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Research paper

On fixed point convergence results for class of nonexpansive mappings in hyperbolic spaces via PJ iteration process

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Abstract

In this paper, we provide certain fixed point results for a mean nonexpansive mapping, as well as a new iterative algorithm called PJ-iteration for approximating the fixed point of this class of mappings in the setting of hyperbolic spaces. Furthermore, we establish strong and Δ-convergence theorem for mean nonexpansive mapping in hyperbolic space. Finally, we present a numerical example to illustrate our main result and then display the efficiency of the proposed algorithm compared to different iterative algorithms in the literature. Our results obtained in this paper improve, extend and unify some related results in the literature.

Keywords: *hyperbolic spaces; mean nonexpansive mapping; strong and Δ-convergence theorems.*

1. Introduction

The concept of a fixed point is important in many areas, including mathematics. Fixed point outcomes define the conditions for map solutions. Fixed-point techniques have been applied in a variety of disciplines, including biology, chemistry, economics, engineering, and informatics. Finding the exact value of a fixed point is often the final step in solving a problem, even though confirming its existence is an important initial step. An iterative procedure is one of the common ways to obtain the intended fixed point. In the last 65 years, many authors have been interested in these areas and established many iterative processes to approximate fixed points for nonexpansive mappings and a broader class of nonexpansive mappings.
The existence results for nonexpansive mappings have been established by Browder [6], Gohde [12] and Krik [22]. After that many researchers have obtained numerous generalization from their results. Suppose that $(\mathfrak{J}, ||.||)$ is real Banach spaces and $\mathfrak{D}$ is a nonempty subset of $\mathfrak{J}$. The mapping $\mathfrak{T}\colon \mathfrak{D} \to \mathfrak{D}$ is self mapping. Assume that $F(\mathfrak{T})$ is a set of all fixed points of mapping $\mathfrak{T}$. Then mapping $\mathfrak{T}$ is called
(i) nonexpansive, if

$$||\mathfrak{T}x - \mathfrak{T}y|| \leq ||x - y|| \ \forall x, y \in \mathfrak{D};$$

(ii) quasi nonexpansive, if

$$||\mathfrak{T}x - \rho|| \leq ||x - y|| \ \forall x \in \mathfrak{D}, \ \rho \in \mathfrak{F}(\mathfrak{T});$$

(iii) Suzuki generalized nonexpansive [31], if

$$(1/2)||x - \mathfrak{T}x|| \leq ||x - y|| \implies ||\mathfrak{T}x - \mathfrak{T}y|| \leq ||x - y|| \ \forall x, y \in \mathfrak{D}.$$

In [35], Zhang provided the following class of mapping.

Definition 1.1. Suppose that $(\mathfrak{J}, ||.||)$ is real Banach spaces and $\mathfrak{T}$ is a nonempty subset of $\mathfrak{J}$. the mapping $\mathfrak{T}\colon \mathfrak{D} \to \mathfrak{D}$ is self mapping. Then mapping $\mathfrak{T}$ is called mean nonexpansive mapping, if there are non-negative real numbers α, β such that $\alpha + \beta \leq 1$, we

$$||\mathfrak{T}x - \mathfrak{T}y|| \leq \alpha||x - y|| + \beta||x - \mathfrak{T}y|| \ \forall x, y \in \mathfrak{D}.$$

Remark 1.2. In [27], Nakprasit gave an example of a mapping that is mean nonexpansive but not Suzuki generalized nonexpansive and showed that increasing mean nonexpansive mappings are Suzuki generalized nonexpansive mappings.

In the literature, there are various mathematicians who worked in the direction of mean nonexpansive mappings to study their elementary properties [36] and to approximate fixed points of this class of mappings [1, 10, 34, 36]. It is easy to observe that each nonexpansive mapping is mean nonexpansive, but the converse is not true in general (see [36]).

2. Preliminaries

Throughout this paper, we consider the following definition of a hyperbolic space introduced by Kohlenbach [23].

Definition 2.1. A metric space $(\mathfrak{J},d)$ is said to be a hyperbolic space if there exists a map $\mathscr{V}: \mathfrak{J}^2 \times [0,1] \to \mathfrak{J}$ satisfying
(i) $d(\rho,\mathscr{V}(x,y,\alpha)) \leq \alpha d(\rho,x)+(1-\alpha)d(\rho,y)$,
(ii) $d(\mathscr{V}(x,y,\alpha),\mathscr{V}(x,y,\beta)) = |\alpha-\beta|d(x,y)$,
(iii) $\mathscr{V}(x,y,\alpha) = \mathscr{V}(y,x,(1-\alpha))$,
(iv) $d(\mathscr{V}(x,z,\alpha),\mathscr{V}(y,w,\alpha)) \leq \alpha d(x,y)+(1-\alpha)d(z,w)$,
for all $x,y,z,w \in \mathfrak{J}$ and $\alpha,\beta \in [0,1]$.

Many researchers attracted in the direction of approximating the fixed points of nonexpansive mapping and its generalized form [4, 5, 9, 14, 16, 17, 19, 20, 21, 30] in a hyperbolic space.

Definition 2.2. [32] A metric space is said to be convex, if a triple $(\mathfrak{J},d,\mathscr{V})$ satisfy only (i) in Definition 2.1.

