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Research paper

Error propagation in the Abel hill inverse problem

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Abstract

Abel's Hill considers the problem of how to determine the shape of the hill from information on the return time of a rolled ball. Though this problem has been solved for piecewise monotonic functions, the issue of error propagation has not been addressed. This paper considers how errors in time measurement correlate to errors in determining the shape of the hill.

Keywords: *Abel's Hill; Inverse Problems; Error Propagation.*

1. Introduction

Abel's Hill is one of the earliest, seriously studied, inverse problems. [1-2] Abel considered if the shape of a hill could be determined if one knew the time it took for a ball rolled up the hill to return. By varying the initial energy of the ball, one gets different return times. The question then is if one can vary the energies from zero to infinity, for what cases can the shape of the hill be determined. Abel showed that if the hill was monotonic this was possible [1] and later authors showed that the solution is possible for hills that are piecewise monotonic. [3]

This paper considers a related issue. The equations to solve the Abel Hill problem are non-linear, so there may be an issue associated with error propagation. In other words, could the non-linearity cause small errors in return time measurement to lead to large errors in determining the hill structure? Here is presented a solution that shows this cannot be a concern for monotonic hills.

First is presented a summary of the known solution for a piecewise monotonic hill. Then by considering a simple assumption of the hill structure the associated error in the hill's monotonic dependence is shown not to give large error results for small errors in time.

Though only the classic hill problem is discussed here, more general applications of Abel's Hill have been studied. [3] The conclusions of this paper equally apply to those studies since they use the same fundamental relationships.

2. Solving the Abel Hill problem

A particle is slid up a frictionless hill with an initial energy E. The time for the particle to return to its original position is T(E). The question posed by Abel, is given a full range of initial energies, can the shape of the hill, s, be determined. Keller [3] gives a thorough solution which will not be detailed here.

The solution, for a monotonic hill is:

$$s = \frac{1}{(2m)^{1/2}\pi}\int_0^V (V-E)^{-\frac{1}{2}}\, T(E)\, dE \qquad (1)$$

Where m is the particle's mass and V is the potential energy of the hill. When V is allowed to vary from zero to infinity there is a complete solution to the problem.

Keller [3] further shows that a piecewise monotonic hill has a solution with a very similar form. This is not of interest to this paper since the same nonlinear dependence exists for both solutions.

3. Error propagation

Now consider the case in which there is an associated error in the measurement of the return time. Since this time is a function of the initial energy, E, an error in the time would depend on the error in that initial energy, δE. The time plus its associate error would then be:

$$T(E) = T + \frac{\partial T}{\partial E}\delta E \qquad (2)$$

Putting this expression in for the equation (1) gives what should be the hill shape plus its associated error:

$$s + \delta s = \frac{1}{(2m)^{1/2}\pi}\int_0^V (V-E)^{-\frac{1}{2}}\left(T + \frac{\partial T}{\partial E}\delta E\right) dE \qquad (3)$$

The right-hand side of the equation can be rewritten to give:

$$s + \delta s = \frac{1}{(2m)^{1/2}\pi}\int_0^V (V-E)^{-\frac{1}{2}}\, T\, dE + \frac{1}{(2m)^{1/2}\pi}\int_0^V (V-E)^{-\frac{1}{2}} \frac{\partial T}{\partial E}\delta E\, dE \qquad (4)$$

The first term on the right-hand side is s, so the second term is δs. The relative size of the error in s is the concern here, so the important quantity to consider is δs/s, which is by equation (4):

$$\frac{\delta s}{s} = \frac{I_2}{I_1} \qquad (5)$$

Where I_1 is the first integral on the right-hand side of equation (4) and I_2 is the second integral.
Now each of these integrals is to be found. Since we are dealing with monotonic solution is reasonable to assume that the time looks like E^n, where n is an integer. (Actually, this is a very reasonable assumption since the problem could also be solved by looking at the expansion of T in a Taylor series of E.) So, the assumed form of T is then:

$$T = \alpha E^n \qquad (6)$$

Where α is a constant.
Combining (6) with the I_1 gives:

$$I_1 = \alpha \int_0^V (V-E)^{-\frac{1}{2}} E^n\, dE \qquad (7)$$

Integral tables [4] give that:

$$I_1 = \frac{\alpha\, V^{n+1/2} n!\, \sqrt{\pi}}{\gamma(n + \frac{3}{2})} \qquad (8)$$

Where g is the Gamma Function.

Likewise, it is straightforward to show that:

$$I_2 = \frac{\alpha\, n\, \delta E\, V^{n-\frac{1}{2}} (n-1)!\, \sqrt{\pi}}{\gamma(n + \frac{1}{2})} \qquad (9)$$

So, by equations (5), (8) and (9), the relative size of the error is, after some simplification:

$$\frac{\delta s}{s} = \delta E\, V^{-1} \left(n + \frac{1}{2}\right) \qquad (10)$$

This is a surprisingly simple result that is easy to interpret. Recall the concern here is how errors in T, which are δE dependent, propagate through to errors in s. Here we see that this is not a concern since the relative error s is proportional to δE.

4. Conclusion

Since the solution to the Abel Hill problem is highly non-linear, there may be concern that errors in time measurement would lead to disproportional errors in the solution. This paper has shown that this is not a concern for the known solution involving monotonic hill shapes.

References

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