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Research paper

Convergence Analysis of Picard Thakur Hybrid Iterative Scheme for α-Nonexpansive Mappings in Uniformly Convex Banach Spaces

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Abstract

In this study, we investigate the convergence behavior of fixed points for generalized α-nonexpansive mappings using the Picard-Thakur hybrid iterative scheme. We obtain weak and strong convergence results for generalized α-nonexpansive mappings in a uniformly convex Banach space. Numerically, we demonstrate that the Picard-Thakur hybrid iterative scheme converges more rapidly than other well-known schemes. Additionally, we present findings on data dependence and provide a numerical example to illustrate the concept. The obtained results are expanded and generalized to be consistent with relevant findings in the existing literature.

Keywords: *Banach space; Fixed Point; Generalized α-Nonexpansive Mapping; Numerical Example; Picard-Thakur hybrid iterative scheme*

1. Introduction

Let $\mathscr{M}$ be a nonempty subset of a Banach space $\mathscr{B}$. A mapping $\mathscr{G}:\mathscr{M}\to\mathscr{M}$ is;
(1) a contraction mapping if for all $a,b\in\mathscr{M}$ and $\zeta\in(0,1)$ such that

$$\|\mathscr{G}a-\mathscr{G}b\|\leq\zeta\|a-b\|.$$

(2) a nonexpansive mapping if for all $a,b\in\mathscr{M}$

$$\|\mathscr{G}a-\mathscr{G}b\|\leq\|a-b\|.$$

(3) a quasi-nonexpansive if there exists an element $a^*\in F(\mathscr{G})$ such that

$$\|\mathscr{G}a-a^*\|\leq\|a-a^*\|,\text{ for all } a\in\mathscr{M}.$$

We denote the set of all fixed points of the mapping $\mathscr{G}:\mathscr{M}\to\mathscr{M}$ by $F(\mathscr{G})$.
In [7] Browder showed that, for any nonempty closed convex subset $\mathscr{M}$ of uniformly convex Banach space (UCBS) $\mathscr{B}$ has a fixed point for a nonexpansive self mapping.
Suzuki [28] presented a new kind of mapping called *Condition (C)* in 2008, which is weaker than nonexpansive mappings but stronger than quasi-nonexpansive mappings. Suzuki discussed the fixed point existence results for a mapping that satisfies the *condition (C)*.
$\mathscr{G}:\mathscr{M}\to\mathscr{M}$ is said to satisfy *Condition (C)* if

$$\frac{1}{2}\|a-\mathscr{G}a\|\leq\|a-b\|\Rightarrow\|\mathscr{G}a-\mathscr{G}b\|\leq\|a-b\|,\ \forall\, a,b\ \in\mathscr{M}. \tag{1}$$

In 2011, Aoyama and Kohshaka [3] defined a new type of mappings in Banach spaces, known as α-nonexpansive mapping, and explored its fixed points. A mapping $\mathscr{G}$ from $\mathscr{M}$ to $\mathscr{M}$ is said to be α-nonexpansive if for all $a,b\in\mathscr{M}$ there is an $\alpha<1$ such that

$$\|\mathscr{G}a-\mathscr{G}b\|^2\leq\alpha\|b-\mathscr{G}a\|^2+\alpha\|a-\mathscr{G}b\|^2+(1-2\alpha)\|a-b\|^2. \tag{2}$$

A nonexpansive mapping is clearly a 0-nonexpansive mapping. An example of a discontinuous α-nonexpansive mapping (with $\alpha>0$) has been given in [3]. In general, an α-nonexpansive mapping and a mapping that satisfies the *Condition (C)* are not continuous, as demonstrated in [21].

Pant and Shukla [19] defined a new type of mappings known as the generalized α-nonexpansive mapping, which is defined in such a way that:

$$\frac{1}{2}\|a-\mathscr{G}a\| \leq \|a-b\|$$

implies

$$\|\mathscr{G}(a)-\mathscr{G}(b)\| \leq \alpha\|b-\mathscr{G}(a)\|+\alpha\|a-\mathscr{G}(b)\|+(1-2\alpha)\|a-b\|, \tag{3}$$

for all $a,b \in \mathscr{M}$ where $\alpha \in (0,1)$.
Several researchers used this mapping to approximate the fixed points in Banach spaces. For instance, we refer to [4, 8, 19, 25].
In 1922, Banach [5] introduced the Banach Contraction Principle which states that fixed points of a contraction mapping can be approximated by Picard iterative scheme [20]. The Picard sequence $\{a_n\}$ defined as follows

$$\begin{cases} a_1 \in \mathscr{M}, \quad n \in \mathbb{Z}^+ \\ a_{n+1} = \mathscr{G}a_n. \end{cases} \tag{4}$$

The above sequence generated by the Picard scheme is not converging to a fixed point of nonexpansive mappings. For more details, we refer the reader to [6].
Many novel iterative techniques have been developed by authors in order to obtain the relative effective rate of convergence and overcome this kind of difficulty (see:[16, 12, 1, 29]) and many more. Among these iterative schemes, some authors introduced hybrid schemes that converge faster than simple schemes.
In 2013, Khan [14] gave the concept of Picard-Mann hybrid iterative scheme. This scheme is defined as follows:

$$\begin{cases} a_1 = a \in \mathscr{M} \\ a_{n+1} = \mathscr{G}b_n & n \in \mathbb{Z}^+ \\ b_n = (1-\alpha_n)a_n + \alpha_n \mathscr{G}a_n \end{cases} \tag{5}$$

where $\{\alpha_n\} \in (0,1)$.
In 2019, following Khan, Okeke [17] gave the Picard-Ishikawa hybrid iterative scheme which is defined as:

$$\begin{cases} a_1 = a \in \mathscr{M} \\ a_{n+1} = \mathscr{G}b_n \\ b_n = (1-\alpha_n)a_n + \alpha_n \mathscr{G}c_n & n \subseteq \mathbb{Z}^+ \\ c_n = (1-\beta_n)a_n + \beta_n \mathscr{G}a_n \end{cases} \tag{6}$$

where $\{\alpha_n\},\{\beta_n\} \subseteq (0,1)$.
Recently, Srivastava [27] introduced Picard-S hybrid iterative scheme which is defined as:

$$\begin{cases} a_1 = a \in \mathscr{M} \\ a_{n+1} = \mathscr{G}b_n \\ b_n = (1-\alpha_n)\mathscr{G}a_n + \alpha_n \mathscr{G}c_n & n \in \mathbb{Z}^+ \\ c_n = (1-\beta_n)a_n + \beta_n \mathscr{G}a_n \end{cases} \tag{7}$$

where $\{\alpha_n\},\{\beta_n\} \subseteq (0,1)$.
Also, Lamba and Panwar [15] introduced the Picard -S^*-iterative scheme which is defined as:

$$\begin{cases} a_1 = a \in \mathscr{M} \\ a_{n+1} = \mathscr{G}b_n \\ b_n = (1-\alpha_n)\mathscr{G}a_n + \alpha_n \mathscr{G}c_n & n \in \mathbb{Z}^+ \\ c_n = (1-\beta_n)\mathscr{G}a_n + \beta_n \mathscr{G}d_n \\ d_n = (1-\gamma_n)a_n + \gamma_n \mathscr{G}a_n \end{cases} \tag{8}$$

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\} \subseteq (0,1)$.

Recently, Jia Jie et al. [13] proposed Picard-Thakur-iterative scheme which is defined as:

$$\begin{cases} a_1 \in \mathscr{M} \\ a_{n+1} = \mathscr{G}b_n \\ b_n = (1-\alpha_n)\mathscr{G}d_n + \alpha_n \mathscr{G}c_n \\ c_n = (1-\beta_n)d_n + \beta_n \mathscr{G}d_n, \ n \in \mathbb{Z}^+ \\ d_n = (1-\gamma_n)a_n + \gamma_n \mathscr{G}a_n \end{cases} \tag{9}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0,1)$.
By using the iterative scheme (9) we prove some fixed point results and discuss a numerical example for generalized $\alpha-$nonexpansive mappings that demonstrates how (9) converges more quickly than all Picard hybrid schemes. Data dependence results for almost contraction mappings are also presented.

2. Preliminaries

Definition 2.1. *Let $\mathscr{M}$ be a nonempty closed convex subset of a Banach space $\mathscr{B}$. A mapping $\mathscr{G}:\mathscr{M}\to\mathscr{M}$ is called demiclosed w.r.t $b\in\mathscr{B}$, if for each $\{a_n\}\subseteq\mathscr{M}$ and $a\in\mathscr{M}$, $\{a_n\}\rightharpoonup a$ and $\{\mathscr{G}a_n\}\to b\Rightarrow\mathscr{G}a=b$.*

Sentor and Dotson [24] gave the concept of *Condition (I)* which is as follows:

Definition 2.2. *Let $\mathscr{G}:\mathscr{M}\to\mathscr{M}$ be a self mapping. It is said to satisfy Condition (I), if there exists an increasing function r from $[0,\infty)$ to $[0,\infty)$ with $r(0)=0$ and $r(s)>0$, for all, $s>0$ such that*

$$d(a,\mathscr{G}a^*)\geq r(d(a,F(\mathscr{G}))),\ \text{for all}\ \ a\in\mathscr{M},$$

where $d(a,F(\mathscr{G}))=\inf\{d(a,a^*):a^*\in F(\mathscr{G})\}$.