Definition 2.3. [32] A subset $\mathfrak{D}$ of a hyperbolic space $\mathfrak{J}$ is said to be convex, if $\mathscr{V}(x,y,\alpha) \in \mathfrak{D}$ for all $x,y \in \mathfrak{D}$ and $\alpha \in [0,1]$.
If $x,y \in \mathfrak{J}$ and $\lambda \in [0,1]$, then we use the notation $(1-\lambda)x \oplus \lambda y$ for $\mathscr{V}(x,y,\lambda)$. The following holds even for more general setting of convex metric space [32] : for all $x,y \in \mathfrak{J}$ and $\lambda \in [0,1]$,

$$d(x,(1-\lambda)x \oplus \lambda y) = \lambda d(x,y)$$

and

$$d(y,(1-\lambda)x \oplus \lambda y) = (1-\lambda)d(x,y).$$

Thus

$$1x \oplus 0y = x,\ 0x \oplus 1y = y$$

and

$$(1-\lambda)x \oplus \lambda x = \lambda x \oplus (1-\lambda)x = x.$$

Definition 2.4. [24] A hyperbolic space $(\mathfrak{J},\partial,\mathscr{V})$ is said to be uniformly convex, if for any $\rho,x,y \in \mathfrak{J},\ r>0$ and $\varepsilon \in (0,2]$, there exists a $\delta \in (0,1]$ such that

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y,\rho\right) \leq (1-\delta)r,$$

whenever $d(x,\rho) \leq r,\ d(y,\rho) \leq r$ and $d(x,y) \geq \varepsilon r$.

Definition 2.5. A map $\eta: (0,\infty)\times(0,2] \to (0,1]$ which provides such a $\delta = \eta(r,\varepsilon)$ for given $r>0$ and $\varepsilon \in (0,2]$, is known as modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ε).
In [24], Luestean proved that every CAT(0) space is a uniformly convex hyperbolic space with modulus of uniform convexity $\eta(r,\varepsilon) = \frac{\varepsilon^2}{8}$ quadratic in ε.

Now we give the concept of Δ-convergence and some of its basic properties.
Let $\mathfrak{D}$ be a nonempty subset of metric space $(\mathfrak{J},d)$ and $\{y_n\}$ be any bounded sequence in $\mathfrak{J}$ while $diam(\mathfrak{D})$ denotes the diameter of $\mathfrak{D}$. Consider a continuous functional $r_a(.,\{y_n\}): \mathfrak{J} \to R^+$ defined by

$$r_a(y,\{y_n\}) = \limsup_{n\to\infty} d(y_n,y),\ y \in \mathfrak{J}.$$

The infimum of $r_a(.,\{y_n\})$ over $\mathfrak{D}$ is said to be an asymptotic radius of $\{y_n\}$ with respect to $\mathfrak{D}$ and it is denoted by $r_a(\mathfrak{D},\{y_n\})$. A point $z \in \mathfrak{D}$ is said to be an asymptotic center of the sequence $\{y_n\}$ with respect to $\mathfrak{D}$ if

$$r_a(z,\{y_n\}) = inf\{r_a(y,\{y_n\}): y \in \mathfrak{D}\}.$$

The set of all asymptotic center of $\{y_n\}$ with respect to $\mathfrak{D}$ is denoted by $AC(\mathfrak{D},\{y_n\})$. The set $AC(\mathfrak{D},\{y_n\})$ may be empty, singleton or have infinitely many points. If the asymptotic radius and asymptotic center are taken with respect to whole space $\mathfrak{J}$, then they are denoted by $r_a(\mathfrak{J},\{y_n\}) = r_a(\{y_n\})$ and $AC(\mathfrak{J},\{y_n\}) = AC(\{y_n\})$, respectively. We know that for $y \in \mathfrak{J},\ r_a(y,\{y_n\}) = 0$ if and only if $\lim_{n\to\infty} y_n = y$ and every bounded sequence has a unique asymptotic center with respect to closed convex subset in uniformly convex Banach spaces.

Definition 2.6. The sequence $\{y_n\}$ in $\mathfrak{J}$ is said to be Δ-convergent to $y \in \mathfrak{J}$, if y is unique asymptotic center of the every subsequence $\{u_n\}$ of $\{y_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} y_n = y$ and call y is the Δ-limit of $\{y_n\}$.

Lemma 2.7. *[25] Let $(\mathfrak{J},d,\mathscr{V})$ be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in $\mathfrak{J}$ has a unique asymptotic center with respect to any nonempty closed convex subset $\mathfrak{D}$ of $\mathfrak{J}$.*

Consider the following lemma of Khan et al. [18] which we use in the sequel.

Lemma 2.8. *Let $(\mathfrak{J},d,\mathscr{V})$ be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in \mathfrak{J}$ and $\{t_n\}$ be a sequence in $[a,b]$ for some $a,b \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathfrak{J}$ such that*

$$\limsup_{n\to\infty} d(x_n,x) \leq c,$$

$$\limsup_{n\to\infty} d(y_n,x) \leq c$$

and

$$\limsup_{n\to\infty} d(\mathscr{V}(x_n,y_n,t_n),x) = c$$

for some $c \geq 0$, then $\lim_{n\to\infty} d(x_n,y_n) = 0$.

