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Research paper

Common Fixed Point Theorems in Extended Rectangular b-Metric Spaces

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Abstract

In this paper, we establish common fixed point theorems for quadruple weakly compatible mappings satisfying a new generalized contraction condition. Our results generalize the corresponding result of Budi Nurwahyu et al. [6]. Non-trivial examples are further provided to support the hypotheses of our results.

Keywords: *Compatible mapping; Common Fixed point; Extended rectangular b-metric; Fixed point; Weak contraction mapping.*

1. Introduction

The term common fixed point theory refers on those fixed points theoretic results in which geometric conditions on the underlying spaces and for mappings play a crucial role. For the past several years metric fixed point theory has been flourishing area for many mathematicians. The first famous result (Banach contraction principle) is due to Banach [5] in 1922.

In 1989, A well-known generalization of the concept of a metric space is introduced by Bakhtin [4], is that of a b-metric space. In 1993, Czerwik [8] used the concept of b-metric space and generalized the renowned Banach fixed point theorem in b-metric spaces. However, several authors have investigated heavily the space for fixed point results in different type of contraction mapping [9, 19]. In 2016, George et al. [11] developed a new idea as an extension of b-metric, that is said a rectangular b-metric space. By utilizing this space, some authors, such as [12, 15, 20, 22], have yielded some fixed point theorems on different types of contraction mappings. Besides, in 2017, Kamran [16] generalized b-metric space to ended up extended b-metric space. Many authors have utilized the space for fixed point results about such as, [1, 2, 25]. Recently, Mustafa et al. [22] obtained some fixed point theorems for a new generalization in extended rectangular b-metric space. Asim et al. [3] worked for fixed point results and its applications in this space. For more on generalized metric spaces (see also [7, 21, 28]). During this period, Banach's contraction principle was mainly used to test the fixed points of many kinds of contraction diagrams. Many authors have summarized Banach contraction in various ways [13, 18, 23, 26]. Generalized weak contraction mapping is one of the interesting studies in recent years, such as the generalization of Banach contraction [17, 24, 27].

We first present some important definitions and notations which will be used in the main results as follows:

Definition 1.1. *[10]. A mapping $T : S \to S$ where (S,d) is a complete metric space is said to be generalized weakly contraction if*

$$d(Ts,Tt) \leq \Psi(d(s,t)) - \varphi(d(s,t)),$$

where $\Psi, \varphi : [0,+\infty) \to [0,+\infty)$ are continuous and monotone nondecreasing functions with $\Psi(t) = \varphi(t) = o$ if and only if $t = 0$. Dutta [10] pointed out that when the function satisfies these conditions, T has a unique fixed point.

Definition 1.2. *([7], [8]). Let S be a non-empty set. A mapping $d_b : S \times S \to [0,+\infty)$ is said to be a b-metric, if there exists $b \geq 1$ such that d_b satisfies the following conditions:*

1. $d_b(s,t) = 0,\ if\ and\ only\ if\ s = t,$
2. $d_b(s,t) = d_b(t,s),$
3. $d_b(s,t) \leq b[d_b(s,r) + d_b(r,t)],$

for all $s,t,r \in S$. The pair (S,d_b) is called a b-metric space.

Definition 1.3. *[16]. Let S be a non-empty set. A mapping $d_b : S\times S \to [0,+\infty)$ is said to be an extended b-metric, if there exists a function $b : S\times S \to [1,+\infty)$ such that d_b satisfies the following conditions:*

1. $d_b(s,t) = 0$, *if and only if* $s = t$,
2. $d_b(s,t) = d_b(t,s)$,
3. $d_b(s,t) \le b(s,t)[d_b(s,r)+d_b(r,t)]$,

for all $s,t,r \in S$. The pair (S,d_b) is called an extended b-metric space.

Definition 1.4. *[11].Let S be a non-empty set. A mapping $d_b : S\times S \to [0,+\infty)$ is said to be a rectangular b-metric, if there is $b \ge 1$ such that d_b satisfies the following conditions:*

1. $d_b(s,t) = 0$, *if and only if* $s = t$,
2. $d_b(s,t) = d_b(t,s)$,
3. $d_b(s,t) \le b[d_b(s,r)+d_b(r,p)+d_b(p,t)]$,

for all $s,t \in S$ and $r,p \in S\setminus\{s,t\}$. The pair (S,d_b) is called rectangular b-metric space.

Definition 1.5. *[3]. Let S be a non-empty set. A mapping $d_b : S\times S \to [0,+\infty)$ is said to be an extended rectangular b-metric, if there exists a function $b : S\times S \to [1,+\infty)$ such that d_b satisfies the following conditions:*

1. $d_b(s,t) = 0$, *if and only if* $s = t$,
2. $d_b(s,t) = d_b(t,s)$,
3. $d_b(s,t) \le b(s,t)[d_b(s,r)+d_b(r,p)+d_b(p,t)]$,

for all $s,t \in S$ and $r,p \in S\setminus\{s,t\}$. The pair (S,d_b) is called an extended rectangular b-metric space.

Definition 1.6. *[14]. Let s_n be a sequence in S and (S,d_b) be an extended rectangular b-metric space.*

- *$\{s_n\}$ is called convergent to $s \in S$ iff $d_b(s_n,s) \longrightarrow 0$, as $n \longrightarrow \infty$.*
- *$\{s_n\}$ is called Cauchy iff $d_b(s_n,s_m) \longrightarrow 0$, as $n,m \longrightarrow \infty$.*

Definition 1.7. *[14]. Let s_n be a sequence in S and (S,d_b) be an extended rectangular b-metric space. Self- mapping f and g on S is said to be compatible, if $d_b(fs_n,u) \longrightarrow 0$ and $d_b(gs_n,u) \longrightarrow 0$, then $d_b(fgs_n,gfs_n) \longrightarrow 0$, as $n \longrightarrow \infty$.*

Definition 1.8. *[14]. Let S be a non-empty set and $T_1,T_2 : S \longrightarrow S$ be self-mappings. T_1,T_2 is called weakly compatible, for every $s \in S$, if $T_1s = T_2s$ then $T_2T_1s = T_1T_2s$.*

Recently, Budi Nurwahyu et al. [6] proved some common fixed point on generalized weak contraction mappings in extended rectangular b-metric spaces. Inspired by their work, we prove generalized common fixed point theorem in extended rectangular b-metric spaces. We use an example to prove our theorem and clarify the definitions.

