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Research paper

Analyzing the Stability of Lanchester Warfare Models for Symmetric Warfare Scenarios

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Abstract

This study analyzes the stability of Lancaster-type ODE models in symmetric warfare situations. In symmetric warfare situations, when the lethality coefficients (K) are equal for both forces in battle, the system exhibits marginal stability, characterized by poles at $\pm K$, indicating that the model is stable in some regions and unstable in others. The present research illustrates a controlled rhythmicity, and these forces show a harmonized balance between the two battling forces in order to develop a strong strategy and decision for military.
Keywords: *Lanchester models, symmetric warfare, stability analysis, mathematical modeling, conflict dynamics*

1. Introduction

Lanchester (1916) developed the simple kinetics leading to analysis the combat dynamics through Ordinary Differential Equations (ODEs). Lanchester's mathematical approach of using models for providing insights into warfare was a revolutionary concept realised in military science. His studies, based on the simulations of air combat from the World War I, were supposed to reveal a relationship between focus and attrition, which would make it possible to discuss different examples of conflict. The articles by Lanchester were especially informative during the era of the World War I when use of technology and technique was starting to emerge. By stressing the qualitative approach to resolve the issues of aerial warfare, the quantitative vision of opportunities, strengths comparative, and death rates became crucial. His models showed the capacity of mathematics to solve odds of war – an understanding that was very useful not only in his day but also in the modern day military strategies.

Subsequent development in Lanchester's models has further sought to meet these developments in warfare form. In another study, Ji (2022) concreted practical conflict planning of unmanned aerial vehicle swarm confrontations by employing Lanchester's law and game theory, combining new defensive measures. Some of the major problems highlighted in this work include complex interactions between the combatants while fighting and about the use of Unmanned Aerial Vehicles in the current battle fields. Likewise, Kress (2018) expanded Lanchester's conflict with more power structure, particularly, three-centred power conflict models in order to analyze the three-sided conflicts. This discovery is in fit with other recent studies, which demonstrates the continued applicability of Lanchester's seminal work. To address these issues, Chen et al. (2012) have applied agent-based modeling (ABM) to combats simulations, better portraying the action of combatants and their decision-making processes. This shift of paradigm helps in bringing Lanchester's equations closer to more modern day warfare application, which again brings up the question for further evolution of these models. Additional developments in the Lanchester-type models show the valuable trends of today's and past warfare strategies. Bracken (1995) examined the Ardennes campaign, applying Lanchester models to dissect and understand the strategic decisions and outcomes of this pivotal World War II battle. In the same vein, Cangiotti (2023) proposed an extension of Lanchester's work incorporating a variety of unaimed fire models which are fundamental to studying the characteristics of multi-battle warfare. This makes it possible to analyze combat situations where several fights take place at the same time. In general, the study by Kalloniatis (2020) centered on the optimization of structure in networked Lanchester models with regards to fire and manoeuvre in war, underlining the necessity of idealizing network theories with current combat theories to increase the Organization's efficiency. In addition to this, Spradlin (2007) employed the enhanced Lanchester equations to study various dimensions of combat to find out that the present complex warfare demands more strategies and depth to be incorporated in a model. Together these works demonstrate the dynamics and progressive applicability of Lanchester-type models in military research and operation planning.

In symmetric warfare, Lanchester's Linear Law serves as a foundational mathematical framework. The core equations are expressed as follows:

$$\begin{aligned}\frac{dx}{dt} &= -ky \\ \frac{dy}{dt} &= -kx\end{aligned} \tag{1}$$

where x and y denote the troop counts of Force X and Force Y, respectively, and k is lethality coefficient. These equations emphasize the critical roles of troop size and combat effectiveness in shaping the outcomes of battles. By building on Lanchester's legacy and adapting his ideas to contemporary challenges, this study seeks to elucidate the stability of these models. Conducting a stability analysis is crucial to ensure that predictions about the system are realistic and reliable. In symmetric conflicts, it is important to understand whether a system will return to equilibrium after a disturbance or if it will devolve into increasing disorder.A stable system implies that the system is manageable and controllable, in the sense that strategic measures taken will be able to move the system to a equilibrium state. On the other hand, an unstable system means that the situation is out of control and despite the measures taken or strategies it formulates. Furthermore, marginal stability can also be deduced from the above analysis which means that the system is stable in one set of conditions and unstable in other conditions, which become apparent that control measures could be only effective under some conditions.

2. Methodology

2.1. Linearization of Lanchester's Linear Law for Symmetric Warfare Scenarios

The nonlinear Lanchester's Linear Law model is represented by the following system of ordinary differential equations:

$$\begin{aligned}\frac{dx}{dt} &= -ky \\ \frac{dy}{dt} &= -kx\end{aligned} \tag{2}$$

where, x and y represent the combatant numbers for forces A and B, respectively. The parameter k denotes the lethality coefficient for both forces A and B.