Definition 2.9. Let $\mathfrak{D}$ be a nonempty convex closed subset of a hyperbolic space $\mathfrak{J}$ and $\{x_n\}$ be a sequence in $\mathfrak{J}$. Then $\{x_n\}$ is said to be Fejér monotone with respect to M if for all $x \in \mathfrak{D}$ and $n \in \mathbb{N}$,

$$d(x_{n+1}, x) \leq d(x_n, x).$$

Proposition 2.10. Let $\{x_n\}$ be a sequence in $\mathfrak{J}$ and $\mathfrak{D}$ be a nonempty subset of $\mathfrak{J}$. Let $\mathfrak{T}\colon \mathfrak{D} \to \mathfrak{D}$ be a nonexpansive mapping with $F(\mathfrak{T}) \neq \emptyset$. Suppose that $\{x_n\}$ is Fejér monotone with respect to $\mathfrak{D}$. Then we have the followings:
(1) $\{x_n\}$ is bounded.
(2) The sequence $\{d(x_n, p)\}$ is decreasing and converges for all $p \in F(\mathfrak{T})$.
(3) $\lim_{n\to\infty} D(x_n, F(\mathfrak{T}))$ exists, where $D(x,A) = \inf_{y\in A} d(x,y)$.

Now, we present some fundamental properties of mean nonexpansive mapping (see [10]).

Definition 2.11. Assume that $\mathfrak{D}$ is a nonempty subset of a hyperbolic space $\mathfrak{J}$ and $\mathfrak{T}\colon \mathfrak{D} \to \mathfrak{D}$ satisfies the mean nonexpansive mapping with $F(\mathfrak{T}) \neq \emptyset$. Then $\mathfrak{T}$ is quasi-nonexpansive.

Lemma 2.12. *[30] Let $\mathfrak{J}$ be complete uniformly convex hyperbolic space with monotone modulus of convexity η, $\mathfrak{D}$ be a nonempty closed convex subset of $\mathfrak{J}$ and $\mathfrak{T}\colon \mathfrak{D} \to \mathfrak{D}$ satisfies the mean nonexpansive mapping. If $\{x_n\}$ is a bounded sequence in $\mathfrak{D}$ such that $\lim_{n\to\infty} d(x_n, \mathfrak{T}x_n) = 0$, then $\mathfrak{T}$ has a fixed point in $\mathfrak{D}$.*

Lemma 2.13. *[30] Let $\mathfrak{D}$ be a nonempty, bounded, closed and convex subsetof a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and $\mathfrak{T}$ satisfies the mean nonexpansive mapping on $\mathfrak{D}$. Suppose that $\{x_n\}$ is a sequence in $\mathfrak{D}$, with $d(x_n, \mathfrak{T}x_n) \to 0$. If $AC(\mathfrak{D}, \{x_n\}) = \rho$, then ρ is a fixed point of $\mathfrak{T}$. Moreover, $F(\mathfrak{T})$ is closed and convex.*

3. Main result

In the literature of fixed-point iterations, one can search for many iterative methods that converge in the case of nonexpansive operators and also suggest better accuracy as compared to the Picard iteration method. If $\mathfrak{D}$ is a closed and convex subset of a Banach space, $n \in \mathbb{N}$ and $\alpha_n, \beta_n, \gamma_n \in (0,1)$. Then for $x_1 \in \mathfrak{D}$, Mann [26], Ishikawa [15], Noor [28], Agarwal [3], Abbas [2], Thakur [33] and Ullah [13] iterative methods.
A natural question arises: does there exist an iterative method that is essentially better than all of the above iterative methods, including the Ullah iterative method [13]? To answer this question, we introduced and studied four step iteration process called PJ-iterative method as follows:
For convex subset $\mathfrak{D}$ of normed linear space $\mathfrak{J}$ and a mapping $\mathfrak{T}\colon \mathfrak{D} \to \mathfrak{D}$, $x_1 \in \mathfrak{D}$, construct a sequence $\{x_n\}$ in $\mathfrak{D}$ as follows:

$$\begin{cases} x_1 \in \mathfrak{D}, \\ w_n = \mathfrak{T}((1-\alpha_n)x_n + \alpha_n \mathfrak{T}x_n), \\ z_n = \mathfrak{T}((1-\beta_n)w_n + \beta_n \mathfrak{T}w_n), \\ y_n = \mathfrak{T}((1-\gamma_n)z_n + \gamma_n \mathfrak{T}z_n), \\ x_{n+1} = \mathfrak{T}(y_n),\ n \geq 1 \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in (0,1). Now, we establish the convergence results for PJ-iteration process for mean nonexpansive mapping in hyperbolic spaces, as follows: Let $\mathfrak{D}$ be a nonempty, closed and convex subset of a hyperbolic space $\mathfrak{J}$ and $\mathfrak{T}$ be a mean nonexpansive mapping on $\mathfrak{D}$. For any $x_1 \in \mathfrak{D}$ the sequence $\{x_n\}$ is defined by

$$\begin{cases} w_n = \mathscr{V}(\mathfrak{T}\sigma_n, 0, 0), \\ \sigma_n = \mathscr{V}(x_n, \mathfrak{T}x_n, \alpha_n), \\ z_n = \mathscr{V}(\mathfrak{T}\nu_n, 0, 0), \\ \nu_n = \mathscr{V}(z_n, \mathfrak{T}w_n, \beta_n), \\ y_n = \mathscr{V}(\mathfrak{T}\rho_n, 0, 0), \\ \rho_n = \mathscr{V}(y_n, \mathfrak{T}y_n, \gamma_n), \\ x_{n+1} = \mathscr{V}(\mathfrak{T}y_n, 0, 0)\ \forall n \in \mathbb{N}. \end{cases} \tag{3.2}$$

This section establishes some significant strong and Δ-convergence results for operators with mean nonexpansive mapping.