2. Main Results

Now, we present our main results as follows:

Theorem 2.1. *Let (S,d_b) be a complete extended rectangular b-metric space. Let $A,B,C,D : S \to S$ be continuous mappings such that $A(S) \subseteq B(S)$ and $C(S) \subseteq D(S)$ and satisfy the following conditions:-*

$$
\begin{aligned}
&b(r,y)\psi[d_b(Ar,Ay)+d_b(Cr,Cy)]\\
&\le \psi\big[\lambda\big(d_b(Ar,Br)+d_b(Cr,Dr)+d_b(Ay,By)+d_b(Cy,Dy)\big)\\
&\quad +\gamma\,\frac{d_b(Br,By)+d_b(Dr,Dy)}{b(Br,By)+b(Dr,Dy)}\big]\\
&\quad -\beta\varphi\big(d_b(Ar,By)d_b(Ay,Br)+d_b(Cr,Dy)d_b(Cy,Dr)\big), \qquad (2.1)
\end{aligned}
$$

where $\beta \ge 0, 0<\lambda<1, \gamma<1, \frac{\lambda+\frac{\gamma}{2}}{1-\lambda}<1$, $\psi,\varphi : [0,+\infty) \to [0,+\infty)$ are continuous and ψ is nondecreasing with $\psi(t)=\varphi(t)=0$ if and only if $t=0$.
If A, B, C and D are compatible and $\lim_{n\longrightarrow\infty}[b(A^2r_{2n},BAr_{2n})+b(C^2r_{2n},DCr_{2n})] < \dfrac{1}{\lambda}$, then A, B, C, D have a unique common fixed point in S.

Proof. Let $r_0 \in S$ and since $A(S) \subseteq B(S)$ and $C(S) \subseteq D(S)$, then we can define a sequence $\{y_n\}$, where $y_{2n} = Ar_{2n} = Br_{2n+1}$ and $y_{2n+1} = Cr_{2n+1} = Dr_{2n+2}$. Since ψ is nondecreasing, now using (2.1) we have,

$$
\begin{aligned}
&b(r_{2n},r_{2n+1})\psi\big[d_b(Ar_{2n},Ar_{2n+1})+d_b(Cr_{2n},Cr_{2n+1})\big]\\
&\le \psi\Big[\lambda\Big\{d_b(Ar_{2n},Br_{2n})+d_b(Cr_{2n},Dr_{2n})+d_b(Ar_{2n+1},Br_{2n+1})\\
&\quad +d_b(Cr_{2n+1},Dr_{2n+1})\Big\}+\gamma\Big(\frac{d_b(Br_{2n},Br_{2n+1})+d_b(Dr_{2n},Dr_{2n+1})}{b(Br_{2n},Br_{2n+1})+b(Dr_{2n},Dr_{2n+1})}\Big)\\
&\quad -\beta\varphi\Big(d_b(Ar_{2n},Br_{2n+1})d_b(Ar_{2n+1},Br_{2n})+d_b(Cr_{2n},Dr_{2n+1})\\
&\quad d_b(Cr_{2n+1},Dr_{2n})\Big)\Big]
\end{aligned}
$$

or

$$b(r_{2n},r_{2n+1})\psi\big[d_b(y_{2n},y_{2n+1})+d_b(y_{2n},y_{2n+1})\big]$$
$$\leq \psi\Big[\lambda\Big\{d_b(y_{2n},y_{2n-1})+d_b(y_{2n+1},y_{2n})+d_b(y_{2n},y_{2n-1})+d_b(y_{2n+1},y_{2n-1})\Big\}$$
$$+\gamma\Big(\frac{d_b(y_{2n-1},y_{2n})+d_b(y_{2n-1},y_{2n})}{b(y_{2n-1})+b((y_{2n-1})}\Big)$$
$$-\beta\varphi\Big(d_b(y_{2n},y_{2n})d_b(y_{2n+1},y_{2n-1}))+d_b(y_{2n},y_{2n})d_b(y_{2n+1},y_{2n-1})\Big)\Big]$$

$$\leq \frac{1}{b(r_{2n},r_{2n+1})}\Big[\psi\Big\{2\lambda\left(d_b(y_{2n},y_{2n-1})+d_b(y_{2n+1},y_{2n})\right)\Big\}+\gamma\Big\{\frac{d_b(y_{2n-1},y_{2n})}{b(y_{2n-1},y_{2n})}\Big\}\Big]$$
$$\leq \frac{1}{b(r_{2n},r_{2n+1})}\Big[\psi\Big\{2\lambda\left(d_b(y_{2n},y_{2n-1})+d_b(y_{2n+1},y_{2n})\right)\Big\}+\gamma\Big(d_b(y_{2n-1},y_{2n})\Big)\Big]$$
$$\leq \psi\Big[2\lambda\left(d_b(y_{2n},y_{2n-1})+d_b(y_{2n+1},y_{2n})\right)+\gamma(d_b(y_{2n-1},y_{2n})\Big].$$