2.1.1. Taylor Series Expansion

To carryout linearization, we define $f_1(x,y)$ and $f_2(x,y)$ as the functions showcase the rate of change of combatant numbers for forces A and B, respectively, in the model (2):

$$\begin{aligned}f_1(x,y) &= -ky \\ f_2(x,y) &= -kx\end{aligned} \tag{3}$$

Before going to linearize the system further, a Taylor series expansion of the Nonlinear functions $f_1(x,y)$ and $f_2(x,y)$ are to be carried out based on the operating point. The Taylor series expansion is performed around the operating point (x_0,y_0) to linearize the functions. The Taylor series expansion is as follows:

$$\begin{aligned}f_1(x,y) &\approx f_1(x_0,y_0) + \left.\frac{\partial f_1}{\partial x}\right|_{(x_0,y_0)}(x-x_0) + \left.\frac{\partial f_1}{\partial y}\right|_{(x_0,y_0)}(y-y_0) \\ &\approx (-ky_0)|_{(x_0,y_0)} + 0|_{(x_0,y_0)}(x-x_0) + (-k)|_{(x_0,y_0)}(y-y_0) \\ &\approx -ky_0 - k(y-y_0) \\ f_2(x,y) &\approx f_2(x_0,y_0) + \left.\frac{\partial f_2}{\partial x}\right|_{(x_0,y_0)}(x-x_0) + \left.\frac{\partial f_2}{\partial y}\right|_{(x_0,y_0)}(y-y_0) \\ &\approx (-kx_0)|_{(x_0,y_0)} + (-k)|_{(x_0,y_0)}(x-x_0) + 0|_{(x_0,y_0)}(y-y_0) \\ &\approx -kx_0 - k(x-x_0)\end{aligned} \tag{4}$$

After obtaining the Taylor series expansions, linearize the functions $f_1(x,y)$ and $f_2(x,y)$ around the operating point (x_0,y_0). These linearized functions describe the attrition rates for forces A and B and are given by:

$$\begin{aligned}f_{1_{lin}}(x,y) &= -ky_0 - k(y-y_0) \\ f_{2_{lin}}(x,y) &= -kx_0 - k(x-x_0)\end{aligned} \tag{5}$$

Equation 5 represents the linearized form in symmetric warfare.

2.2. Replace the Nonlinear Functions

In the obtained model replace the original nonlinear functions with their linearized versions to get the linearized equations of Lanchester warfare model. The linearized equations for the dynamics of forces A and B are as follows:

$$\begin{aligned}\frac{dx}{dt} &\approx f_{2_{lin}}(x,y) \\ \frac{dy}{dt} &\approx f_{1_{lin}}(x,y)\end{aligned} \tag{6}$$

2.3. Laplace Transform For Pole Analysis

Applying the Laplace transform to the linearized equations with initial conditions $x(0) = x_0$ and $y(0) = y_0$, we obtain:

$$sX(s) - x(0) = -k \cdot y_0 - k \cdot (Y(s) - y_0)$$
$$sY(s) - y(0) = -k \cdot x_0 - k \cdot (X(s) - x_0)$$

Solving these equations, we derive the Laplace transforms $X(s)$ and $Y(s)$:

$$X(s) = \frac{s \cdot x_0 + k \cdot y_0}{k^2 - s^2}$$
$$Y(s) = \frac{s \cdot y_0 + k \cdot x_0}{k^2 - s^2}$$

2.4. Transfer Function

The transfer functions $X(s)$ and $Y(s)$ describe the frequency domain representation of the system's response:

$X(s)$ relates the Laplace transform of $x(t)$ to $y(t)$.
$Y(s)$ relates the Laplace transform of $y(t)$ to $x(t)$.

2.5. Poles Analysis

The poles of the transfer functions, determined from the characteristic equation $k^2 - s^2 = 0$, are:

$$s = \pm k$$

These poles indicate marginal stability and dynamics of the symmetric warfare scenario:

- For $k > 0$, the system exhibits controlled oscillations around equilibrium points, reflecting the balanced attrition between forces A and B.
- The pole at $s = k$ (and $s = -k$) signifies marginal stability, suggesting sustained but controlled dynamics without exponential growth or decay.

These poles indicate the system's marginal stability, characterized by controlled oscillations and balanced dynamics between the opposing forces.

2.6. Numerical Representation

2.6.1. Pole-Zero Map

The pole-zero map for the Lanchester model with a lethality coefficient $k = 0.2$ is shown in Figure 1.