Theorem 3.1. *Let $\mathfrak{D}$be a nonempty, closed and convex subset of a hyperbolic space $\mathfrak{J}$ and $\mathfrak{T}\colon \mathfrak{D} \to \mathfrak{D}$ satisfies the mean nonexpansive mapping. If $\{x_n\}$ is a sequence defined by (3.2), then $\{x_n\}$ is Fejér monotone with respect to $F(\mathfrak{T})$.*

Proof. Since $\mathfrak{T}$ is a mean nonexpansive mapping, for $\rho \in F(\mathfrak{T})$, we have
Using Definition 1.1, 2.11 and (3.2), we get

$$\begin{aligned} d(w_n, p) &= d(\mathscr{V}(\mathfrak{T}\sigma_n, 0, 0)), \rho) \\ &= d(\mathfrak{T}\sigma_n, \rho) \\ &\leq \alpha d(\sigma_n, \rho) + \beta d(\sigma_n, \rho) \\ &\leq \alpha d(\mathscr{V}(x_n, \mathfrak{T}x_n, \alpha_n), \rho) + \beta d(\mathscr{V}(x_n, \mathfrak{T}x_n, \alpha_n), \rho) \\ &\leq \alpha[(1-\alpha_n)d(x_n, \rho) + \alpha_n d(\mathfrak{T}x_n, \rho)] + \beta[(1-\alpha_n)d(x_n, \rho) + \alpha_n d(\mathfrak{T}x_n, \rho)] \\ &\leq \alpha[(1-\alpha_n)d(x_n, \rho) + \alpha_n(\alpha d(x_n, \rho) + \beta d(x_n, \rho))] \\ &\quad + \beta[(1-\alpha_n)d(x_n, \rho) + \alpha_n(\alpha d(x_n, \rho) + \beta d(x_n, \rho))] \\ &\leq \alpha(d(x_n, \rho)) + \beta(d(x_n, \rho)) \\ &\leq d(x_n, \rho). \end{aligned} \tag{3.3}$$

Using Definition 1.1, 2.11, (3.2) and (3.3), we get

$$\begin{aligned}
d(z_n,p) &= d(\mathscr{V}(\mathfrak{T}v_n,0,0)),\rho) \\
&= d(\mathfrak{T}v_n,\rho) \\
&\leq \alpha d(v_n,\rho)+\beta d(v_n,\rho) \\
&\leq \alpha d(\mathscr{V}(w_n,\mathfrak{T}w_n,\beta_n),\rho)+\beta d(\mathscr{V}(w_n,\mathfrak{T}w_n,\beta_n),\rho) \\
&\leq \alpha[(1-\beta_n)d(w_n,\rho)+\beta_n d(\mathfrak{T}w_n,\rho)]+\beta[(1-\beta_n)d(w_n,\rho)+\beta_n d(\mathfrak{T}w_n,\rho)] \\
&\leq \alpha[(1-\beta_n)d(w_n,\rho)+\beta_n(\alpha d(w_n,\rho)+\beta d(w_n,\rho))] \\
&+\beta[(1-\beta_n)d(w_n,\rho)+\beta_n(\alpha d(w_n,\rho)+\beta d(w_n,\rho))] \\
&= \alpha(d(w_n,\rho))+\beta(d(w_n,\rho)) \\
&\leq \alpha(d(x_n,\rho))+\beta(d(x_n,\rho)) \\
&\leq d(x_n,\rho).
\end{aligned} \tag{3.4}$$

Using Definition 1.1, 2.11, (3.2), (3.3) and (3.4), we get

$$\begin{aligned}
d(y_n,p) &= d(\mathscr{V}(\mathfrak{T}\rho_n,0,0)),\rho) \\
&= d(\mathfrak{T}\rho_n,\rho) \\
&\leq \alpha d(\rho_n,\rho)+\beta d(\rho_n,\rho) \\
&\leq \alpha d(\mathscr{V}(z_n,\mathfrak{T}z_n,\gamma_n),\rho)+\beta d(\mathscr{V}(z_n,\mathfrak{T}z_n,\gamma_n),\rho) \\
&\leq \alpha[(1-\gamma_n)d(z_n,\rho)+\gamma_n d(\mathfrak{T}z_n,\rho)]+\beta[(1-\gamma_n)d(z_n,\rho)+\gamma_n d(\mathfrak{T}z_n,\rho)] \\
&\leq \alpha[(1-\gamma_n)d(z_n,\rho)+\gamma_n(\alpha d(z_n,\rho)+\beta d(z_n,\rho))] \\
&+\beta[(1-\gamma_n)d(z_n,\rho)+\gamma_n(\alpha d(z_n,\rho)+\beta d(z_n,\rho))] \\
&= \alpha(d(z_n,\rho))+\beta(d(z_n,\rho)) \\
&\leq \alpha(d(x_n,\rho))+\beta(d(x_n,\rho)) \\
&\leq d(x_n,\rho).
\end{aligned} \tag{3.5}$$

Using Definition 1.1, 2.11, (3.2), (3.3), (3.4) and (3.5), we get

$$\begin{aligned}
d(x_{n+1}) &= d(\mathscr{V}(\mathfrak{T}y_n,0,0)\rho) \\
&= d(\mathfrak{T}y_n,\rho) \\
&\leq \alpha(d(y_n,\rho))+\beta(d(y_n,\rho)) \\
&\leq d(y_n,\rho) \\
&\leq d(x_n,\rho).
\end{aligned} \tag{3.6}$$