Again since ψ is nondecreasing, we have

$$2d_b(y_{2n},y_{2n+1})\leq 2\lambda\left(d_b(y_{2n},y_{2n-1})+d_b(y_{2n+1},y_{2n})\right)+\gamma(d_b(y_{2n-1},y_{2n}),$$

or

$$d_b(y_{2n},y_{2n+1})\leq (\frac{2\lambda+\gamma}{2-2\lambda})(d_b(y_{2n-1},y_{2n})$$
$$\leq (\frac{\lambda+\frac{\gamma}{2}}{1-\lambda})(d_b(y_{2n-1},y_{2n}).$$

Let $\alpha=\frac{\lambda+\frac{\gamma}{2}}{1-\lambda}$, then we have

$$d_b(y_{2n},y_{2n+1})\leq \alpha(d_b(y_{2n-1},y_{2n}),$$

and by using recursively, we get

$$d_b(y_{2n},y_{2n+1})\leq \alpha^n(d_b(y_0,y_1). \tag{2.2}$$

Since $0<\alpha<1$, then

$$\lim_{n\longrightarrow\infty} d_b(y_{2n},y_{2n+1})\leq \lim_{n\longrightarrow\infty}\alpha^n(d_b(y_0,y_1)=0$$
$$i.e.\quad d_b(y_{2n},y_{2n+1})\longrightarrow 0 \quad as\ \ n\longrightarrow\infty. \tag{2.3}$$

Now we show that $\{y_n\}$ is a Cauchy sequence. By using (2.1), we have

$$b(r_{2m},r_{2n})\psi\big[d_b(Ar_{2m},Ar_{2n})+d_b(Cr_{2m},Cr_{2n})\big]$$
$$\leq \psi\Big[\lambda\Big\{d_b(Ar_{2m},Br_{2m})+d_b(Cr_{2m},Dr_{2m})+d_b(Ar_{2n},Br_{2n})$$
$$+d_b(Cr_{2n},Dr_{2n})\Big\}+\gamma\Big(\frac{d_b(Br_{2m},Br_{2n})+d_b(Dr_{2m},Dr_{2n})}{b(Br_{2m},Br_{2n})+b(Dr_{2m},Dr_{2n})}\Big)$$
$$-\beta\varphi\Big(d_b(Ar_{2m},Br_{2n})d_b(Ar_{2n},Br_{2m})+d_b(Cr_{2m},Dr_{2n})d_b(Cr_{2n},Dr_{2m})\Big)\Big]$$

or

$$b(r_{2m},r_{2n})\psi\big[d_b(y_{2m},y_{2n})+d_b(y_{2m},y_{2n})\big]$$
$$\leq \psi\Big[\lambda\Big\{d_b(y_{2m},y_{2m-1})+d_b(y_{2m},y_{2m-1})+d_b(y_{2n},y_{2n-1})+d_b(y_{2n},y_{2n-1})\Big\}$$
$$+\gamma\Big(\frac{d_b(y_{2m-1},y_{2n-1})+d_b(y_{2m-1},y_{2n-1})}{b(y_{2m-1},y_{2n-1})+b(y_{2m-1},y_{2n-1})}\Big)$$
$$-\beta\varphi\Big(d_b(y_{2m},y_{2n-1})d_b(y_{2n},y_{2m-1})+d_b(y_{2m},y_{2n-1})d_b(y_{2n},y_{2m-1})\Big)\Big]$$

or

$$b(r_{2m},r_{2n})\psi\big[2d_b(y_{2m},y_{2n})\big]$$
$$\leq \psi\Big[2\lambda\Big\{d_b(y_{2m},y_{2m-1})+d_b(y_{2n},y_{2n-1})\Big\}+\gamma\Big(\frac{d_b(y_{2m-1},y_{2n-1})}{b(y_{2m-1},y_{2n-1})}\Big)\Big]$$
$$\leq \psi\Big[2\lambda\Big\{d_b(y_{2m},y_{2m-1})+d_b(y_{2n},y_{2n-1})\Big\}+\gamma\Big(d_b(y_{2m-1},y_{2n-1})\Big)\Big]$$

$$\leq \frac{1}{b(r_{2m},r_{2n})}\Big[\psi 2\lambda\Big\{d_b(y_{2m},y_{2m-1})+d_b(y_{2n},y_{2n-1})\Big\}+\gamma\Big(d_b(y_{2m-1},y_{2n-1})\Big)\Big]$$
$$\leq \psi\Big[2\lambda\Big\{d_b(y_{2m},y_{2m-1})+d_b(y_{2n},y_{2n-1})\Big\}+\gamma\Big(d_b(y_{2m-1},y_{2n-1})\Big)\Big]$$
$$\leq \psi\Big[2\lambda\Big\{d_b(y_{2m},y_{2m-1})+d_b(y_{2n},y_{2n-1})\Big\}$$
$$+\gamma\Big(d_b(y_{2m-1},y_{2m})+d_b(y_{2m},y_{2n})+d_b(y_{2n},y_{2n-1})\Big)\Big].$$

Since ψ is nondecreasing, we get

$$2d_b(y_{2m},y_{2n})\leq 2\lambda\Big\{d_b(y_{2m},y_{2m-1})+d_b(y_{2n},y_{2n-1})\Big\}$$
$$+\gamma\Big(d_b(y_{2m-1},y_{2m})+d_b(y_{2m},y_{2n})+d_b(y_{2n},y_{2n-1})\Big).$$

By using (2.2) and (2.3), we get $\lim_{m,n\longrightarrow\infty} d_b(y_{2m},y_{2n})=0$. i.e. $d_b(y_{2m},y_{2n})\longrightarrow 0\ \ as\ \ m,n\longrightarrow\infty$. Hence $\{y_n\}$ is a Cauchy sequence in S. Since S is complete, then $\exists\, u^*\in S$ such that $d_b(y_{2n},u^*)\longrightarrow\infty$ i.e.

$$d_b(Ar_{2n},u^*)\longrightarrow 0, d_b(Br_{2n},u^*)\longrightarrow 0, d_b(Cr_{2n},u^*)\longrightarrow 0,$$
$$d_b(Dr_{2n},u^*)\longrightarrow 0\ \ as\ \ n\longrightarrow\infty. \tag{2.4}$$