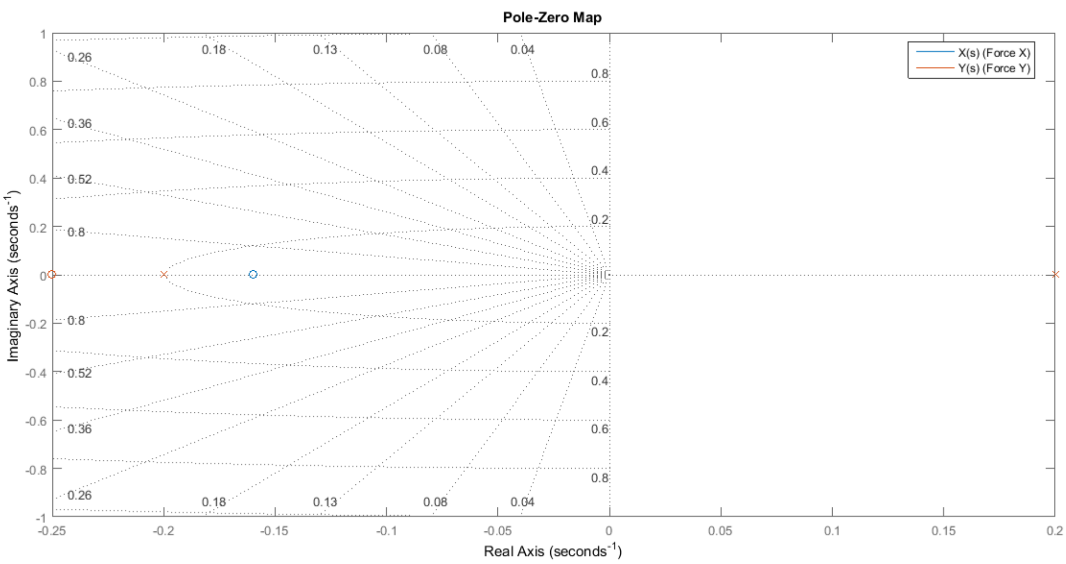

Figure 1: Pole-Zero Map for $k = 0.2$

2.7. Stability Analysis

The position of the poles on the real axis at $s = \pm 0.2$ indicates marginal stability. This suggests that any perturbations in the system will result in sustained oscillations rather than exponential growth or decay. Consequently, the system exhibits controlled dynamics where the forces experience balanced attrition over time. Understanding this behavior is crucial for strategic planning and conflict management, as it implies that the forces will neither naturally stabilize nor escalate without external influence.

3. Results and Discussion

3.1. Result

The stability analysis of the Lanchester-type model provides significant insights into the behavior of symmetric warfare scenarios. Through linearization and the application of the Laplace transform, we derived the characteristic equations that describe the system dynamics. Specifically, the poles of the transfer functions $X(s)$ and $Y(s)$ were found to be at $s = \pm k$, where k is the lethality coefficient. For $k > 0$, the system is marginally stable, indicating that it will experience sustained oscillations around the equilibrium point without exponential growth or decay. This is illustrated in the pole-zero map for $k = 0.2$, where the poles are located at $s = \pm 0.2$. The numerical example reinforces the theoretical analysis, showing that the forces will exhibit controlled dynamics and balanced attrition over time. For $k < 0$, the system would be unstable, as the poles would lie on the real axis outside the unit circle, leading to exponential divergence from equilibrium. Conversely, for $k = 0$, the system is neutrally stable, and perturbations would not grow but also would not decay, leading to constant oscillations.

3.2. Discussion

The stability analysis of the Lanchester-type model provides significant insights into the behavior of symmetric warfare scenarios. Specifically, the analysis shows that for $k > 0$, the system is marginally stable, leading to sustained oscillations around the equilibrium point. Such behavior is in agreement with the results of Kress et al. (2018), who observed that stability could stabilize the dynamics of combat models in a way that does not cause exponential increasing or decreasing with time. Similarly, Lin et al. (2014) discussed the instability in heterogeneous Lanchester models, noting that negative lethality coefficients may cause uncontrolled dynamics, which supports the necessity to avoid negative lethality for stability. The numerical example with $k = 0.2$ illustrates the theoretical findings, showing balanced attrition, aligning with the conflict resolution models proposed by Deitchman (1962), who highlighted stability's role in achieving controlled outcomes in guerrilla warfare. Insights from the stability properties of the Lanchester-type model can be useful to military strategists. Thus, the specific behaviors of forces can be anticipated depending on various states of lethality, allowing commanders to devise the best plans that will enhance stability without the Carrot and Stick Element. This approach supports the development of more sophisticated and realistic conflict simulations, as highlighted by Ji et al. (2022) in their work on swarm confrontation methods based on Lanchester's laws. Ji et al. emphasized that stability in combat models allows for better planning and execution of military strategies, consistent with the strategic implications of our stability analysis. Therefore, recognizing the pattern of stability as a key factor that can influence the outcomes of conflicts and the approaches to creating an effective military strategy represents one of the key advantages of referring to the literature in the field.

4. Conclusion

In conclusion, the evaluation of Lanchester-type models in contexts of symmetric warfare draws significant insights into the interactions of warfare. Positive lethality coefficients are vital for maintaining stability in both symmetric and heterogeneous models, ensuring that combat dynamics remain controlled and predictable.It is in this understanding that the factor of stability and military strategy becomes relevant with the view of enabling the commander to anticipate and manage conflict in the most remunerative manner. Therefore, preserving stability as a concept decreases the chances of having possible escalating situations and makes possible strategic practical thinking and planning in different auspices and conflicts.

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