Hence, $\{x_n\}$ is Fejér monotone with respect to $F(\mathfrak{T})$. □

Theorem 3.2. *Let $\mathfrak{D}$ be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space $\mathfrak{J}$ with monotone modulus of uniform convexity η and $\mathfrak{T}\colon \mathfrak{D}\to\mathfrak{D}$ satisfies the mean nonexpansive mapping. If $\{x_n\}$ is a sequence defined by (3.2), then $F(\mathfrak{T})$ is nonempty if and only if the sequence $\{x_n\}$ is bounded and* $\lim_{n\to\infty} d(x_n,\mathfrak{T}x_n)=0$.

Proof. Due to Theorem 3.1, $\lim_{n\to\infty} d(x_n,\rho)$ exists for each $\rho\in F(\mathfrak{T})$. Assume that $\lim_{n\to\infty} d(x_n,\rho)=l$. If $l=0$,

$$\begin{aligned}
\lim_{n\to\infty} d(x_n,\mathfrak{T}x_n) &\leq \lim_{n\to\infty} d(x_n,\rho)+\lim_{n\to\infty} d(\rho,\mathfrak{T}x_n) \\
&\leq \lim_{n\to\infty} d(x_n,\rho)+\lim_{n\to\infty}[\alpha d(x_n,\rho)+\beta d(x_n,\rho)] \\
&= 0
\end{aligned}$$

If $l>0$, then using (3.2), we have

$$d(w_n,\rho)\leq d(x_n,\rho)$$

Therefore, we have

$$\limsup_{n\to\infty} d(w_n,\rho)\leq l.$$

Using (3.6), we get

$$\begin{aligned}
\liminf_{n\to\infty} d(x_{n+1},\rho) &\leq \liminf_{n\to\infty} d(w_n,\rho) \\
l &\leq \liminf_{n\to\infty} d(w_n,\rho).
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} d(w_n,\rho)=l.$$

Hence,

$$
\begin{aligned}
l &= \lim_{n\to\infty} d(w_n,\rho)\\
&= \lim_{n\to\infty} d(\mathscr{V}(\mathfrak{T}\sigma_n,0,0)),\rho)\\
&= \lim_{n\to\infty} d(\mathfrak{T}\sigma_n,\rho)\\
&\le \lim_{n\to\infty} [\alpha d(\sigma_n,\rho)+\beta d(\sigma_n,\rho)]\\
&\le \alpha \lim_{n\to\infty} d(\mathscr{V}(x_n,\mathfrak{T}x_n,\alpha_n),\rho)+\beta \lim_{n\to\infty} d(\mathscr{V}(x_n,\mathfrak{T}x_n,\alpha_n),\rho)\\
&\le \alpha[(1-\alpha_n)\lim_{n\to\infty} d(x_n,\rho)+\alpha_n \lim_{n\to\infty} d(\mathfrak{T}x_n,\rho)]+\\
&\beta[(1-\alpha_n)\lim_{n\to\infty} d(x_n,\rho)+\alpha_n \lim_{n\to\infty} d(\mathfrak{T}x_n,\rho)]\\
&\le \alpha[(1-\alpha_n)\lim_{n\to\infty} d(x_n,\rho)+\alpha_n(\alpha d(x_n,\rho)+\beta \lim_{n\to\infty} d(x_n,\rho))]\\
&+\beta[(1-\alpha_n)\lim_{n\to\infty} d(x_n,\rho)+\alpha_n(\alpha d(x_n,\rho)+\beta \lim_{n\to\infty} d(x_n,\rho))]\\
&\le \alpha(\lim_{n\to\infty} d(x_n,\rho))+\beta(\lim_{n\to\infty} d(x_n,\rho))\\
&\le \lim_{n\to\infty} d(x_n,\rho).\\
&= l.
\end{aligned} \tag{3.7}
$$

From Lemma 2.8,

$$\lim_{n\to\infty} d(x_n,\mathfrak{T}x_n)=0.$$

Hence, the proof is complete. □

Theorem 3.3. *Let $\mathfrak{D}$ be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space $\mathfrak{J}$ with monotone modulus of uniform convexity η. Let $\mathfrak{T}\colon \mathfrak{D}\to\mathfrak{D}$ satisfies the mean nonexpansive mapping with $F(\mathfrak{T})\neq\emptyset$. Then the sequence $\{x_n\}$ defined in (3.2), is Δ-convergent to a fixed point of $\mathfrak{T}$.*

Proof. From Theorem 3.1, we observe that $\{x_n\}$ is a bounded sequence, therefore $\{x_n\}$ has a Δ-convergent subsequence. Now we will prove that every Δ-convergent subsequence of $\{x_n\}$ has a unique $\Delta-limit$ in $F(\mathfrak{T})$. For this, let y and z be $\Delta-limit$ of the subsequences $\{y_n\}$ and $\{z_n\}$ of $\{x_n\}$ respectively.
Now by Lemma 2.7, $AC(\mathfrak{D},\{y_n\})=\{y_n\}$ and $AC(\mathfrak{D},\{z_n\})=\{z_n\}$. By Theorem 3.2, we have $\lim_{n\to\infty} d(y_n,\mathfrak{T}y_n)=0$.
Now we will prove that y and z are fixed points of $\mathfrak{T}$ and they are same. If not, then by the uniqueness of the asymptotic center

$$
\begin{aligned}
\limsup_{n\to\infty} d(x_n,y) &= \limsup_{n\to\infty} d(y_n,y)\\
&< \limsup_{n\to\infty} d(y_n,z)\\
&= \limsup_{n\to\infty} d(x_n,z)\\
&= \limsup_{n\to\infty} d(z_n,z)\\
&< \limsup_{n\to\infty} d(z_n,y)\\
&= \limsup_{n\to\infty} d(x_n,y)
\end{aligned}
$$

which is a contradiction. Hence $y=z$ and sequence $\{x_n\}$ is Δ-convergent to a unique fixed point of $\mathfrak{T}$. □