Since A, B, C and D are compatible mapping, then from (2.4), we have

$$d_b(A^2r_{2n},Au^*)\longrightarrow 0, d_b(ABr_{2n},Au^*)\longrightarrow 0, d_b(BAr_{2n},Bu^*)\longrightarrow 0\ \ and$$
$$d_b(C^2r_{2n},Cu^*)\longrightarrow 0, d_b(CDr_{2n},Cu^*)\longrightarrow 0, d_b(DCr_{2n},Du^*)\longrightarrow 0$$
$$as\ \ n\longrightarrow\infty. \tag{2.5}$$

Now

$$b(Ar_{2n},r_{2n})\psi\big[d_b(A^2r_{2n},Ar_{2n})\big]+b(Cr_{2n},r_{2n})\psi\big[d_b(C^2r_{2n},Cr_{2n})\big]$$
$$\leq \psi\Big[\lambda\Big\{d_b(A^2r_{2n},BAr_{2n})+d_b(C^2r_{2n},Dr_{2n})+d_b(Ar_{2n},Br_{2n})$$
$$+d_b(Cr_{2n},Dr_{2n})\Big\}+\gamma\Big(\frac{d_b(BAr_{2n},Br_{2n})+d_b(DCr_{2n},Dr_{2n})}{b(BAr_{2n},Br_{2n})+b(DCr_{2n},Dr_{2n})}\Big)$$
$$-\beta\varphi\Big(d_b(A^2r_{2n},Bx_{2n})d_b(Ar_{2n},BAr_{2n})+d_b(C^2r_{2n},Dr_{2n})$$
$$d_b(Cr_{2n},DCr_{2n})\Big)\Big]$$

or

$$\frac{1}{b(Cr_{2n},r_{2n})}\psi\big[d_b(A^2r_{2n},Ar_{2n})\big]+\frac{1}{b(Ar_{2n},r_{2n})}\psi\big[d_b(C^2r_{2n},Cr_{2n})\big]$$
$$\leq\frac{1}{b(Ar_{2n},r_{2n})+b(Cr_{2n},r_{2n})}\Big[\psi\Big\{\lambda\Big(b(A^2r_{2n},BAr_{2n})$$
$$+b(C^2r_{2n},DCr_{2n})\Big)\Big(d_b(A^2r_{2n},Au^*)+d_b(Au^*,ABr_{2n})$$
$$+d_b(ABr_{2n},BAr_{2n})+d_b(C^2r_{2n},Cu^*)+d_b(Cu^*,CDr_{2n})$$
$$+d_b(CDr_{2n},DCr_{2n})\Big)+d_b(Ar_{2n},Br_{2n})+d_b(Cr_{2n},Dr_{2n})\Big\}$$
$$+\gamma\Big(\frac{d_b(BAr_{2n},Br_{2n})+d_b(DCr_{2n},Dr_{2n})}{b(BAr_{2n},Br_{2n})+b(DCr_{2n},Dr_{2n})}\Big)\Big]$$

$$\leq\frac{1}{b(Ar_{2n},r_{2n})+b(Cr_{2n},r_{2n})}\Big[\psi\Big\{\lambda\Big(b(A^2r_{2n},BAr_{2n})+b(C^2r_{2n},DCr_{2n})\Big)$$
$$\Big(d_b(A^2r_{2n},Au^*)+d_b(Au^*,ABr_{2n})+d_b(ABr_{2n},BAr_{2n})$$
$$+d_b(C^2r_{2n},Cu^*)+d_b(Cu^*,CDr_{2n})+d_b(CDr_{2n},DCr_{2n})\Big)$$
$$+d_b(Ar_{2n},Br_{2n})+d_b(Cr_{2n},Dr_{2n})\Big\}+\gamma\Big(d_b(BAr_{2n},Br_{2n})+d_b(DCr_{2n},Dr_{2n})\Big)\Big]$$

$$\leq\Big[\psi\Big\{\lambda\Big(b(A^2r_{2n},BAr_{2n})+b(C^2r_{2n},DCr_{2n})\Big)\Big(d_b(A^2r_{2n},Au^*)+d_b(Au^*,ABr_{2n})$$
$$+d_b(ABr_{2n},BAr_{2n})+d_b(C^2r_{2n},Cu^*)+d_b(Cu^*,CDr_{2n})+d_b(CDr_{2n},DCr_{2n})\Big)$$
$$+d_b(Ar_{2n},Br_{2n})+d_b(Cr_{2n},Dr_{2n})\Big\}+\gamma\Big(d_b(BAr_{2n},Br_{2n})+d_b(DCr_{2n},Dr_{2n})\Big)\Big]$$

$$\leq \Big[\psi\Big\{\lambda\Big(b(A^2r_{2n},BAr_{2n})+b(C^2r_{2n},DCr_{2n})\Big)\Big(d_b(A^2r_{2n},Au*)+d_b(Au^*,ABr_{2n})$$
$$+d_b(ABr_{2n},BAr_{2n})+d_b(C^2r_{2n},Cu^*)+d_b(Cu^*,CDr_{2n})+d_b(CDr_{2n},DCr_{2n})\Big)$$
$$+d_b(y_{2n},y_{2n-1})+d_b(y_{2n},y_{2n-1})\Big\}+\gamma\Big(d_b(BAr_{2n},Br_{2n})+d_b(DCr_{2n},Dr_{2n})\Big)\Big]$$
$$\leq \Big[\psi\Big\{\lambda\Big(b(A^2r_{2n},BAr_{2n})+b(C^2r_{2n},DCr_{2n})\Big)\Big(d_b(A^2r_{2n},Au^*)+d_b(Au^*,ABr_{2n})$$
$$+d_b(ABr_{2n},BAr_{2n})+d_b(C^2r_{2n},Cu^*)+d_b(Cu^*,CDr_{2n})+d_b(CDr_{2n},DCr_{2n})\Big)$$
$$+2d_b(y_{2n},y_{2n-1})\Big\}+\gamma\Big(d_b(BAr_{2n},Br_{2n})+d_b(DCr_{2n},Dr_{2n})\Big)\Big].$$