Definition 2.3. *Let $\mathscr{B}$ be a Banach space and $\{a_n\}$ be bounded in $\mathscr{B}$. Define a mapping $r_a(\{a_n\},a):\mathscr{B}\to\mathbb{R}^+$ by*

$$r_a(\{a_n\},a)=\limsup_{n\to\infty}\|a_n-a\|.$$

For $a\in\mathscr{B}$, the value $r_a(\{a_n\},a)$ is the asymptotic radius of $\{a_n\}$ at a.
The asymptotic radius of $\{a_n\}$ w.r.t. to $\mathscr{M}\subset\mathscr{B}$ is defined as:

$$r_a(\mathscr{M},\{a_n\})=\inf\{r_a(\{a_n\},a):a\in\mathscr{M}\}.$$

The asymptotic center of $\{a_n\}$ w.r.t. to $\mathscr{M}$ is

$$A(\mathscr{M},\{a_n\})=\{a\in\mathscr{M}:r_a(\mathscr{M},\{a_n\})=r_a(a,\{a_n\})\}.$$

The asymptotic centre of $\{a_n\}$ with respect to $\mathscr{M}$ is nonempty and convex whenever $\mathscr{M}$ is weakly compact [2, 10]. One of the known properties of the set $A(\mathscr{M},\{a_n\})$ is the *singleton property* in a UCBS $\mathscr{B}$ [9].

Definition 2.4. *[18]*
A Banach space $\mathscr{B}$ obeys the Opial's property if for each weakly convergent sequence $\{a_n\}$ to $a\in\mathscr{B}$,

$$\liminf_{n\to\infty}\|a_n-a\|<\liminf_{n\to\infty}\|a_n-b\|$$

holds, for all $b\in\mathscr{B}$ with $a\neq b$.

Proposition 2.1. *[19] Every mapping that satisfies Condition (C) is also a generalized α-nonexpansive mapping, but the converse is not true.*

Proposition 2.2. *[19] let $\mathscr{M}$ be a nonempty subset of Banach space $\mathscr{B}$ and $\mathscr{G}:\mathscr{M}\to\mathscr{M}$ is a generalized α-nonexpansive mapping. Then*

$$\|a-\mathscr{G}(a)\|\leq\frac{(3+\alpha)}{(1-\alpha)}\|a-\mathscr{G}(a)\|+\|a-b\|.\ \forall\,a,b\in\mathscr{M}.$$

Theorem 2.1. *[28] Let $\mathscr{G}$ be a mapping on $\mathscr{M}$, where $\mathscr{M}$ is a convex and weakly compact subset of a uniformly Banach space $\mathscr{B}$.*
Let $\mathscr{G}$ satisfies Condition (C). Then $\mathscr{G}$ has a fixed point.

Lemma 2.1. *[23] Let $0<x\leq\alpha_n\leq y<1$ for all $n\in\mathbb{Z}^+$ and $\mathscr{B}$ be a UCBS and $\{a_n\}$ and $\{b_n\}$ are sequences such that $\limsup_{n\to\infty}\|a_n\|\leq d$, $\limsup_{n\to\infty}\|b_n\|\leq d$ and $\limsup_{n\to\infty}\|(1-\alpha_n)a_n+\alpha_nb_n\|=d$ holds for some $d\geq0$. Then $\lim_{n\to\infty}\|a_n-b_n\|=0$.*

Lemma 2.2. *[19] Let $\mathscr{G}$ be a generalized α-nonexpansive mapping satisfying the Opial's property. If $\{a_n\}\rightharpoonup c$ and $\lim_{n\to\infty}\|a_n-\mathscr{G}(a_n)\|=0$, then $\mathscr{G}(c)=c$, i.e. $I-\mathscr{G}$ is demiclosed at zero, where $I:\mathscr{B}\to\mathscr{B}$ is the identity mapping.*

Definition 2.5. *Let $\mathscr{G}$ be an α-nonexpansive mapping and $\mathscr{M}$ be a non-empty subset of Banach space $\mathscr{B}$ such that $\mathscr{G}:\mathscr{M}\to\mathscr{M}$, then $F(\mathscr{G})$ is closed. In addition, $F(\mathscr{G})$ is convex provided that $\mathscr{B}$ is strictly convex and $\mathscr{M}$ is convex.*

Proposition 2.3. *[22] Let $\mathscr{G}:\mathscr{M}\to\mathscr{M}$ be a Generalized α-nonexpansive mapping. Then the following holds.*

i) If condition (C) is satisfied by $\mathscr{G}$, then condition (C_α) is satisfied as well.
ii) If $F(\mathscr{G})\neq\emptyset$ and condition (C_α) holds for $\mathscr{G}$, then $\mathscr{G}$ is quasi-nonexpansive.

Next, we review some definitions and lemmas that helped us to demonstrate the validity of our data dependence results. The following general definition for contractive-like operators was taken into consideration by Imoru and Olatinwo [11].

Definition 2.6. *An operator $\mathscr{G}$ is called a contractive-like operator if there exist a constant $q\in[0,1)$ and a strictly increasing continuous function $\phi:[0,\infty)\to(0,\infty)$ with $\phi(0)=0$, such that for each $a,b\in\mathscr{B}$,*

$$\|\mathscr{G}a-\mathscr{G}b\|\leq\phi\left(\|a-\mathscr{G}a\|\right)+q\|a-b\|. \tag{10}$$

Definition 2.7. *[26] Let $\mathscr{G},\mathscr{S}:\mathscr{M}\to\mathscr{M}$ be two operators. We say that $\mathscr{S}$ is an appropriate operator of $\mathscr{G}$ if for fixed $\varepsilon>0$, and for all $a\in\mathscr{M}$, we have*

$$\|\mathscr{G}a-\mathscr{S}a\|\leq\varepsilon.$$

Lemma 2.3. *[26] Let $\{x_n\}$ be a nonnegative sequence for which there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$, one has the following inequality,*

$$x_{n+1}\leq(1-\lambda_n)x_n+\lambda_nq_n,$$

where $\lambda_n\in(0,1)$, $\forall\,n\in\mathbb{N}$, $\sum_{n=1}^{\infty}\lambda_n=\infty$ and $q_n\geq0$, for all $n\in\mathbb{N}$. Then,

$$0\leq\limsup_{n\to\infty}x_n\leq\limsup_{n\to\infty}q_n.$$

3. Rate of Convergence

By using (9), we demonstrate convergence results for generalized α-nonexpansive mappings in UCBS $\mathscr{B}$.

Lemma 3.1. *Let $\mathscr{M}$ be a nonempty closed convex subset of UCBS $\mathscr{B}$. Let $\mathscr{G}:\mathscr{M}\to\mathscr{M}$ be a generalized α-nonexpansive mapping with $F(\mathscr{G})\neq\emptyset$. Let $\{a_n\}$ is defined by the Picard-Thakur hybrid iterative scheme (9), then $\lim_{n\to\infty}\|a_n-a^*\|$ exists $\forall\ a^*\in F(\mathscr{G})$.*

Proof. Since $\mathscr{G}$ satisfies condition (C_α), therefore by proposition 2.3, $\mathscr{G}$ is α-nonexpansive mapping that is

$$\begin{aligned}\frac{1}{2}\|a-\mathscr{G}(a)\| \leq& \|a-b\| \text{ implies}\\ \|\mathscr{G}(a)-\mathscr{G}(b)\| \leq& \alpha\|b-\mathscr{G}(a)\|+\alpha\|a-\mathscr{G}(b)\|+(1-2\alpha)\|a-b\|\end{aligned}$$

Let $a^*\in F(\mathscr{G})$. By (9)

$$d_n=(1-\gamma_n)a_n+\gamma_n\mathscr{G}(a_n).$$

So,

$$\begin{aligned}\|d_n-a^*\| =& \|(1-\gamma_n)a_n+\gamma_n\mathscr{G}(a_n)-a^*\|\\ =& \|(1-\gamma_n)a_n+\gamma_n\mathscr{G}(a_n)-a^*+\gamma_n a^*-\gamma_n a^*\|\\ =& \|(1-\gamma_n)(a_n-a^*)+\gamma_n(\mathscr{G}(a_n)-a^*)\|\\ \leq& (1-\gamma_n)\|a_n-a^*\|+\gamma_n\|\mathscr{G}(a_n)-a^*\|\end{aligned}$$

and

$$\|\mathscr{G}(a_n)-a^*\| \leq \alpha\|a_n-\mathscr{G}(a^*)\|+\alpha\|\mathscr{G}(a_n)-a^*\|+(1-2\alpha)\|a_n-a^*\|$$

that is,

$$\|\mathscr{G}(a_n)-a^*\| = \alpha\|a_n-a^*\|+\alpha\|\mathscr{G}(a_n)-a^*\|+(1-2\alpha)\|a_n-a^*\|$$