Theorem 3.4. *Let $\mathfrak{D}$ be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space $\mathfrak{J}$ with monotone modulus of uniform convexity η and $\mathfrak{T}\colon \mathfrak{D}\to\mathfrak{D}$ satiesfies the mean nonexpansive mapping with $F(\mathfrak{T})\neq\emptyset$. Then the sequence $\{x_n\}$ which is defined by (3.2), converges strongly to some fixed point of $\mathfrak{T}$ if and only if $\liminf_{n\to\infty} D(x_n,F(\mathfrak{T}))=0$, where $D(x_n,F(\mathfrak{T}))=\inf_{y\in F(\mathfrak{T})} d(x_n,y)$.*

Proof. Assume that $\{x_n\}$ converges strongly to $y\in F(\mathfrak{T})$. Therefore we have $\lim_{n\to\infty} d(x_n,y)=0$. Since $0\le D(x_n,F(\mathfrak{T}))\le d(x_n,y)$, we have

$$\liminf_{n\to\infty} D(x_n,F(\mathfrak{T}))=0.$$

Next, we prove sufficient part. From Lemma 2.13, the fixed point set $F(\mathfrak{T})$ is closed. Suppose that

$$\liminf_{n\to\infty} D(x_n,F(\mathfrak{T}))=0.$$

Then, from (3.5), we have

$$D(x_{n+1},F(\mathfrak{T}))\le D(x_n,F(\mathfrak{T})).$$

From Theorem 3.1 and Proposition 2.10, we have $\lim_{n\to\infty} d(x_n,F(\mathfrak{T}))$ exists. Hence

$$\lim_{n\to\infty} D(x_n,F(\mathfrak{T}))=0.$$

Consider the subsequence $\{x_{n_k}\}$of $\{x_n\}$ such that $d(x_{n_k}, p_k) < \frac{1}{2^k}$ for all $k \geq 1$, where $\{p_k\}$ is in $F(\mathfrak{T})$. From (3.4), we have

$$d(x_{n_{k+1}}, p_k) \leq d(x_{n_k}, p_k) < \frac{1}{2^k},$$

which implies that

$$\begin{aligned} d(p_{k+1}, p_k) &\leq d(p_{k+1}, x_{n_{k+1}}) + d(x_{n_{k+1}}, p_k) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned}$$

This shows that $\{p_k\}$ is a Cauchy sequence. Since $F(\mathfrak{T})$is closed, $\{p_k\}$ is a convergent sequence. Let $\lim_{k\to\infty} p_k = p$. Then we know that $\{x_n\}$ converges to y. Since

$$d(x_{n_k}, y) \leq d(x_{n_k}, p_k) + d(p_k, y),$$

we have

$$\lim_{k\to\infty} d(x_{n_k}, y) = 0.$$

Since $\lim_{n\to\infty} d(x_n, y)$ exists, the sequence $\{x_n\}$ converges to y. □

Recall that a mapping $\mathfrak{T}$ from a subset of a hyperbolic space $\mathfrak{J}$ into itself with $F(\mathfrak{T}) \neq \emptyset$ is said to satisfy condition (I) if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(t) > 0$ for $t \in (0,\infty)$ such that

$$d(x, \mathfrak{T}x) \geq f(D(x, F(\mathfrak{T}))),$$

for all $x \in \mathfrak{D}$.

Theorem 3.5. *Let $\mathfrak{D}$ be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space $\mathfrak{J}$ with monotone modulus of uniform convexity η and $\mathfrak{T}\colon \mathfrak{D} \to \mathfrak{D}$ satisfies the mean nonexpansive mapping. Moreover, $\mathfrak{T}$ satisfies the condition (I) with $F(\mathfrak{T}) \neq \emptyset$. Then the sequence $\{x_n\}$ which is defined by (3.2), converges strongly to some fixed point of $\mathfrak{T}$.*

Proof. From Lemma 2.13, we have $F(\mathfrak{T})$ is closed. Observe that by Theorem 3.2, we have $\lim_{n\to\infty} d(x_n, \mathfrak{T}x_n) = 0$. It follows from the condition (I) that

$$\lim_{n\to\infty} f(D(x_n, F(\mathfrak{T}))) \leq \lim_{n\to\infty} d(x_n, \mathfrak{T}x_n).$$

Thus, we get $\lim_{n\to\infty} f(D(x_n, F(\mathfrak{T}))) = 0$. Since $f\colon [0,1) \to [0,1)$ is a nondecreasing mapping with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$, we have $\lim_{n\to\infty} D(x_n, F(\mathfrak{T})) = 0$. Rest of the proof follows in lines of Theorem 3.4. Hence the sequence $\{x_n\}$ is convergent to $p \in F(\mathfrak{T})$. This completes the proof. □

