.e.$V(G)= V(P_n) \cup V(K^1{}_n)\cup V(K^2{}_n) \cup V(K^3{}_n)\cup\ldots\ldots\ V(K^n{}_n)$.

where $V(P_n) = \{ v_1,v_2,v_3....v_n \}$ and $V(K^i{}_n) = \{u_i^j : 1 \le i \le n,,1 \le j \le n\}$.
By observation, we see that there are n copies of disjoint subgraph which induces a clique of order $n+1$ (say K_{n+1}). Therefore we can assign more than or equal to $n+1$ colours to every corona product of path graph with complete graph. Consider the colour class $C=\{c_1,c_2,c_3,c_4...c_n,c_{n+1}\}$. Now assign a proper colouring to these vertices as follows. Suppose if we assign more than $n+1$colours, it contradicts the

definition of b-chromatic colouring. Due to this condition, we cannot assign more than $n+1$colours. Hence we have $\varphi(P_n o K_n) \leq n+1$.Therefore $\varphi(P_n o K_n) = n+1$.
Thus by the colouring Procedure the above said colouring is maximal and b-chromatic colouring.

Example

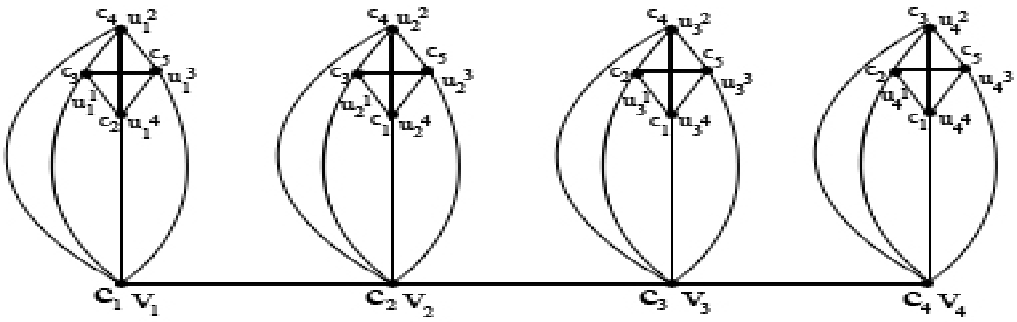

Figure 8:$\varphi(P_4\ oK_4) = 5$

6.2 Structural Properties of ($P_n\ o\ K_n$)

The Number of vertices in $P_n o K_n$ i.e. $p(P_n o K_n) = n(n+1)$, number of edges in the $P_n o K_n$ i.e. $q(P_n o K_n) = \left[\frac{n^3+n^2+2n-2}{2}\right]$. The Maximum and Minimum degree of $P_n o K_n$ is denoted as $\Delta = n+2$ and $\delta = n$ respectively. The number of vertices having maximum and minimum degree in $P_n o C_n$ is denoted by $n(p_\Delta) = n-2$ and $n(p_\delta) = n^2$.

Theorem 6.3: *For any path P_n and complete graph graph K_n the number of edges in corona product of P_n with K_n is*

$$q(P_n o\ K_n) = \left[\frac{n^3+n^2+2n-2}{2}\right]$$

Proof

$$\begin{aligned} q(P_n o\ K_n) &= \text{Number of edges in all } K_{n+1} + \text{Number of edges not in any of the } K_{n+1} \\ &= n \times q(K_{n+1}) + \text{Number of edges not in anyof the } K_{n+1} \\ &= n \times \binom{n+1}{2} + n-1 \\ &= n\left[\frac{n(n+1)}{2}\right] + n-1 \\ &= n\left[\frac{n^2+n}{2}\right] + n-1 \\ &= \left[\frac{n^3+n^2}{2}\right] + n-1 \\ &= \left[\frac{n^3+n^2+2n-2}{2}\right] \end{aligned}$$

Therefore $q(P_n o\ K_n) = \left[\frac{n^3+n^2+2n-2}{2}\right]$

7 b-Chromatic Number of Corona Product K_n with Fan Graph

Theorem 7.1:

$$\varphi(F_{1,n}\, oK_2)=\begin{cases} n+1 \; for\; every\; 2 \le n \le 4 \\ 5 \; for\; every\; n > 5 \end{cases}$$

Theorem 7.2: φ $(F_{1,n}oK_n)= n+1$ for every $n>2$.

Proof
The Proof of the theorem is similar to theorem (6.1).

8 b-Chromatic Number of Corona Product $K_{1,n}$ with K_2

Theorem 8.1: *If* $K_{1,n}$*and* K_2 *are Star graph and Complete graphs respectively, then*

$$\varphi(K_{1,n}\, oK_2) = \begin{cases} n+1 \; for\; every\; n \le 3 \\ 4 \; for\; every\; n \ge 4 \end{cases}$$

Theorem 8.2: φ $(K_{1,n}oK_n) =n+1$ for every $n>2$

Proof

The Proof of the theorem is similar to theorem (6.1).

9 Conclusion

In this paper, we discussed about b-chromatic number of Transformation graph G^{++-} of Cycle, Path and Star graph and the Corona product of Path graph with Cycle and Complete graph.

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