Hence,

$$(1-\alpha)\|\mathscr{G}(a_n)-a^*\| \leq (1-\alpha)\|a_n-a^*\|.$$

Therefore,

$$\|\mathscr{G}(a_n)-a^*\| \leq \|a_n-a^*\|.$$

So,

$$\begin{aligned}\|d_n-a^*\| \leq& (1-\gamma_n)\|a_n-a^*\|+\gamma_n\|a_n-a^*\|\\ \leq& \|a_n-a^*\|.\end{aligned} \tag{11}$$

As, $c_n=(1-\beta_n)d_n+\beta_n\mathscr{G}(d_n)$, then

$$\begin{aligned}\|c_n-a^*\| =& \|(1-\beta_n)d_n+\beta_n\mathscr{G}(d_n)-a^*\|\\ \leq& (1-\beta_n)\|d_n-a^*\|+\beta_n\|\mathscr{G}(d_n)-a^*\|.\end{aligned}$$

From (11), we have

$$\|d_n-a^*\| \leq \|a_n-a^*\|$$

and

$$\begin{aligned}\|\mathscr{G}(d_n)-a^*\| \leq& \alpha\|d_n-\mathscr{G}(a^*)\|+\alpha\|\mathscr{G}(d_n)-a^*\|+(1-2\alpha)\|d_n-a^*\|.\\ \leq& \alpha\|d_n-a^*\|+\alpha\|\mathscr{G}(d_n)-a^*\|+(1-2\alpha)\|d_n-a^*\|.\end{aligned}$$

Hence,

$$(1-\alpha)\|\mathscr{G}(d_n)-a^*\| \leq (1-\alpha)\|d_n-a^*\|.$$

Therefore,

$$\begin{aligned}\|\mathscr{G}(d_n)-a^*\| \leq& \|d_n-a^*\|\\ \leq& \|a_n-a^*\|,\end{aligned}$$

which implies

$$\|c_n-a^*\| \leq \|a_n-a^*\|. \tag{12}$$

For third step of (9), as $b_n=(1-\alpha_n)\mathscr{G}(d_n)+\alpha_n\mathscr{G}(c_n)$, then we have

$$\begin{aligned}\|b_n-a^*\| =& \|(1-\alpha_n)\mathscr{G}(d_n)+\alpha_n\mathscr{G}(c_n)-a^*\|\\ \leq& (1-\alpha_n)\|\mathscr{G}(d_n)-a^*\|+\alpha_n\|\mathscr{G}(c_n)-a^*\|.\end{aligned}$$

As, from above $\|\mathscr{G}(d_n)-a^*\| \leq \|a_n-a^*\|$ and

$$\|\mathscr{G}(c_n)-a^*\| \leq \alpha\|\mathscr{G}(c_n)-a^*\|+\alpha\|c_n-\mathscr{G}(a^*)\|+(1-2\alpha)\|c_n-a^*\|,$$

so,

$$(1-\alpha)\|\mathscr{G}(c_n)-a^*\| \leq (1-\alpha)\|c_n-a^*\|,$$

which implies that

$$\begin{aligned}\|\mathscr{G}(c_n)-a^*\| \leq & \|c_n-a^*\| \\ \leq & \|a_n-a^*\|,\end{aligned}$$

which implies

$$\|b_n-a^*\| \leq \|a_n-a^*\|. \tag{13}$$

Next

$$\|a_{n+1}-a^*\| \leq \|\mathscr{G}(b_n)-a^*\|$$

We know that

$$\|\mathscr{G}(b_n)-a^*\| \leq \alpha\|b_n-\mathscr{G}(a^*)\|+\alpha\|\mathscr{G}(b_n)-a^*\|+(1-2\alpha)\|b_n-a^*\|$$

so,

$$(1-\alpha)\|\mathscr{G}(b_n)-a^*\| \leq (1-\alpha)\|b_n-a^*\|$$

and hence

$$\begin{aligned}\|\mathscr{G}(b_n)-a^*\| \leq & \|b_n-a^*\| \\ \leq & \|a_n-a^*\|.\end{aligned}$$

This implies,

$$\|a_{n+1}-a^*\| \leq \|a_n-a^*\|. \tag{14}$$

Equation (14) show that $\{a_n\}$ is non decreasing and bounded. Hence $\lim_{n\to\infty}\|a_n-a^*\|$ exists for every a^* belongs to $F(\mathscr{G})$. □

Lemma 3.2. *Let $\mathscr{M}$, $\mathscr{B}$, $\mathscr{G}$ and $\{a_n\}$ are as in Lemma 3.1. Then $F(\mathscr{G})\neq\emptyset$ if and only if $\{a_n\}$ is bounded and $\lim_{n\to\infty}\|a_n-\mathscr{G}a_n\|=0$.*

Proof. Let $a^*\in F(\mathscr{G})$, $\{a_n\}$ be bounded and from Lemma 3.1, the limit exists for $a^*\in F(\mathscr{G})$. Let

$$\lim_{n\to\infty}\|a_n-a^*\|=d. \tag{15}$$

Using the proof of Lemma 3.1 and keeping equation (11), we obtain

$$\|d_n-a^*\| \leq \|a_n-a^*\|. \tag{16}$$

By applying lim sup as $n\to\infty$, we get

$$\limsup_{n\to\infty}\|d_n-a^*\| \leq \limsup_{n\to\infty}\|a_n-a^*\|=d. \tag{17}$$

By above Lemma 3.1, we have

$$\|\mathscr{G}(a_n)-\mathscr{G}(a^*)\| \leq \|a_n-a^*\|.$$

Applying limsup as $n\to\infty$, we get

$$\limsup_{n\to\infty}\|\mathscr{G}(a_n)-\mathscr{G}(a^*)\| \leq \limsup_{n\to\infty}\|a_n-a^*\| \leq d. \tag{18}$$

From Lemma 3.1, we also have

$$\|a_{n+1}-a^*\| \leq \|d_n-a^*\|.$$

Applying further liminf as $n\to\infty$ we get

$$d \leq \liminf_{n\to\infty}\|d_n-a^*\|. \tag{19}$$

From (18) and (17), we have

$$\lim_{n\to\infty}\|d_n-a^*\|=d.$$

Hence,

$$\begin{aligned}d &= \lim_{n\to\infty}\|d_n-a^*\| \\ &= \lim_{n\to\infty}\|(1-\gamma_n)a_n+\gamma_n\mathscr{G}a_n-a^*\| \\ &= \lim_{n\to\infty}\|(1-\gamma_n)(a_n-a^*)+\gamma_n(\mathscr{G}a_n-a^*)\|.\end{aligned} \tag{20}$$

From (20), (18), (15) and Lemma 2.1 we have

$$\lim_{n\to\infty}\|a_n-\mathscr{G}a_n\|=0.$$

Conversely, if $\{a_n\}$ is bounded and $\lim_{n\to\infty}\|a_n-\mathscr{G}a_n\|=0$, then $F(\mathscr{G})\neq\emptyset$.
Let $a^*\in A(\mathscr{M},\{u_k\})$; by Proposition 2.2 we have

$$\begin{aligned}r_a(\mathscr{G}(a^*),\{u_k\}) =& \limsup_{n\to\infty}\|\mathscr{G}(a^*)-a_n\| \\ \leq & \limsup_{n\to\infty}(\frac{(3+\alpha)}{(1-\alpha)}\|a_n-\mathscr{G}(a_n)\|+\|a_n-a^*\| \\ \leq & \limsup_{n\to\infty}\|a_n-a^*\|\end{aligned}$$

so,

$$r_a(\mathscr{M},\{a_n\})=r_a(\{a_n\},a^*)$$

which implies $\mathscr{G}a^*\in A(\mathscr{M},\{a_n\})$.
Since $\mathscr{B}$ is uniformly convex then $A(\mathscr{M},\{a_n\})$ is singleton. Hence, $\mathscr{G}a^*=a^*$. □

Theorem 3.1. *Let $\mathscr{M}$, $\mathscr{B}$, $\mathscr{G}$ and $\{a_n\}$ be are as in Lemma 3.1. Let $\mathscr{B}$ satisfies Opial condition. Then $\{a_n\}$ converges weakly to a point of $F(\mathscr{G})$.*