4. Numerical example

Example 4.1. Let $\mathfrak{D} = [0,1]$ which is a closed, and convex subset of the hyperbolic space $\mathfrak{J} = \mathbb{R}$, endowed with the usual metric. Define a mapping $\mathfrak{T}\colon \mathfrak{D} \to \mathfrak{D}$

$$\mathfrak{T}x = \begin{cases} \frac{x}{5}, & if \;\; x \in [0, \frac{1}{2}), \\ \frac{x}{6}, & if \;\; x \in [\frac{1}{2}, 1]. \end{cases}$$

Then $\mathfrak{T}$ is mean nonexpansive with $\alpha = \frac{1}{3}$, $\beta = \frac{2}{3}$, but nut continuous at $x = \frac{1}{2}$. Thus, $\mathfrak{T}$ is not a nonexpansive mapping. By using example 4.1, we tried to show that the rate of convergence of the PJ iteration is better then some known iteration processes for mean nonexpansive mapping. Parameters are $\alpha_n = 1 - \frac{1}{(2n+8)}$, $\beta_n = \frac{n}{16n+1}$, $\gamma_n = \frac{n}{(n+5)}$, $\forall n \in \mathbb{N}$.

Table 1: Convergence of PJ iteration for fixed point 0.

No. of iteration	Agrawal	Thakur	K	PJ
1	0.75	0.75	0.75	0.75
2	0.11868721	0.02373744	0.00540538	0.00000001
3	0.02258659	0.00090346	0.00004569	0
4	0.00429831	0.00000343	0.00000038	0
5	0.00081798	0.00000130	0	0
6	0.00015566	0.00000004	0	0
7	0.00002962	0	0	0
8	0.00000563	0	0	0
9	0.00000010	0	0	0
10	0.00000002	0	0	0

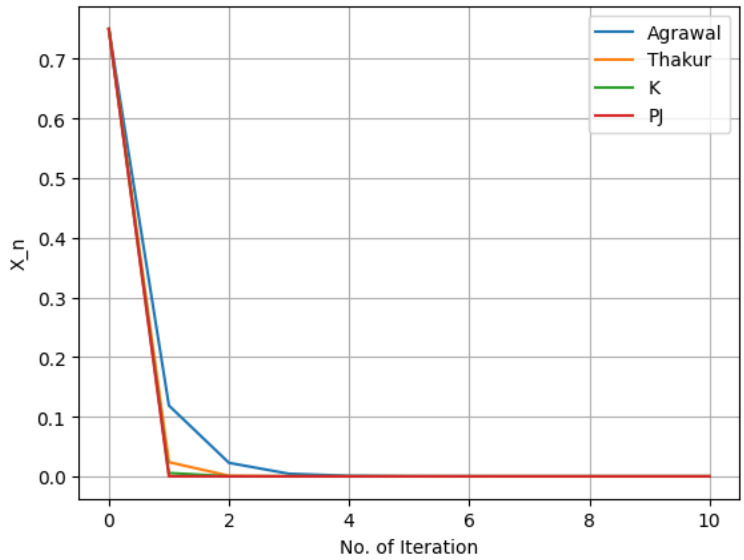

Figure 1: Convergence of Agrawal, Thakur, K and PJ iterations

Clearly $\rho = 0$ is a fixed point of mean nonexpansive mapping. Table 1 shows that behaviour of some iteration processes to fixed point of $\mathfrak{T}$ for initial value 0.75.

5. Conclusion

In this work, we present some fixed point results for a mean nonexpansive mapping and also used a PJ iterative algorithm for approximating the fixed point of this class of mappings in the framework of hyperbolic spaces. Our numerical experiment shows that our iterative algorithm is better compare to some existing iterative algorithms in the literature.