Using(2.2), we have

$$\leq \Big[\psi\Big\{\lambda\Big(b(A^2r_{2n},BAr_{2n})+b(C^2r_{2n},DCr_{2n})\Big)\Big(d_b(A^2r_{2n},Au^*)+d_b(Au^*,ABr_{2n})$$
$$+d_b(ABr_{2n},BAr_{2n})+d_b(C^2r_{2n},Cu^*)+d_b(Cu^*,CDr_{2n})+d_b(CDr_{2n},DCr_{2n})\Big)$$
$$+2\alpha^{n-1}d_b(y_0,y_1)\Big\}+\gamma\Big(d_b(BAr_{2n},Br_{2n})+d_b(DCr_{2n},Dr_{2n})\Big)\Big].$$

Since ψ is nondecreasing, we get

$$\frac{1}{b(Cr_{2n},r_{2n})}(d_b(A^2r_{2n},Ar_{2n})+\frac{1}{b(Ar_{2n},r_{2n})}d_b(C^2r_{2n},Cr_{2n})$$
$$\leq \Big\{\lambda\Big(b(A^2r_{2n},BAr_{2n})+b(C^2r_{2n},DCr_{2n})\Big)\Big(d_b(A^2r_{2n},Au^*)$$
$$+d_b(Au^*,ABr_{2n})+d_b(ABr_{2n},BAr_{2n})+d_b(C^2r_{2n},Cu^*)$$
$$+d_b(Cu^*,CDr_{2n})+d_b(CDr_{2n},DCr_{2n})\Big)+2\alpha^{n-1}d_b(y_0,y_1)\Big\}$$
$$+\gamma\Big(d_b(BAr_{2n},Br_{2n})+d_b(DCr_{2n},Dr_{2n})\Big). \tag{2.6}$$

Since

$$\lim_{n\longrightarrow\infty}\Big[b(A^2r_{2n},BAr_{2n})+b(C^2r_{2n},DCr_{2n})\Big]<\frac{1}{\lambda}$$

.

Using (2.4), (2.5), (2.6) and for $n\longrightarrow\infty$, we get

$$\lim_{n\longrightarrow\infty}\Big[\frac{1}{b(Cr_{2n},r_{2n})}(d_b(A^2r_{2n},Ar_{2n})+\frac{1}{b(Ar_{2n},r_{2n})}d_b(C^2r_{2n},Cr_{2n})\Big]=0$$

.

This implies that

$$\lim_{n\longrightarrow\infty}\Big[\frac{1}{b(Cr_{2n},r_{2n})}(d_b(A^2r_{2n},Ar_{2n})\Big]=0 \ \ and \ \lim_{n\longrightarrow\infty}\Big[\frac{1}{b(Ar_{2n},r_{2n})}d_b(C^2r_{2n},Cr_{2n})\Big]=0$$

$$yields \ \lim_{n\longrightarrow\infty}(d_b(A^2r_{2n},Ar_{2n})=0 \ \ and \ \lim_{n\longrightarrow\infty}d_b(C^2r_{2n},Cr_{2n})=0. \tag{2.7}$$

Now

$$d_b(Au^*,u^*)\leq b(Au^*,u^*)\Big[d_b(Au^*,A^2r_{2n})+d_b(A^2r_{2n},Ar_{2n})+d_b(Ar_{2n},u^*)\Big]$$
$$and \ d_b(Cu^*,u^*)\leq b(Cu^*,u^*)\Big[d_b(Cu^*,C^2r_{2n})+d_b(C^2r_{2n},Cr_{2n})+d_b(Cr_{2n},u^*)\Big]. \tag{2.8}$$

Using (2.4), (2.5), (2.7) and (2.8) for $n\longrightarrow\infty$, we obtain $d_b(Au^*,u^*)=0$ and $d_b(Cu^*,u^*)=0$. Thus we have $Au^*=u^*$ and $Cu^*=u^*$. Now

$$d_b(u^*,Bu^*)=d_b(Au^*,Bu^*)\leq b(Au^*,Bu^*)\Big[d_b(Au^*,ABr_{2n})$$
$$+d_b(ABr_{2n},BAr_{2n})+d_b(BAr_{2n},u^*)\Big]$$
$$and \ \ d_b(u^*,Du^*)=d_b(Cu^*,Du^*)\leq b(Cu^*,Du^*)\Big[d_b(Cu^*,CDr_{2n})$$
$$+d_b(CD_{2n},DCr_{2n})+d_b(DCr_{2n},u^*)\Big]. \tag{2.9}$$

By using (2.4), (2.5) and taking $n\longrightarrow\infty$ in (2.8), we get $d_b(u^*,Bu^*)=0$ and $d_b(u^*,Du^*)=0$. Hence we get $Au^*=Bu^*=u^*$ and $Cu^*=Du^*=u^*$. Thus u^* is common fixed point of A, B, C and D.