Proof. Let $a^* \in F(\mathscr{G})$. Then by Lemma 3.1 $\lim_{n\to\infty} \|a_n - a^*|$ exists. In the next step, we demonstrate that $F(\mathscr{G})$ has a unique weak subsequential limit for $\{a_n\}$.
Suppose l and m are the weak limits of $\{a_{n_i}\}$ and $\{a_{n_j}\}$ as subsequences of $\{a_n\}$. By Lemma 3.2, $\lim_{n\to\infty} \|a_n - \mathscr{G}(a_n)\| = 0$ and $I - \mathscr{G}$ is demiclosed at zero.
Using Lemma 2.2 we can conclude that $(I - \mathscr{G})l = 0 \Rightarrow l = \mathscr{G}(l)$, similarly $m = \mathscr{G}(m)$.
Further, we show the uniqueness of the solution. If $l \neq m$, using the Opial condition, we have

$$\begin{aligned}
\lim_{n\to\infty} \|a_n - l\| &= \lim_{l\to\infty} \|a_{n_l} - l\| \\
&< \lim_{l\to\infty} \|a_{n_l} - l\| \\
&= \lim_{n\to\infty} \|a_n - l\| \\
&= \lim_{m\to\infty} \|a_{n_m} - l\| \\
&< \lim_{n\to\infty} \|a_{n_m} - m\| \\
&= \lim_{n\to\infty} \|a_n - l\|.
\end{aligned}$$

Hence,

$$\lim_{n\to\infty} \|a_n - l\| \leq \lim_{n\to\infty} \|a_n - l\|$$

which is a Contradiction; then $l = m$. Consequently, $\{a_n\} \rightharpoonup F(\mathscr{G})$. ☐

Theorem 3.2. *Let $\mathscr{M}$, $\mathscr{B}$, $\mathscr{G}$ and $\{a_n\}$ be as in Lemma 3.1. Then $\{a_n\}$ converges to a point of $F(\mathscr{G}) \Leftrightarrow \liminf_{n\to\infty} d(a_n, F(\mathscr{G})) = 0$ or $\limsup_{n\to\infty} d(a_n, F(\mathscr{G})) = 0$, where $d(a_n, F(\mathscr{G})) = \inf\{\|a_n - a^*\| : a^* \in F(\mathscr{G})\}$.*

Proof. If the sequence $\{a_n\} \to a^* \in F(\mathscr{G})$, then it is evident that $\liminf_{n\to\infty} d(a_n, F(\mathscr{G})) = 0$ or $\limsup_{n\to\infty} d(a_n, F(\mathscr{G})) = 0$.
On the other hand, suppose that $\liminf_{n\to\infty} d(a_n, F(\mathscr{G})) = 0$. From Lemma 3.1, $\lim_{n\to\infty} \|a_n - a^*\|$ exists, for all $a^* \in F(\varphi)$. Therefore, by assumption $\lim_{n\to\infty} d(a_n$, we get $F(\mathscr{G})) = 0$.
Now we show $\{a_n\}$ is Cauchy in $\mathscr{G}$. As

$$\lim_{n\to\infty} d(a_n, F(\mathscr{G})) = 0,$$

for $\varepsilon > 0$, there is $m_0 \in Z^+$ such that for all $n \geq m_0$, we have

$$d(a_n, F(\mathscr{G}) < \frac{\varepsilon}{2}$$

which yields that

$$\inf\{\|a_n - a^*\| : a^* \in F(\mathscr{G}) < \frac{\varepsilon}{2}.$$

In particular, $d(a_n, F(\mathscr{G})) = \inf\{\|a_n - a^*\| : a^* \in F(\mathscr{G}))\} < \frac{\varepsilon}{2}$.
Therefore, there is $a^* \in F(\mathscr{G}))$ such that

$$\|a_{m_0} - a^*\| < \frac{\varepsilon}{2}.$$

For $m, n \geq m_0$,

$$\begin{aligned}
\|a_{n+m} - a_n\| &\leq \|a_{m+n} - a^*\| + \|a_n - a^*\| \\
&\leq \|a_{m_0} - a^*\| + \|a_{m_0} - a^*\| \\
&= 2\|a_{m_0} - a^*\| \\
&< \varepsilon,
\end{aligned}$$

which implies $\{a_n\}$ is Cauchy in $\mathscr{M}$. As $\mathscr{M}$ is closed in $\mathscr{B}$, so there is $k \in \mathscr{M}$ such that, $\lim_{n\to\infty} a_n = k$. $\lim_{n\to\infty} d(a_n, F(\mathscr{G})) = 0$ gives that $\lim_{n\to\infty} d(a_n, F(\mathscr{G})) = 0$ implies $k \in F(\mathscr{G})$. ☐

Theorem 3.3. *Let $\mathscr{M}$, $\mathscr{B}$, $\mathscr{G}$ be as in Lemma 3.1. If $F(\mathscr{G})) \neq \emptyset$ and $\liminf_{n\to\infty} \infty\|a_n - F(\mathscr{G}))\| = 0$, then $\{a_n\}$ defined by (9) converges strongly to $F(\mathscr{G})$.*

Proof. Assume that

$$\liminf_{n\to\infty} \|a_n - F(\mathscr{G})\| = 0. \tag{21}$$

Then there is, $\{g_n\}$ a subsequence of $\{a_n\}$ such that:

$$\liminf_{n\to\infty} \|g_n - F(\mathscr{G})\| = 0.$$

Suppose, $\{g_{n_j}\}$ is again a subsequence of g_n for which

$$\|g_{n_j} - f_j\| \leq \frac{1}{2^j},\ \forall\ j \geq 1.$$

such that $\{f_j\} \subset F(\mathscr{G})$, then by Lemma 3.3:

$$\|g_{n_{j+1}} - f_j\| \leq \|g_{n_j} - f_j\| \leq \frac{1}{2^j}.$$

Now, we show $\{f_j\}$ is Cauchy in $F(\mathscr{G})$. By (21) and triangular inequality:

$$\|f_{j+1} - f_j\| \leq \|f_{j+1} - g_{n_{j+1}}\| \leq \|g_{n_{j+1}} - f_j\| < \frac{1}{2^{j-1}}.$$

Hence, the above argument implies that $\{f_j\}$ is Cauchy in $F(\mathscr{G})$. By definition 2.5, $F(\mathscr{G})$ is a closed convex subset of the Banach space $\mathscr{B}$. Thus $\{f_j\}$ converges to the fixed point f, we have

$$\|g_{nj} - f\| \leq \|g_{nj} - f_j\| + \|f_j - f\|.$$

Let $j \to \infty$ implies $\{g_{nj}\}$ converges strongly to f.
Accordingly, $\lim_{n\to\infty} \|u_n - f\|$ exists. For $f \in F(\mathscr{G})$, by Theorem 3.1 we get $\{a_n\}$ converges strongly to $F(\mathscr{G})$. □

Further, we demonstrate the strong convergence result due to *Condition (I)*.

Theorem 3.4. *Let $\mathscr{M}$, $\mathscr{B}$, $\mathscr{G}$ be as in Lemma 3.1 and satisfying the property (I). Then the sequence $\{a_n\}$ defined by (9) converges strongly to $F(\mathscr{G})$.*

Proof. From Lemma 3.2, we can conclude that

$$\liminf_{n\to\infty} \|\mathscr{G}(a_n) - a_n\| = 0. \tag{22}$$

Since $\mathscr{G}$ fulfills *Condition (I)*, we have

$$\liminf_{n\to\infty} \|a_n - F(\mathscr{G})\| = 0.$$

Since, the conditions of theorem 3.2 are all met, its conclusion shows strong the convergence to $F(\mathscr{G})$. □

4. Numerical Example

Now, we provide a numerical example to support our results.

Example 4.1. *Let $\mathscr{M}$=[0,4] endowed with the usual norm $|\cdot|$ in Banach space $\mathbb{R}$. Let $\mathscr{G} : \mathscr{M} \to \mathscr{M}$, such that*

$$\mathscr{G}(a) = \begin{cases} \frac{2a+3}{2} & \text{if } a \in [0, 2.50) \\ \frac{a+3}{2} & \text{if } a \in [2.50, 4]. \end{cases}$$

Then

1. *$\mathscr{G}$ does not satisfies Condition (C).*
2. *$\mathscr{G}$ is a generalized α-nonexpansive mapping.*

For (1), let $a = 2.85$ and $b = 2.37$. Then we get

$$\frac{1}{2}\left|a - \mathscr{G}(a)\right| = \frac{1}{2}\left|2.85 - \frac{a+3}{2}\right|$$
$$= \frac{1}{4}\left|2(2.85) - 2.85 - 3\right| = \frac{0.15}{4} = 0.375$$

$$\textit{and,} \qquad \left|a - b\right| = \left|2.85 - 2.37\right| = 0.48.$$

So, $\frac{1}{2}\left|a - \mathscr{G}(a)\right| < \left|a - b\right|$.

