International Journal of Applied Mathematical Research, 11 (1) (2022) 11-13

International Journal of Applied Mathematical Research

Website: www.sciencepubco.com/index.php/IJAMR

Research paper

Separability and the 3d Gelfand Levitan equation

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Abstract

The 1D Gelfand-Levitan equation has been well studied with respect to the separability of the spectral measure function. The analytic solution has been shown to be associated with reflectionless potentials. This paper considers the 3D version of this equation to see if an analytic solution can be found for a separable spectral measure function and if it also corresponds to known reflectionless potentials. Though the analytic solution is shown, it does not correspond to reflectionless potentials.

Keywords: *Inverse Scattering; Gelfand-Levitan Equation; Reflection Coefficient; One-Dimensional Scattering.*

1. Introduction

The 1D problem of inverse scattering is well studied. A potential V(r) can be determined from the reflection coefficient R(k) of a scattered wave using the Gelfand-Levitan equation [1-3]:

$$K(r,s) + G(r,s) + \int_{-\infty}^{r} K(r,t)G(t,s)dt = 0 \tag{1}$$

Where G(r,s), the spectral measure function, is the Fourier transform of the reflection coefficient R(k):

$$G(r,s) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R(k)e^{-ik(r+s)}dk + \text{ Bound State Terms} \tag{2}$$

And K(r,s) is related to the potential by:

$$V(r) = -2\frac{dK(r,r)}{dr}. \tag{3}$$

Finding V(r) is done by taking the Fourier transform of R(k) to get G(r,s) and then solving (1) for K(r,s). Then (3) can be used to find the potential. The challenge arises in solving (1). This is typically done by successive iteration [4].

Of interest here is how the separability of G(r,s) leads to an analytic solution that corresponds to reflectionless potentials. This has been considered in several papers [5-7] including demonstrating the equivalence to reflectionless potentials like the Bargmann Potential. [8]

This paper considers whether a similar separability condition leads to interesting results for the 3D Gelfand-Levitan equation. Kay and Moses developed and verified the 3D Gelfand-Levitan equation [9] given by:

$$K(\vec{r},\vec{s}) + G(\vec{r},\vec{s}) + \int_{r}^{\infty} dr' \int_{0}^{\pi} \sin\theta' d\theta' \int_{0}^{2\pi} d\varphi' \, K\left(\vec{r},\vec{r'}\right) G\left(\vec{r'},\vec{s}\right) = 0 \tag{4}$$

Where $\vec{r'}$ has the standard spherical coordinates r', θ' and φ'. By following the techniques outlined in previous work [5], this paper can see how separability applies to this 3D equation.

2. Separability and the 3d Gelfand Levitan equation

Starting with equation (4) it is straightforward to follow the method outlined in [5] to show that given a separable $G(\vec{r},\vec{s})$ a separable $K(\vec{r},\vec{s})$ follows. Since $G(\vec{r},\vec{s})$ is symmetric with respect to r and s this leads to the equivalence of the conditions that:

$$K(\vec{r},\vec{s}) = g(\vec{s})L(\vec{r}) \tag{8}$$

$$G(\vec{r},\vec{s}) = g(\vec{s})g(\vec{r}) \tag{9}$$

Which is the same form as the 1D result. This is not surprising if one considers the forms of equations (1) and (4).
However, the form of $L(\vec{r})$ in (8) is much different from that found for the 1D case. To find $L(\vec{r})$ consider the 3D relationship between $K(\vec{r},\vec{s})$ and the associated potential, $V(\vec{r})$. [9] (This relationship is not angle dependent, so the vector notation has been dropped.)

$$\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}r^2 - \frac{1}{s^2}\frac{\partial^2}{\partial s^2}s^2\right] K(r,s) = V(r)K(r,s) \tag{10}$$

Using equation (8) this becomes:

$$\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}r^2 - \frac{1}{s^2}\frac{\partial^2}{\partial s^2}s^2\right] L(r)g(s) = V(r)L(r)g(s) \tag{11}$$

Which can be separated to give:

$$\frac{1}{L(r)}\frac{1}{r^2}\frac{\partial^2}{\partial r^2}r^2L(r) - V(r) = \frac{1}{g(s)}\frac{1}{s^2}\frac{\partial^2}{\partial s^2}s^2g(s) \tag{12}$$

With the variables separated, and assuming each side is equal to the constant c^2, one can solve the right had side of (12) to get that:

$$g(s) = A\,\frac{e^{cs}}{s^2} + B\frac{e^{-cs}}{s^2} \tag{13}$$

where A and B are constants. As noted by Kay and Moses [9] it must be true that:

$$\lim_{s\to\infty} K(r,s) = 0 \tag{14}$$

This, by relation (8), is equivalent to requiring that:

$$\lim_{s\to\infty} g(s) = 0 \tag{15}$$

Applying this to equation (13), yields that A = 0, so:

$$g(s) = B\frac{e^{-cs}}{s^2} \tag{16}$$

As shown for the 1D case [5], L(r) is related to g(r) by:

$$L(r) = \frac{-g(r)}{1+\int_{-\infty}^{r} g^2(t)dt} \tag{17}$$

Combining (16) and (17) gives:

$$L(r) = \frac{-B\frac{e^{-cr}}{r^2}}{1+\int_{-\infty}^{r}\frac{e^{-2ct}}{t^4}dt} \tag{18}$$

The integral in the denominator can be expressed in terms of the Incomplete Gamma Function, G:

$$\int_{-\infty}^{r}\frac{e^{-ct}}{t^4}dt = -\,G\,(-3, ct)\,c^3 \tag{19}$$

Combining (18) and (19) gives the form of L(r) and therefore K(r,s) which would then give V(r). This is obviously a complicated potential and, importantly here, does not look like the reflectionless potentials identified in other work. [5, 9]

3. Conclusion

Though the separability techniques used to solve the 1D Gelfand-Levitan equation can be applied to the 3D equation the results have a significant difference. In the case of the 1D equation, separability has been shown to be equivalent to the condition that the potential is reflectionless. In the 3D case, the potential has a much more complicate mathematical form that clearly is not a reflectionless potential.
Further research might consider how other techniques used in studying the 1D Gelfand-Levitan equation transfer to the 3D case.

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