Next, we show the uniqueness of common fixed point of A, B, C and D. Suppose that t^* is another common fixed point of A, B, C and D. i.e. $t^* = At^* = Bt^* = Ct^* = Dt^*$. From 2.1, we have

$$
\begin{aligned}
& b(u^*,t^*)\psi\big[d_b(Au^*,At^*)+d_b(Cu^*,Ct^*)\big] \\
& \le \psi\Big[\lambda\Big\{d_b(Au^*,Bu^*)+d_b(Cu^*,Du^*)+d_b(At^*,Bt^*)+d_b(Ct^*,Dt^*)\Big\} \\
& \quad +\gamma\Big(\frac{d_b(Bu^*,Bt^*)+d_b(Du^*,Dt^*)}{b(Bu^*,Bt^*)+b(Du^*,Dt^*)}\Big) \\
& \quad -\beta\varphi\Big(d_b(Au^*,Bt^*)d_b(At^*,Bu^*)+d_b(Cu^*,Dt^*)d_b(Ct^*,Du^*)\Big)\Big]
\end{aligned}
$$

or

$$
\begin{aligned}
& \psi\big[d_b(Au^*,At^*)+d_b(Cu^*,Ct^*)\big] \\
& \le \frac{1}{b(u^*,t^*)}\psi\Big[\lambda\Big\{d_b(Au^*,Bu^*)+d_b(Cu^*,Du^*)+d_b(At^*,Bt^*) \\
& \quad +d_b(Ct^*,Dt^*)\Big\}+\gamma\Big(\frac{d_b(Bu^*,Bt^*)+d_b(Du^*,Dt^*)}{b(Bu^*,Bt^*)+b(Du^*,Dt^*)}\Big)\Big] \\
& \le \psi\Big[\lambda\Big\{d_b(Au^*,Bu^*)+d_b(Cu^*,Du^*)+d_b(At^*,Bt^*)+d_b(Ct^*,Dt^*)\Big\} \\
& \quad +\gamma\Big(d_b(Bu^*,Bt^*)+d_b(Du^*,Dt^*)\Big)\Big].
\end{aligned}
$$

Since ψ is nondecreasing, we have

$$
\begin{aligned}
\big[d_b(Au^*,At^*)+d_b(Cu^*,Ct^*)\big] & \le \Big[\lambda\Big\{d_b(Au^*,Bu^*)+d_b(Cu^*,Du^*)+d_b(At^*,Bt^*)+d_b(Ct^*,Dt^*)\Big\} \\
& \quad +\gamma\Big(d_b(Bu^*,Bt^*)+d_b(Du^*,Dt^*)\Big)\Big] \\
& \le \Big[\lambda\Big\{d_b(u^*,u^*)+d_b(u^*,u^*)+d_b(t^*,t^*)+d_b(t^*,t^*)\Big\} \\
& \quad +\gamma\Big(d_b(u^*,t^*)+d_b(u^*,t^*)\Big)\Big]
\end{aligned}
$$

or

$$
\begin{aligned}
\big[d_b(u^*,t^*)+d_b(u^*,t^*)\big] & \le \Big[\lambda\Big\{d_b(u^*,u^*)+d_b(u^*,u^*)+d_b(t^*,t^*)+d_b(t^*,t^*)\Big\} \\
& \quad +\gamma\Big(d_b(u^*,t^*)+d_b(u^*,t^*)\Big)\Big]
\end{aligned}
$$

or

$$
\begin{aligned}
\big[2d_b(u^*,t^*)\big] & \le \Big[2\lambda\Big\{d_b(u^*,u^*))+d_b(t^*,t^*)\Big\}+2\gamma\Big(d_b(u^*,t^*)\Big)\Big] \\
& \le 2\gamma\Big(d_b(u^*,t^*)\Big)
\end{aligned}
$$

Thus

$$(1-\gamma)(d_b(u^*,t^*) \le 0. \textit{ Since } 1-\gamma>0, \textit{ we have } d_b(u^*,t^*)=0.$$

This implies that

$$u^* = t^*.$$

□

Now, we provide an example of our proven result.

Example 1. *Let $S=[0,1]$, define $d_b(r,p)=\frac{1}{9}(r-p)^2$ and $b(r,p)=4^{(r-p)^2}$ on $S\times S$. Define A,B,C,D to be self-mapping on S as follows: $A(r)=\frac{r}{2^3}, B(r)=\frac{r}{2^2}, C(r)=\frac{r}{2}, andD(r)=r$, and define $\psi(t)=\frac{9}{2}t, \varphi=\frac{t}{2^9}$ for $t\in[0,\infty)$ and take $\lambda=\frac{1}{4}, \beta=\frac{1}{2}, \gamma=\frac{1}{2^9}$.*
In fact, it is clear that $d_b(r,p)=\frac{1}{9}(r-p)^2$ is an extended rectangular $b-$metric with $b(r,p)=4^{(r-p)^2}$. Now we have

$$
\begin{aligned}
\lim_{n\longrightarrow\infty}[b(A^2r_n,BAr_n)+b(C^2r_n,DCr_n)] & = \lim_{n\longrightarrow\infty}[b\big(A(\frac{r_n}{2^3}),B(\frac{r_n}{2^3})\big)+b\big(C(\frac{r_n}{2}),D(\frac{r_n}{2})\big)] \\
& = \lim_{n\longrightarrow\infty}[b\big(\frac{r_n}{2^6},\frac{r_n}{2^5}\big)+b\big(\frac{r_n}{2^2},\frac{r_n}{2}\big)] \\
& = \lim_{n\longrightarrow\infty}[4^{(\frac{r_n}{2^6}-\frac{r_n}{2^5})^2}+4^{(\frac{r_n}{2^2}-\frac{r_n}{2})^2}] \\
& = 1+1=2<\frac{1}{\lambda}=4.
\end{aligned}
$$