Now,

$$
\begin{aligned}
\left|\mathscr{G}(a)-\mathscr{G}(b)\right| &= \left|\frac{a+3}{2}-\frac{2b+3}{2}\right| \\
&= \left|\frac{a+3-2b-3}{2}\right| \\
&= \left|\frac{a-2b}{2}\right| \\
&= \left|\frac{2.85-2(2.37)}{2}\right| \\
&= \frac{1.89}{2} = 0.945.
\end{aligned}
$$

Hence, $|\mathscr{G}(a)-\mathscr{G}(b)| \geq |a-b|$. *Then* $\mathscr{G}$ *does not fulfill the Condition (C).*
The following step is to demonstrate that $\mathscr{G}$ *is an generalized* α*-generalized mapping. The following cases are taken into account for this.*
$Case-1:$ *When* $a,b \in [0,\frac{1}{6})$, *then*

$$
\|\mathscr{G}(a)-\mathscr{G}(b)\| = |\frac{2a+3}{2}-\frac{2b+3}{2}| = |a-b|
$$

and

$$
\begin{aligned}
\alpha\|b-\mathscr{G}(a)\|+\alpha\|a-\mathscr{G}(b)\|+(1-2\alpha)\|a-b\| &= \frac{1}{3}|b-\frac{2a+3}{2}|+\frac{1}{3}|a-\frac{2b+3}{2}|+\frac{1}{3}|a-b| \\
&\geq \frac{1}{3}[|b-\frac{2a+3}{2}-a+\frac{b+3}{2}|]+\frac{1}{3}|a-b| \\
&= \frac{1}{3}[|b-\frac{2a}{2}-\frac{3}{2}-a+\frac{2b}{2}+\frac{3}{2}|]+\frac{1}{3}|a-b| \\
&= \frac{1}{3}|\frac{a-2b-b+2a}{2}|+\frac{1}{3}|a-b| \\
&= \frac{1}{3}|b-a-a+b|+\frac{1}{3}|a-b| \\
&= \frac{1}{3}|2a-2b|+\frac{1}{3}|a-b|
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2}{3}|a-b|+\frac{1}{3}|a-b| \\
&= |a-b| = \|\mathscr{G}(a)-\mathscr{G}(b)\|.
\end{aligned}
$$

$Case-2:$ *If* $a,b \in [\frac{5}{2},4)$, *then*

$$
\|\mathscr{G}(a)-\mathscr{G}(b)\| = |\frac{a+3}{2}-\frac{b+3}{2}| = \frac{1}{2}|a-b|.
$$

Then we have the following estimation:

$$
\begin{aligned}
\alpha\|b-\mathscr{G}(a)\|+\alpha\|a-\mathscr{G}(b)\|+(1-2\alpha)\|a-b\| &= \frac{1}{3}|b-\frac{a+3}{2}|+\frac{1}{3}|a-\frac{b+3}{2}|+\frac{1}{3}|a-b| \\
&\geq \frac{1}{3}|\frac{a+3}{2}-b-\frac{b+3}{2}-a|+\frac{1}{3}|a-b| \\
&= \frac{1}{3}|b-\frac{a}{2}-\frac{3}{2}-a+\frac{b}{2}+\frac{3}{2}|+\frac{1}{3}|a-b| \\
&= \frac{1}{3}|\frac{-a-2a}{2}+\frac{b+2b}{2}|+\frac{1}{3}|a-b| \\
&= \frac{1}{3}|\frac{3a}{2}-\frac{3b}{2}|+\frac{1}{3}|a-b| \\
&= \frac{5}{6}|a-b| \geq \frac{1}{2}|a-b| = \|\mathscr{G}(a)-\mathscr{G}(b)\|.
\end{aligned}
$$

Case -3 : *If* $a \in [0, \frac{1}{6})$ *and* $b \in [\frac{1}{6}, 4)$, *then*

$$\begin{aligned}
&\alpha\|b-\mathscr{G}(a)\|+\alpha\|a-\mathscr{G}(b)\|+(1-2\alpha)\|a-b\| \\
=&\frac{1}{3}|b-\frac{2a+3}{2}|+\frac{1}{3}|a-\frac{b+3}{2}|+\frac{1}{3}|a-b| \\
=&\frac{1}{3}|b-\frac{2a}{2}-\frac{3}{2}-a+\frac{b}{2}+\frac{3}{2}|+\frac{1}{3}|a-b| \\
=&\frac{1}{3}|\frac{3b}{2}-\frac{4a}{2}|+\frac{1}{3}|a-b| \\
=&\frac{1}{3}|\frac{3b}{2}-\frac{4a}{2}|-(a-b)| \\
=&\frac{1}{3}|\frac{3b-2a-2a+2b}{2}| \\
=&\frac{1}{3}|\frac{-4a+5b}{2}| \\
=&\frac{1}{3}|\frac{4a-5b}{2}| \\
\geq&\frac{1}{2}|2a-b|=\|\mathscr{G}(a)-\mathscr{G}(b)\|.
\end{aligned}$$

$\mathscr{G}$ is therefore a generalized $\frac{1}{3}-$nonexpansive mapping.

Next, we perform some experiment to compare convergence behaviour of the scheme (9). In the Table 1 we perform some tests for the convergence behaviour of an iterative scheme for various initial points. Let the parameters are $\alpha_n = 1-\frac{n}{n+2}$, $\beta_n = \frac{n}{n^2+n+1}$ and $\gamma_n = \sqrt{\frac{n}{n+7}}$ and fix the stopping criteria to $\|a-a^*\| < 10^{-10}$ where a^* is a fixed point. Here, we found that the fixed point of the generalized α-nonexpansive mapping is reached more quickly by the iterative scheme (9).

Table 1: Comparison of the convergence of iteration processes for various starting points

Initial points	Hybrid Iteration Processes				
	Picard-Mann	Picard-Ishikawa	Picard-S iteration	Picard-S^*	Picard-Thakur
0.2	29	27	15	14	**12**
0.4	28	27	14	14	**12**
1.6	27	26	14	13	**12**
2.7	25	24	13	12	**12**
3.8	26	25	15	14	**12**

In the Figure 1, we perform the convergence for different choices of parameters. For this, assume $a_1 = 0.5$ and we observe that iterative scheme (9) reached faster to the fixed point of the generalized $\alpha-$nonexpansive mapping.

5. Data Dependence

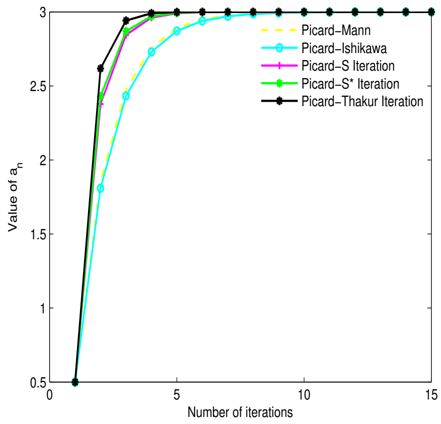
We show the data dependent result for iterative scheme (9) in this section, and use a numerical example to support our theoretical result.

Theorem 5.1. *Let $\mathscr{B}$ be a Banach space and $\mathscr{G}$ be a contractive-like operator on a nonempty closed convex subset $\mathscr{M}$ of $\mathscr{B}$, with $F(\mathscr{G}) \neq \emptyset$. Let $\{a_n\}$ is generated by* (9). *Then $\{a_n\} \to F(\mathscr{G})$.*

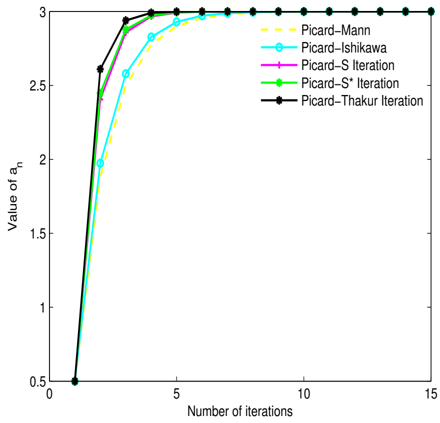
Proof. Let $a^* \in F(\mathscr{G})$ and from (9), we have

$$\begin{aligned}
\|d_n-a^*\| =&\|(1-\gamma_n)a_n+\gamma_n\mathscr{G}(a_n)-a^*\| \\
\leq&(1-\gamma_n)\|a_n-a^*\|+\gamma_n\|\mathscr{G}(a_n)-\mathscr{G}(a^*)\| \\
\leq&(1-\gamma_n)\|a_n-a^*\|+\delta\gamma_n\|a^*-a_n\|+\gamma_n\phi\|a^*-\mathscr{G}(a^*)\| \\
=&(1-\gamma_n)\|a_n-a^*\|+\delta\gamma_n\|a_n-a^*\| \\
\|d_n-a^*\| \leq&(1-\gamma_n(1-\delta))\|a_n-a^*\|,
\end{aligned}$$