References

[1] J. Ahmad, K. Ullah, M. Arshad and M. Sen, Approximation fixed points for mean nonexpansive mapping in Banach spaces, *J. Func. Spaces*, **Article ID 1934274**(2021), 6.
[2] M. Abbas and T. Nazir, A new faster iteration process applied to constrained minimization and feasibility problems, *Matematichki Vesnik*, **67(2)**(2014), 223–234.
[3] R.P. Agarwal, D. O'Regan and D.R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, *J. of Nonlinear and Conv. Anal.*, **8(1)**(2007), 61–79.
[4] S. Aggarwal, I. Uddin and S. Mujahid, Convergence theorems for SP-iteration scheme in a ordered hyperbolic metric space, *Nonlinear Funct. Anal. Appl.*, **26(5)**(2021), 961-969.
[5] F. Akutsah and O.K. Narain, On generalized (α,β)-nonexpansive mappings in Banach spaces with applications, *Nonlinear Funct. Anal. Appl.*, **26(4)**(2021), 663-684..
[6] F. E. Browder, Nonexpansive nonlinear operators in a Banach space, *P. Natl. A. Sci.*, **54**(1965), 1041–1044.
[7] V. Berinde, Picard iteration converges faster than Mann iteration for a class of quasi contractive operators, *Fixed Point Theory Appl.*, **2**(2004), 97-105.
[8] S.S. Chang, G. Wang, L. Wang, Y.K. Tang and G.L. Ma, Δ-convergence theorems for multi-valued nonexpansive mapping in hyperbolic spaces, *Appl. Math. Comput.*, **249**(2014), 535–540.
[9] S. Dashputre, Padmavati and K. Sakure, Strong and Δ-convergence results for generalized nonexpansive mapping in hyperbolic space, *Comm. Math. Appl.*, **11(3)**(2020), 389-401.
[10] J. N. Ezeora, C. Izuchukwu, A. Mebawondu and O. Mewomo, Approximating common fixed points of mean nonexpansive mapping in hyperbolic space, *Int. J. of Nonl. Ana. and Appl.*, **13(2)**(2022), 459-471.
[11] J. García-Falset, E. Llorens-Fuster and T. Suzuki, Fixed point theory for a class of generalized nonexpansive mappings, *J. Math. Anal. Appl.*, **375**(2011), 185-195.
[12] D. Gohde, Zum prinzip der kontraktiven abbildung, *Math. Nachr.*, **30(3-4)**(1965), 251–258.
[13] N. Hussain, K. Ullah and M. Arshad, Fixed point approximation of Suzuki generalized nonexpansive mappings via new faster iteration process, *J. Nonlinear Convex Anal.*, **19**(2018), 1383–1393.
[14] M. Imdad and S. Dashputre, Fixed point approximation of Picard normal S-iteration process for generalized nonexpansive mappings in hyperbolic spaces, *Math. Sci.*, **10(3)**(2016), 131-138.
[15] S. Ishikawa, Fixed points by a new iteration method, *Proc. Amer. Math. Soc.*, **44**(1974), 147–150.
[16] S.M. Kang, S. Dashputre, B.L. Malagar and Y.C. Kwun, Fixed point approximation for asymptotically nonexpansive type mappings in uniformly convex hyperbolic spaces, *J. Appl. Math.*, **2015** Article ID 510798, 7 pages.
[17] S.M. Kang, S. Dashputre, B.L. Malagar and A. Rafiq, On the convergence of fixed points for Lipschitz type mappings in hyperbolic spaces, *Fixed Point Theory Appl.*, **2014**(2014), 229.
[18] A.R. Khan, H. Fukhar-ud-din and M.A. Khan, An implicit algorithm for two finite families of nonexpansive maps in hyperbolic space, *Fixed Point Theory Appl.*, **2012**(2012), 54.
[19] J.K. Kim and S. Dashputre, Fixed point approximation for SKC mappings in hyperbolic spaces, *J. Ineq. Appl.*, **2015(1)**(2015), 1-16.
[20] J.K. Kim, R.P. Pathak, S. Dashputre, S.D. Diwan and R. Gupta, Fixed point approximation of generalized nonexpansive mappings in hyperbolic spaces, *Inter. J. Math. Math. Sci.*, **2015**(2015) Article Id : 368204.
[21] J.K. Kim, R.P. Pathak, S. Dashputre, S.D. Diwan and R. Diwan, Convergence theorems for generalized nonexpansive multivalued mapping in hyperbolic space, *SpringerPlus,*, **5(1)**(2016), 1-16.
[22] W. A. Kirk, A fixed point theorem for mappings which do not increase distance, *Am. Math. Mon.*, **72**(1965), 1004–1006.
[23] U. Kohlenbach, Some logical metatheorems with applications in functional analysis, *Trans. Amer. Math. Soc.*, **357(1)**(2004), 89-128.
[24] L. Leustean, A quadratic rate of asymptotic regularity for CAT(0) spaces, *J. Math. Anal. Appl.*, **325(1)**(2007), 386-399.
[25] L. Leustean, Nonexpansive iteration in uniformly convex W-hyperbolic space, *J. Math. Anal. Appl.*, **513**(2010), 193-209.
[26] W. R. Mann, Mean value methods in iteration, *Proceedings of American Mathematical Society*, **4(3)**(1953), 506–510.

[27] K. Nakprasit, Mean nonexpansive mappings and Suzuki-generalized nonexpansive mappings, *Journal of Mathematical Analysis and Applications*, **1(1)**(2010), 93-96.

[28] M. A. Noor, New approximation schemes for general variational inequalities, *Journal of Mathematical Analysis and Applications*, **251(1)**(2000), 217–229.

[29] E. Picard, Ḿemoire sur la théorie des ´equations aux dérivées partielles et la méthode des approximations successives, *J. Math. Pures Appl.,* **6**(1890), 145-210.

[30] C. Suanoom, K. Sriwichai, C. Klin-Eam and W. Khuangsatung, The generalized α-nonexpansive mappings and related convergence theorems in hyperbolic spaces, *J. Inform. Math. Sci.*, **11(1)**(2019), 1-17.

[31] T. Suzuki, Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, *Journal of Mathematical Analysis and Applications*, **340(4)**(2008), 1088–1095.

[32] W. Takahashi, A convexity in metric space and nonexpansive mappings, *I. Kodai Math. Sem. Rep.*, **22**(1970), 142-149.

[33] B. S. Thakur, D. Thakur and M. Postolache, A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings, *Appl. Math. Comput.*, **275**(2016), 147–155.

[34] Y. Yang and Y. Cui , Viscosity approximation methods for mean nonexpansive mappings in Banach spaces, *Appl. Math. Sci.*, **2(13)**(2008), 627–638.

[35] S. S. Zhang, About fixed point theorem for mean nonexpansive mapping in Banach spaces, *J. of Sichuan University*, **2**(1975), 67–68.

[36] J. Zou, Fixed point theorem for mean nonexpansive mappings in Banach spaces, *Abs. and app. anal.* , **2014**(2014), 6.