Consider

$$\psi\Big[\lambda\big(d_b(Ar,Br)+d_b(Cr,Dr)+d_b(Ap,Bp)+d_b(Cp,Dp)\big)+\gamma\,\frac{d_b(Br,Bp)+d_b(Dr,Dp)}{b(Br,Bp)+b(Dr,Dp)}\Big]$$
$$-\beta\varphi\big(d_b(Ar,Bp)d_b(Ap,Br)+d_b(Cr,Dp)d_b(Cp,Dr)\big)$$
$$\le\psi\Big[\lambda\big(d_b(\frac{r}{2^3},\frac{r}{2^2})+d_b(\frac{r}{2},r)+d_b(\frac{p}{2^3},\frac{p}{2^2})+d_b(\frac{p}{2},p)\big)+\gamma\,\frac{d_b(\frac{r}{2^2},\frac{p}{2^2})+d_b(r,p)}{b(\frac{r}{2^2},\frac{p}{2^2})+b(r,p)}\Big]$$
$$-\beta\varphi\Big(d_b(\frac{r}{2^3},\frac{p}{2^2})d_b(\frac{p}{2^3},\frac{r}{2^2})+d_b(\frac{r}{2},p)d_b(\frac{p}{2},r)\Big)$$

$$=\psi\Big[\frac{1}{4}\Big(\frac{1}{9}(\frac{r}{2^3}-\frac{r}{2^2})^2+\frac{1}{9}(\frac{r}{2}-r)^2+\frac{1}{9}(\frac{p}{2^3}-\frac{p}{2^2})^2+\frac{1}{9}(\frac{p}{2}-p)^2\Big)+\gamma\Big(\frac{\frac{1}{9}(\frac{r}{2^2}-\frac{p}{2^2})^2+\frac{1}{9}(r-p)^2}{4^{(\frac{r}{2^2}-\frac{p}{2^2})^2}+4^{(r-p)^2}}\Big)\Big]$$
$$-\beta\varphi\Big(\frac{1}{9}(\frac{r}{2^3}-\frac{p}{2^2})^2\frac{1}{9}(\frac{p}{2^3}-\frac{r}{2^2})^2+\frac{1}{9}(\frac{r}{2}-p)^2\frac{1}{9}(\frac{p}{2}-r)^2\Big)$$

$$=\frac{9}{2}\Big[\frac{1}{4}\Big(\frac{1}{9}(\frac{r}{2^3}-\frac{r}{2^2})^2+\frac{1}{9}(\frac{r}{2}-r)^2+\frac{1}{9}(\frac{p}{2^3}-\frac{p}{2^2})^2+\frac{1}{9}(\frac{p}{2}-p)^2\Big)+\frac{1}{2^9}\Big(\frac{\frac{1}{9}(\frac{r}{2^2}-\frac{p}{2^2})^2+\frac{1}{9}(r-p)^2}{4^{(\frac{r}{2^2}-\frac{p}{2^2})^2}+4^{(r-p)^2}}\Big)\Big]$$
$$-\frac{1}{2}\Big(\frac{1}{2^9}(\frac{1}{9}(\frac{r}{2^3}-\frac{p}{2^2})^2\frac{1}{9}(\frac{p}{2^3}-\frac{r}{2^2})^2+\frac{1}{9}(\frac{r}{2}-p)^2\frac{1}{9}(\frac{p}{2}-r)^2)\Big)$$

$$=\frac{1}{8}\Big((\frac{r}{2^3}-\frac{r}{2^2})^2+(\frac{r}{2}-r)^2+(\frac{p}{2^3}-\frac{p}{2^2})^2+(\frac{p}{2}-p)^2+\frac{4}{2^9}(\frac{\frac{1}{9}(\frac{r}{2^2}-\frac{p}{2^2})^2+\frac{1}{9}(r-p)^2}{4(\frac{r}{2^2}-\frac{p}{2^2})^2+4(r,p)^2})\Big)$$
$$-\frac{\frac{1}{81}((\frac{r}{2^3}-\frac{p}{2^2})^2(\frac{p}{2^3}-\frac{r}{2^2})^2+(\frac{r}{2}-p)^2(\frac{p}{2}-r)^2)}{2^{10}}$$

$$=\frac{1}{8}\Big((\frac{r}{2^3})^2+(\frac{r}{2})^2+(\frac{p}{2^3})^2+(\frac{p}{2})^2+\frac{4}{2^9}(\frac{\frac{1}{9}(\frac{r}{2^2}-\frac{p}{2^2})^2+\frac{1}{9}(r-p)^2}{4+4})\Big)$$
$$-\frac{\frac{1}{81}((\frac{r}{2^3}-\frac{p}{2^2})^2(\frac{p}{2^3}-\frac{r}{2^2})^2+(\frac{r}{2}-p)^2(\frac{p}{2}-r)^2)}{2^{10}}.$$

Since $r,p\in S=[0,1]$, *then we have*

$$\ge\frac{1}{8}\Big(\frac{17r^2+17p^2}{2^6}+\frac{4}{2^{12}}\big(2(r-p)^2\big)\Big)-\frac{\frac{1}{81}\big((\frac{r}{2^3}-\frac{p}{2^2})^2+(\frac{r}{2}-y)^2\big)}{2^{10}}$$

$$=\frac{1}{8}\Big(\frac{17r^2+17p^2}{2^6}+\frac{(r-p)^2}{2^9}\Big)-\frac{\frac{1}{81}\big((\frac{r}{2^3})^2-\frac{rp}{2^4}+(\frac{p}{2^2})^2+(\frac{r}{2})^2-rp+p^2\big)}{2^{10}}$$
$$\ge\frac{17r^2+17p^2}{2^9}+\frac{(r-p)^2}{2^{12}}-\Big(\frac{r^2+p^2+r^2+p^2}{2^{10}}\Big)$$
$$=\frac{17r^2+17p^2}{2^9}+\frac{(r-p)^2}{2^{12}}-\frac{(r^2+p^2)}{2^9}$$
$$=\frac{136(r^2+p^2)+(r-p)^2-8(r^2+p^2)}{2^{12}}$$
$$=\frac{128(r^2+p^2)-2rp}{2^{12}}$$
$$=\frac{128(r^2+p^2-\frac{2}{128}rp)}{2^{12}}$$
$$\ge\frac{128(r^2+p^2-2rp)}{2^{12}}$$
$$=\frac{128(r-p)^2}{2^{12}}$$
$$=\frac{56\big((r-p)^2+(r-p)^2\big)}{2^{12}}.$$