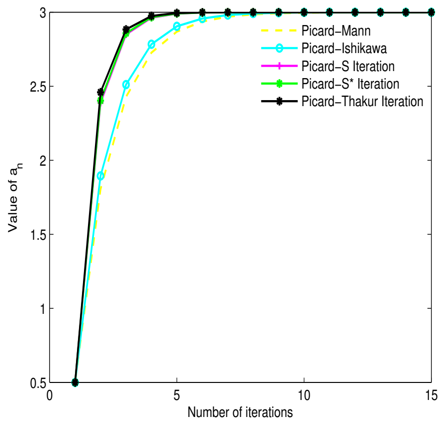
$$\begin{aligned}
\|c_n-a^*\| =&\|(1-\beta_n)d_n+\beta_n\mathscr{G}(d_n)-a^*\| \\
\leq&(1-\beta_n)\|d_n-a^*\|+\beta_n\|\mathscr{G}(d_n)-\mathscr{G}(a^*)\| \\
\leq&(1-\beta_n)\|d_n-a^*\|+\delta\beta_n\|a^*-d_n\|+\beta_n\phi\|a^*-\mathscr{G}(a^*)\| \\
=&(1-\beta_n)\|d_n-a^*\|+\delta\beta_n\|d_n-a^*\| \\
\|c_n-a^*\| \leq&(1-\beta_n(1-\delta_n))\|d_n-a^*\|,
\end{aligned}$$



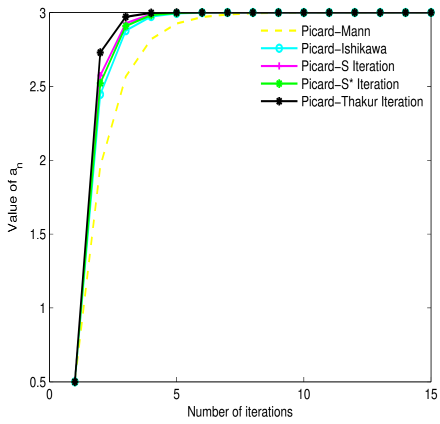
(a) $\alpha_n = 1 - \frac{n}{n+2}$, $\beta_n = \frac{n}{n^2+n+1}$, $\gamma_n = \sqrt{\frac{n}{n+7}}$



(b) $\alpha_n = \sqrt{\frac{n}{\sqrt{n+6}}}$, $\beta_n = \sqrt{\frac{n}{n^2+4}}$, $\gamma_n \sqrt{\frac{n}{n+10}}$



(c) $\alpha_n = 1 - \frac{n}{n+5}$, $\beta_n = 1 - \frac{n}{n+6}$, $\gamma_n = 1 - \sqrt{\frac{n}{n+8}}$



(d) $\alpha_n = 1 - \frac{n}{n+5}$, $\beta_n = 1 - \frac{2n}{9n+8}$, $\gamma_n = 1 - \frac{1}{n^3+5}$

Figure 1: Comparison of the iteration processes for different parameter selections.

$$\begin{aligned}
\|b_n - a^*\| =& \|(1-\alpha_n)\mathscr{G}(d_n) + \alpha_n \mathscr{G}(c_n) - a^*\| \\
\leq& (1-\alpha_n)\|\mathscr{G}(d_n) - a^*\| + \alpha_n \|\mathscr{G}(c_n) - a^*\| \\
\leq& (1-\alpha_n)\delta\|d_n - a^*\| + \alpha_n \delta \|c_n - a^*\| \\
\leq& (1-\alpha_n)\delta(1-\gamma_n(1-\delta))\|a_n - a^*\| + \alpha_n\delta(1-\beta_n(1-\delta))(1-\gamma_n(1-\delta))\|a_n - a^*\| \\
=& (1-\alpha_n)\delta + \alpha_n\delta(1-\beta_n(1-\delta))(1-\gamma_n(1-\delta))\|a_n - a^*\| \\
=& \delta - \delta\alpha_n + \delta\alpha_n(1-\beta_n(1-\delta))(1-\gamma_n(1-\delta))\|a_n - a^*\| \\
=& \delta - \delta\alpha_n(1-(1-\beta_n(1-\delta))(1-\gamma_n(1-\delta))\|a_n - a^*\| \\
=& \delta(1-\alpha_n(1-(1-\beta_n(1-\delta))(1-\gamma_n(1-\delta))\|a_n - a^*\| \\
=& \delta(1-\alpha_n\beta_n(1-\delta))(1-\gamma_n(1-\delta))\|a_n - a^*\|.
\end{aligned}$$

Let $\alpha_n \, \beta_n = \nu_n$. Then we have

$$\begin{aligned}
\|b_n - a^*\| =& \delta(1-\nu_n(1-\delta))(1-\gamma_n(1-\delta))\|a_n - a^*\| \\
\|a_{n+1} - a^*\| =& \|\mathscr{G}(b_n) - a^*\| \\
\leq& \delta\|b_n - a^*\| \\
\leq& \delta(\delta(1-\nu_n(1-\delta))(1-\gamma_n(1-\delta))\|a_n - a^*\|) \\
=& \delta^2(1-\nu_n(1-\delta))(1-\gamma_n(1-\delta))\|a_n - a^*\| \\
& \dots \\
=& \delta^{2n}[(1-\nu(1-\delta))(1-\gamma(1-\delta))]^n\|a_1 - a^*\|
\end{aligned}$$

Since, $0 \leq \delta < 1$, then $\lim_{n\to\infty} \delta^{2n} = 0$ which yields that $\lim\limits_{n\to\infty} \|a_{n+1} - a^*\| = 0$, which completes the proof. Hence, $\{a_n\}$ converges to the $F(\mathscr{G})$. □

Theorem 5.2. *Let $\mathscr{M}$,$\mathscr{B}$, $\mathscr{G}$ and $\{a_n\}$ be as in Theorem 5.1 and $\mathscr{S}$ be an approximate operator of $\mathscr{G}$. Now, define $\{g_n\}$ for $\mathscr{S}$ as follows:*

$$\begin{cases} g_1 \in \mathscr{M} \\ h_n = (1-\gamma_n)g_n + \gamma_n \mathscr{S} g_n \\ e_n = (1-\beta_n)h_n + \beta_n \mathscr{S} h_n \quad n \in \mathbb{N} \\ f_n = (1-\alpha_n)\mathscr{S} h_n + \alpha_n \mathscr{S} e_n \\ g_{n+1} = \mathscr{S} f_n, \end{cases} \tag{23}$$

where $\{\gamma_n\}$, $\{\beta_n\}$ and $\{\alpha_n\} \subset (0,1)$. Let $a^* \in F(\mathscr{G})$ and $a^{**} \in F(\mathscr{S})$. If $\{g_n\} \to a^{**}$ as $n \to \infty$, then we have

$$\|a^* - a^{**}\| \leq \frac{\varepsilon}{1-\delta}.$$

Proof.

$$\begin{aligned} \|d_n - h_n\| =& \|(1-\gamma_n)a_n + \gamma_n \mathscr{G}(a_n) - (1-\gamma_n)g_n - \gamma_n \mathscr{S}(g_n)\| \\ \leq& (1-\gamma_n)\|a_n - g_n\| + \gamma_n \|\mathscr{G}(a_n) - \mathscr{S}(g_n)\| \\ =& (1-\gamma_n)\|a_n - g_n\| + \gamma_n \|\mathscr{G}(a_n) - \mathscr{G}(g_n) + \mathscr{G}(g_n) - \mathscr{S}(g_n)\| \\ \leq& (1-\gamma_n)\|a_n - g_n\| + \gamma_n \|\mathscr{G}(a_n) - \mathscr{G}(g_n)\| + \gamma_n \|\mathscr{G}(g_n) - \mathscr{S}(g_n)\| \\ \leq& (1-\gamma_n)\|a_n - g_n\| + \gamma_n[\delta\|a_n - g_n]\| + \phi\|a_n - \mathscr{G}(a_n)\|] + \gamma_n \varepsilon \\ =& (1-\gamma_n)\|a_n - g_n\| + \gamma_n \delta \|a_n - g_n]\| + \gamma_n \phi \|a_n - \mathscr{G}(a_n)\| + \gamma_n \varepsilon \\ =& (1-\gamma_n + \gamma_n \delta)\|a_n - g_n\| + \gamma_n \phi \|a_n - \mathscr{G}(a_n)\| + \gamma_n \varepsilon \\ =& (1-\gamma_n(1-\delta))\|a_n - g_n\| + \gamma_n \phi \|a_n - \mathscr{G}(a_n)\| + \gamma_n \varepsilon. \end{aligned}$$

Since, $a^* \in F(\mathscr{G})$ and $\mathscr{G}$ is a contractive-like operator, Theorem 5.1 implies
$\lim_{n\to\infty} \|a_n - a^*\| = 0$. Hence

$$\begin{aligned} 0 \leq \|a_n - \mathscr{G}(a_n)\| \leq& \|a_n - a^*\| + \|\mathscr{G}(a^*) - \mathscr{G} a_n\| \\ \leq& \|a_n - a^*\| + \delta \|a_n - a^*\| \\ \leq& (1+\delta)\|a_n - a^*\| \to 0 \ \text{ as } \ n \to \infty. \end{aligned}$$

So,

$$\|d_n - h_n\| \leq (1-\gamma_n(1-\delta))\|a_n - g_n\| + \gamma_n \varepsilon, \tag{24}$$

$$\begin{aligned} \|c_n - e_n\| =& \|(1-\beta_n)d_n + \beta_n \mathscr{G}(d_n) - (1-\beta_n)h_n - \beta_n \mathscr{S}(h_n)\| \\ \leq& (1-\beta_n)\|d_n - h_n\| + \beta_n \|\mathscr{G}(d_n) - \mathscr{S}(h_n)\| \\ \leq& (1-\beta_n)\|d_n - h_n\| + \beta_n[\delta\|d_n - h_n]\| + \phi\|d_n - \mathscr{G}(d_n)\|] + \beta_n \varepsilon \\ =& (1-\beta_n)\|d_n - h_n\| + \beta_n \delta \|d_n - h_n]\| + \beta_n \phi \|d_n - \mathscr{G}(d_n)\|] + \beta_n \varepsilon \\ =& (1-\beta_n(1-\delta))\|d_n - h_n\| + \beta_n \phi \|d_n - \mathscr{G}(d_n)\| + \beta_n \varepsilon. \end{aligned}$$

Since, $a^* \in F(\mathscr{G})$ and $\mathscr{G}$ is a contractive-like operator, Theorem 5.1 implies
$\lim_{n\to\infty} \|a_n - a^*\| = 0$. Hence

$$\begin{aligned} 0 \leq \|d_n - \mathscr{G}(d_n)\| \leq& \|d_n - a^*\| + \|\mathscr{G} a^* - \mathscr{G}(d_n)\| \\ \leq& \|d_n - a^*\| + \delta \|d_n - a^*\| \\ \leq& (1+\delta)\|d_n - a^*\| \\ \leq& (1+\delta)\|(1-\gamma_n)a_n + \gamma_n \mathscr{G}(a_n) - \mathscr{G}(a^*)\| \\ \leq& (1+\delta)\left[(1-\gamma_n)\|a_n - a^*\| + \gamma_n \|\mathscr{G}(a_n) - \mathscr{G}(a^*)\|\right] \\ \leq& (1+\delta)\left[(1-\gamma_n)\|a_n - a^*\| + \delta \gamma_n \|a_n - a^*\|\right] \\ \leq& (1+\delta)(1-(1-\delta)\gamma_n)\|a_n - a^*\| \to 0 \ \text{ as } \ n \to \infty. \end{aligned}$$

So,

$$\|c_n - e_n\| \leq (1-\beta_n(1-\delta))\|d_n - h_n\| + \beta_n \varepsilon, \tag{25}$$

$$\begin{aligned} \|b_n - f_n\| =& (1-\alpha_n)\mathscr{G}(d_n) + \alpha_n \mathscr{G}(c_n) - (1-\alpha_n)\mathscr{S}(h_n) - \alpha_n \mathscr{S}(e_n)\| \\ \leq& (1-\alpha_n)\|\mathscr{G}(d_n) - \mathscr{S}(h_n)\| + \alpha_n \|\mathscr{G}(c_n) - \mathscr{S}(e_n)\| \\ \leq& (1-\alpha_n)\left[\|\mathscr{G}(d_n) - \mathscr{G}(h_n)\| + \|\mathscr{G}(h_n) - \mathscr{S}(h_n)\|\right] + \alpha_n\left[\|\mathscr{G}(c_n) - \mathscr{G}(e_n)\| + \|\mathscr{G}(c_n) - \mathscr{S}(e_n)\|\right] \\ =& (1-\alpha_n)\left[\phi(\|d_n - \mathscr{G}(d_n)\|) + \delta\|d_n - h_n\| + \varepsilon\right] + \alpha_n\left[\phi(\|c_n - \mathscr{G}(c_n)\|) + \delta\|c_n - e_n\| + \varepsilon\right] \\ =& (1-\alpha_n)\left[\phi(\|d_n - \mathscr{G}(d_n)\|) + \delta\|d_n - h_n\|\right] + \alpha_n\left[\phi(\|c_n - \mathscr{G}(c_n)\|) + \delta\|c_n - e_n\|\right] + \varepsilon. \end{aligned}$$

Since, $a^* \in F(\mathscr{G})$ and $\mathscr{G}$ is a contractive-like operator, Theorem 5.1 implies $\lim_{n\to\infty} \|a_n - a^*\| = 0$. Hence

$$\begin{aligned}
0 \le \|c_n - \mathscr{G}(c_n)\| \le \|c_n - a^*\| + \|\mathscr{G}(a^*) - \mathscr{G}(c_n)\| \\
&\le \|c_n - a^*\| + \delta\|c_n - a^*\| \\
&\le (1+\delta)\|c_n - a^*\| \\
&\le (1+\delta)\|(1-\beta_n)d_n + \beta_n\mathscr{G}(d_n) - a^*\| \\
&\le (1+\delta)\big[(1-\beta_n)\|d_n - a^*\| + \beta_n\|\mathscr{G}(d_n) - \mathscr{G}a^*\|\big] \\
&\le (1+\delta)\big[(1-\beta_n)\|d_n - a^*\| + \delta\beta_n\|d_n - a^*\|\big] \\
&\le (1+\delta)(1-(1-\delta)\beta_n)\|d_n - a^*\| \to 0 \ \text{ as } \ n\to\infty.
\end{aligned}$$

So,

$$\|b_n - f_n\| = (1-\alpha_n)\delta\|d_n - h_n\| + \alpha_n\delta\|c_n - e_n\| + \varepsilon, \tag{26}$$

and

$$\begin{aligned}
\|a_{n+1} - g_{n+1}\| &= \|\mathscr{G}(b_n) - \mathscr{S}(f_n)\| \\
&\le \|\mathscr{G}(b_n) - \mathscr{G}(f_n)\| + \|\mathscr{G}(f_n) + \mathscr{S}(f_n)\| \\
&\le \phi(\|b_n - \mathscr{G}(b_n)\|) + \delta\|b_n - f_n\| + \varepsilon.
\end{aligned}$$

Since, $a^* \in F(\mathscr{G})$ and $\mathscr{G}$ is a contractive-like operator, Theorem 5.1 implies $\lim_{n\to\infty} \|a_n - a^*\| = 0$. Hence

$$\begin{aligned}
0 \le \|b_n - \mathscr{G}(b_n)\| \le \|b_n - a^*\| + \|\mathscr{G}(a^*) - \mathscr{G}(b_n)\| \\
&\le \|b_n - a^*\| + \delta\|b_n - a^*\| \\
&\le (1+\delta)\|b_n - a^*\| \\
&\le (1+\delta)\|(1-\alpha_n)\mathscr{G}(d_n) + \alpha_n\mathscr{G}(c_n) - a^*\| \\
&\le (1+\delta)\big[(1-\alpha_n)\|\mathscr{G}(d_n) - a^*\| + \alpha_n\|\mathscr{G}(c_n) - \mathscr{G}(a^*)\|\big] \\
&\le (1+\delta)\big[(1-\alpha_n)\delta\|d_n - a^*\| + \delta\alpha_n\|c_n - a^*\|\big] \\
&\le \delta(1+\delta)\big[(1-\alpha_n)\|d_n - a^*\| + \alpha_n\|c_n - a^*\|\big] \to 0 \ \text{ as } \ n\to\infty,
\end{aligned}$$

and

$$\|a_{n+1} - g_{n+1}\| \le \delta\|b_n - f_n\| + \varepsilon.$$

Put the values from (26), we get

$$\begin{aligned}
\|a_{n+1} - g_{n+1}\| &\le \delta\big[(1-\alpha_n)\delta\|d_n - h_n\| + \alpha_n\delta\|c_n - e_n\| + \varepsilon\big] + \varepsilon \\
&\le \delta^2\big[(1-\alpha_n)\|d_n - h_n\| + \alpha_n\|c_n - e_n\|\big] + \delta\varepsilon + \varepsilon.
\end{aligned}$$

Put the values from (25) and (24), we have

$$\begin{aligned}
\|a_{n+1} - g_{n+1}\| &\le \delta^2\big[(1-\alpha_n)\|d_n - h_n\| + \alpha_n\big\{(1-\beta_n(1-\delta))\|d_n - h_n\| + \beta_n\varepsilon\big\}\big] + \delta\varepsilon + \varepsilon \\
&= \delta^2\big[(1-\alpha_n) + \alpha_n(1-\beta_n(1-\delta))\|d_n - h_n\| + \alpha_n\beta_n\varepsilon\big] + \delta\varepsilon + \varepsilon \\
&= \delta^2\big[(1-\alpha_n + \alpha_n - \alpha_n\beta_n(1-\delta))\|d_n - h_n\| + \alpha_n\beta_n\varepsilon\big] + \delta\varepsilon + \varepsilon \\
&= \delta^2\big[(1-\alpha_n\beta_n(1-\delta))\|d_n - h_n\| + \alpha_n\beta_n\varepsilon\big] + \delta\varepsilon + \varepsilon \\
&\le \delta^2\big[(1-\alpha_n\beta_n(1-\delta))(1-\gamma_n(1-\delta))\|a_n - g_n\| + \gamma_n\varepsilon + \alpha_n\beta_n\varepsilon\big] + \delta\varepsilon + \varepsilon.
\end{aligned}$$

As $1-\gamma_n(1-\delta) < 1$, then:

$$\|a_{n+1} - g_{n+1}\| \le (1-\alpha_n\beta_n(1-\delta))\delta^2\|a_n - g_n\| + \delta^2\gamma_n + (\delta^2\alpha_n\beta_n + \delta)\varepsilon + \varepsilon.$$

Here, $x_n = \|a_n - g_n\|, \lambda_n = \alpha_n\beta_n(1-\delta)$, and $q_n = \frac{\delta^2\alpha_n\beta_n + \delta)\varepsilon + \varepsilon}{1-\delta}$.
All requirements of Lemma 2.3 are fulfilled. Hence, we have

$$0 \le \limsup_{n\to\infty} \|a_n - g_n\| \le \limsup_{n\to\infty} q_n = \frac{\varepsilon}{1-\delta},$$

additionally, Theorem 2.5 yields

$$\|a^* - a^{**}\| \le \frac{\varepsilon}{1-\delta}.$$

□

In addition, we present an example of numerical validation of Theorem 5.2.