It is clear that $r,p \in [0,1]$, *we have* $\frac{56}{2^{12}} \geq \frac{4^{(r-p)^2}}{2^{12}}$, *so*

$$\begin{aligned}
\frac{56\big((r-p)^2+(r-p)^2\big)}{2^{12}} &\geq \frac{4^{(r-p)^2}}{2^{12}}\left[(r-p)^2+(r-p)^2\right] \\
&\geq 4^{(r-p)^2}\left[\frac{(r-p)^2}{2^{12}}+\frac{(r-p)^2}{2^{12}}\right] \\
&\geq 4^{(r-p)^2}\left[\frac{\left(\frac{r}{2^3}-\frac{p}{2^3}\right)^2}{2^6}+\frac{\left(\frac{r}{2}-\frac{p}{2}\right)^2}{2^{10}}\right] \\
&\geq 4^{(r-p)^2}\,\frac{\left[9\left(\frac{1}{9}\left(\frac{r}{2^3}-\frac{p}{2^3}\right)^2+\frac{1}{9}\left(\frac{r}{2}-\frac{p}{2}\right)^2\right)\right]}{2} \\
&\geq 4^{(r-p)^2}\left[\frac{9}{2}\left(d_b\left(\frac{r}{2^3},\frac{p}{2^3}\right)+d_b\left(\frac{r}{2},\frac{p}{2}\right)\right)\right] \\
&\geq 4^{(r-p)^2}\left[\psi\left(d_b\big(Ar,Ap\big)+d_b\big(Cr,Cp\big)\right)\right] \\
&\geq b(r,p)\left[\psi\left(d_b\big(Ar,Ap\big)+d_b\big(Cr,Cp\big)\right)\right].
\end{aligned}$$

Thus the condition:

$$\begin{aligned}
&b(r,p)\psi[d_b(Ar,Ap)+d_b(Cr,Cp)] \\
&\leq \psi\Bigg[\lambda\left(d_b(Ar,Br)+d_b(Cr,Dr)+d_b(Ap,Bp)+d_b(Cp,Dp)\right) \\
&\quad +\gamma\,\frac{d_b(Br,Bp)+d_b(Dr,Dp)}{b(Br,Bp)+b(Dr,Dp)}\Bigg] \\
&\quad -\beta\varphi\big(d_b(Ar,Bp)d_b(Ap,Br)+d_b(Cr,Dp)d_b(Cp,Dr)\big) \text{ holds.}
\end{aligned}$$

Hence, based on theorem 2.1, this implies that A,B,C *and* D *have unique common fixed point.*

Corollary 2.1. *Let* (S,d_b) *be a complete extended rectangular b-metric space. Let* $A,B,C,D : S \to S$ *be continuous mappings such that* $A(S) \subseteq B(S)$ *and* $C(S) \subseteq D(S)$, $B(S)$ *and* $D(S)$ *are closed and satisfy the following conditions*

$$\begin{aligned}
&b(r,p)[d_b(Ar,Ap)+d_b(Cr,Cp)] \\
&\leq \lambda d_b(Ar,Br)+d_b(Cr,Dr)+d_b(Ap,Bp)+d_b(Cp,Dp) \\
&+\gamma\,\frac{d_b(Br,Bp)+d_b(Dr,Dp)}{b(Br,Bp)+b(Dr,Dp)},
\end{aligned} \tag{2.10}$$

where $0<\lambda,\gamma<1$, $\frac{\lambda+\frac{\gamma}{2}}{1-\lambda}<1$. *If A, B, C and D are compatible and* $\lim_{n\longrightarrow\infty}[b(A^2r_n,BAr_n)+b(A^2r_n,DCr_n)] < \frac{1}{\lambda}$, *then A, B, C, D have a unique common fixed point in S.*

Proof. By taking $\psi(t)=t$, $\beta=0$ in Theorem 2.1 . Then we conclude that A, B, C, D have a unique common fixed point. □

Theorem 2.2. *Let* (S,d_b) *be a complete extended rectangular b-metric space and the functions* $g,h:[0,\infty)\longrightarrow[0,\infty)$ *be Riemann integrable on* $[0,\infty)$ *with* $\int_0^{\varepsilon} g(p)\,dp>0$ *for every* $\varepsilon>0$. *If* $A,C:S\longrightarrow S$ *be a self-mapping satisfying the following integral inequality condition*

$$\begin{aligned}
&\int_0^{d_b(Ar,Ay)+d_b(Cr,Cy)} g(p)\,dp \\
&\leq \frac{1}{b(r,y)}\Big(\int_0^{\lambda\left(d_b(Ar,r)+d_b(Cr,r)+d_b(Ay,y)+d_b(Cy,y)\right)+\gamma\,\frac{d_b(r,y)}{b(r,y)}} g(p)\,dp-\beta\int_0^{2d_b(r,y)} h(p)\,dp,
\end{aligned}$$

where $\beta\geq 0, 0<\lambda(\neq 1),\gamma<1$, $\frac{\lambda+\frac{\gamma}{2}}{1-\lambda}<1$, *then A and C has a unique common fixed point.*

Proof. Taking $B(r)=C(r)=r$, $\psi(s)=\int_0^s g(p)\,dp$ *and* $\varphi(s)=\int_0^s h(p)\,dp$, since $g(p)$ *and* $h(p)$ is Riemann integrable on $[0,\infty)$, we have $\psi(s)$ *and* $\varphi(s)$ is continuous and nondecreasing on $[0,\infty)$. Then we immediately conclude that A and C has a unique common fixed point. □

3. Conclusion

In general, extended rectangular b-metric space is not a Hausdorff space. So, we need a Hausdorff property in this results, therefore there exists a unique limit point of sequence as the uniqueness of fixed point.

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