Example 5.1. *Let $\mathscr{B} = \mathbb{R}$ and $\mathscr{M} = [0,6]$. Let $\mathscr{G} : \mathscr{M} \to \mathscr{M}$ be defined as:*

$$\mathscr{G}(a) = \begin{cases} \frac{a}{3}, & if a \in [0,3,) \\ \frac{a}{6}, & if a \in [3,6]. \end{cases}$$

Clearly, $0 = a \in F(\mathscr{G})$. Firstly, we establish $\mathscr{G}$ as a contractive-like operator rather than a contraction. Since the operator $\mathscr{G}$ is not continuous at $a = 3 \in [0,6]$, $\mathscr{G}$ is not a contraction. We demonstrate that the operator $\mathscr{G}$ is a contractive-like.
For this define a continuous and strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ as $\phi(a) = \frac{a}{4}$. We have to prove that

$$\|\mathscr{G}a - \mathscr{G}b\| \leq \phi(\|a - \mathscr{G}a\|) + \delta\|a - b\|, \tag{27}$$

for all $a, b \in [0,6]$ where $\delta \in [0,1)$.
Here, we considered the four cases:

Case: A. Let $a, b \in [0,3)$, then

$$\|a - \mathscr{G}a\| = \|a - \frac{a}{3}\| = \frac{2a}{3}$$

so,

$$\phi(\|a - \mathscr{G}a\|) = \phi(\frac{2a}{3}) = \frac{a}{6}$$

which implies

$$\begin{aligned} \|\mathscr{G}a - \mathscr{G}b\| =& \|\frac{a}{3} - \frac{b}{3}\| \\ \leq& \frac{1}{3}\|a - b\| \\ \leq& \frac{1}{3}\|a - b\| + \|\frac{a}{6}\| \\ \leq& \frac{1}{3}\|a - b\| + \phi(\frac{2a}{3}) \\ \leq& \frac{1}{3}\|a - b\| + \phi(\|a - \mathscr{G}a\|). \end{aligned}$$

Hence, (27) is satisfied with $\delta = \frac{1}{3}$.
Case: B. Let $a, b \in [3,6]$, then

$$\|a - \mathscr{G}a\| = \|a - \frac{a}{6}\| = \frac{5a}{6}$$

so,

$$\phi(\|a - \mathscr{G}a\|) = \phi(\frac{a}{2}) = \frac{5a}{24},$$

which implies

$$\begin{aligned} \|\mathscr{G}a - \mathscr{G}b\| =& \|\frac{a}{6} - \frac{b}{6}\| \\ \leq& \frac{1}{6}\|a - b\| \\ \leq& \frac{1}{6}\|a - b\| + \|\frac{5a}{24}\| \\ \leq& \frac{1}{3}\|a - b\| + \phi(\frac{5a}{6}) \\ \leq& \frac{1}{3}\|a - b\| + \phi(\|a - \mathscr{G}a\|). \end{aligned}$$

Hence, (27) is satisfied with $\delta = \frac{1}{3}$.
Case: C. Let $a \in [0,3), b \in [3,6]$, then

$$\|a - \mathscr{G}a\| = \|a - \frac{a}{3}\| = \frac{2a}{3}$$

so,

$$\phi(\|a - \mathscr{G}a\|) = \phi(\frac{2a}{3}) = \frac{a}{6},$$

which implies

$$\begin{aligned} \|\mathscr{G}a - \mathscr{G}b\| =& \|\frac{a}{3} - \frac{b}{6}\| \\ =& \|\frac{a}{6} + \frac{a}{6} - \frac{b}{6}\| \\ \leq& \frac{1}{6}\|a - b\| + \|\frac{a}{6}\| \\ \leq& \frac{1}{3}\|a - b\| + \phi(\frac{2a}{3}) \\ \leq& \frac{1}{3}\|a - b\| + \phi(\|a - \mathscr{G}a\|). \end{aligned}$$

Hence, (27) is satisfied with $\delta = \frac{1}{3}$.
Case: D. Let $a \in [3,6], b \in [0,3)$, then

$$\|a - \mathscr{G}a\| = \|a - \frac{a}{6}\| = \frac{5a}{6}$$

so,

$$\phi(\|a - \mathscr{G}a\|) = \phi(\frac{5a}{6}) = \frac{5a}{24},$$

which implies

$$\begin{aligned}\|\mathscr{G}a - \mathscr{G}b\| =& \|\frac{a}{6} - \frac{b}{3}\| \\ =& \|\frac{a}{3} - \frac{5a}{24} - \frac{b}{3}\| \\ \leq& \frac{1}{3}\|a-b\| + \|\frac{5a}{24}\| \\ \leq& \frac{1}{3}\|a-b\| + \phi(\frac{5a}{6}) \\ \leq& \frac{1}{3}\|a-b\| + \phi(\|a - \mathscr{G}a\|).\end{aligned}$$

Hence, (27) is satisfied with $\delta = \frac{1}{3}$.
Consequently, for all viable cases, (27) is satisfied. As a result, $\mathscr{G}$ is a mapping which is contractive-like.
Next, we define an operator $S : \mathscr{M} \to \mathscr{M}$ as:

$$S(g) = \begin{cases} \frac{g}{3} + \frac{1}{10{,}000}, & if g \in [0,3), \\ \frac{g}{6} + \frac{1}{10{,}000}, & if g \in [3,6], \quad \forall g \in \mathscr{M}. \end{cases}$$

It is simple to demonstrate that S is an appropriate operator for $\mathscr{G}$ with $\varepsilon = 0.0001$ as $\|\mathscr{G}a - Sa\| < \varepsilon = 0.0001$. Also, $0 = a^* \in F(\mathscr{G})$ and $0.000149999 = a^{**} \in \mathscr{S}$. Table 2 of iterated values is obtained with an initial estimate of 5 and $\alpha_n = \frac{n+4}{n+5}$, $\beta_n = \frac{n}{n+2}$ and $\gamma_n = \frac{1}{n+3}$ for all $n \in \mathbb{N}$. Also, we have $\|a^* - a^{**}\| = \|0 - 0.000149999\| \leq 0.00015 = \frac{\varepsilon}{1-\delta}$. As a result, we can assert that in situations where it is difficult to determine $F(\mathscr{G})$, we can select a mapping $\mathscr{S}$ which is closer to $\mathscr{G}$ so that will also reduce the distance between the two fixed points.

Table 2: Comparative iteration values of the operators $\mathscr{G}$ and $\mathscr{S}$

Step	Operator $\mathscr{G}$	Operator $\mathscr{S}$	Difference
1	5	5	0
2	0.30135460	0.17803375	0.1233208
3	0.02072809	0.01238539	0.0083427
4	0.00133069	$9.35481837 \times 10^{-4}$	0.000395211
5	$8.09245977 \times 10^{-5}$	$1.97768214 \times 10^{-4}$	0.000116844
6	$4.70989722 \times 10^{-6}$	$1.52780161 \times 10^{-4}$	0.000148070
7	$2.64303997 \times 10^{-7}$	$1.50156014 \times 10^{-4}$	0.000149892
8	$1.43814246 \times 10^{-8}$	$1.50008489 \times 10^{-4}$	0.000149994
9	$7.62092586 \times 10^{-10}$	$1.50000450 \times 10^{-4}$	0.000149999
10	$3.94670363 \times 10^{-11}$	$1.50000023 \times 10^{-4}$	0.000149999

6. Conclusion

Our article presents a comprehensive study of the convergence behavior of an iterative scheme (9) applied to α-nonexpansive mappings in UCBS. Some strong and weak convergence results support the effectiveness of our approach. The numerical comparison demonstrates the superior performance of the Picard-Thakur hybrid iterative scheme with other well-known methods in the literature. Additionally, the data dependence analysis enriches our understanding about the behavior of the iterative scheme under varying data conditions. Overall, our findings contribute valuable insights for researchers and practitioners in related fields and underscore the practical utility of our proposed scheme.

Conflict of Interest

The authors declared no conflict of interest.

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