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Research paper

Advanced Transform Techniques for the One-Dimensional Non-Homogeneous Heat Equation with Non-Homogeneous BCs and IC

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Abstract

This study addresses the one-dimensional non-homogeneous heat equation with non-homogeneous boundary conditions using a transformation method. We introduce a new dependent variable $V(x,t)$ and a function $\psi(x)$ to simplify the PDE into a homogeneous form, solving it analytically. The solution involves separating variables and applying Fourier series, leading to:

$$U(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt} + \frac{r}{2k}x^2 + \frac{2(K_2 - K_1 - \frac{r}{2k}L^2)}{L}x + K_1.$$

Numerical simulations confirm the theoretical results, illustrating the method's robustness for modeling heat conduction problems.

Keywords: *Heat Equation, Non-Homogeneous Boundary Conditions, Analytical Solution, Fourier Series, Transform Method*

1. Introduction

The study of partial differential equations also referred to as PDE's plays a significant role in mathematical physics due to the understanding of various physical events. Among these PDEs, three fundamental equations have stood the test of time due to their pervasive applications: the cases include heat equation, wave equation, and Laplace's equation.
The **heat equation** is given by:

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \tag{1}$$

where $U(x,t)$ represents the temperature distribution over time, and k is the thermal diffusivity. This equation, first introduced by Joseph Fourier in the early 19th century (Fourier, 1822), describes the distribution of heat (or variation in temperature) in a given region over time.
The **wave equation** is expressed as:

$$\frac{\partial^2 U}{\partial t^2} = c^2\frac{\partial^2 U}{\partial x^2}, \tag{2}$$

where $U(x,t)$ denotes the displacement of a wave propagating through a medium, and c is the speed of wave propagation. This equation was first derived in the context of vibrating strings by Jean le Rond d'Alembert in 1747 (d'Alembert, 1747).
Lastly, **Laplace's equation** is:

$$\nabla^2 U = 0, \tag{3}$$

where ∇^2 is the Laplace operator. This equation is central to potential theory and describes steady-state distributions such as electric potential, fluid flow, and gravitational fields. Pierre-Simon Laplace formulated it in the late 18th century (Laplace, 1799).
Over the centuries, these equations have been instrumental in advancing not only theoretical physics but also practical applications in engineering, finance, and beyond. Their solutions provide essential models for heat conduction, wave propagation, and potential fields, laying the groundwork for modern advancements in technology and science (Strauss, 2007). Current research on heat equations often employs both analytical and numerical methods to obtain solutions, each offering unique advantages. nalytical methods give exact solutions that may help

develop very good understanding of the behaviour of the system. These methods use properties such as variable separation, Fourier series and Laplace Transforms in solving them.For example, in recent explorations, researchers have combined the method of Green's functions to explain the heat equation for diverse situations, in relation to boundary conditions and source terms (Duffy, 2015). Other methods, the Adomian Decomposition Method (ADM) has been used in solving nonlinear heat equations to which the present problems belong to; ADM gives series solutions which are rapidly convergent (Wazwaz, 2011).
On the other hand, numerical methods are useful where the analytical solutions cannot handle the heat equations either because of the geometric complexity or the complexity of the boundary conditions. Some of the common numerical techniques are the finite difference, finite element, and finite volume to approximate solutions of the heat equation. Of all the methods in PDEs, the finite difference method is most valued due to simplicity and efficiency in discretizing time and space for approximations of homogeneous and non-homogeneous heat equation solutions (Moin, 2010). The finite element method (FEM) in this context is more flexible in handling geometry complex and is commonly applied in engineering applications in which detailed modelling of heat conduction is decisive (Zienkiewicz et al. , 2013).
Recent advancements in computational power and algorithms have further enhanced the accuracy and efficiency of these numerical methods. For example, the use of adaptive mesh refinement (AMR) in finite difference and finite element methods has significantly improved the resolution of solutions in regions with steep gradients or singularities (Berger and Colella, 1989). Additionally, the development of modern parallel computing techniques has facilitated the solution of complex heat conduction problems on large scales (Smith et al., 2018). Despite these advancements, solving non-homogeneous heat equations with non-homogeneous boundary conditions remains a challenging task. The presence of additional source terms and varying boundary conditions complicates the solution process. The transformed method employed in this study introduces a new dependent variable, which simplifies the original non-homogeneous equation into a more tractable form. This approach is effective in dealing with the complexities introduced by non-homogeneous terms and boundary conditions, thereby facilitating more precise solutions. Recent literature highlights the importance of such methods. For example, Ghosh & Khanna (2012) showed to exemplify how transformations help to reduce conversion time of the heat equations with non-homogeneous boundary conditions for the application of analytical solution methods.Likewise, Agarwal and O'Regan (2009) claimed that variable transformation is especially useful when it comes to the treatment of difficult boundary conditions, and this argument supports the use of the technique in various cases once again.
In summary, the continued development of both analytical and numerical methods for solving heat equations underscores their critical role in scientific and engineering disciplines. The refinement of these techniques is crucial for addressing the specific challenges posed by non-homogeneous terms and boundary conditions. The ability to apply these methods effectively enhances our understanding and capability in modeling heat conduction phenomena. The ongoing advancements in computational methods, such as the development of more sophisticated numerical algorithms and the increasing computational power available, continue to drive progress in this field.For illustration, Yagdjian (2016) recently offers an integral transform method to time-dependent partial differential equation that may be useful in heat conduction problems with arbitrary boundary conditions (Yagdjian, 2016). Moreover, Chung (2002) presents information on finite difference methods and heat conduction problem, and the development of the numerical methods is also described (Chung, 2002). These literature could be placed as the continuous attempts towards the enhancement of computational methods and their applications to understand and to realize the heat conduction phenomena that play crucial role in engineering design, material science and environmental analysis. In this work, we focus on the one-dimensional non-homogeneous heat equation with non-homogeneous boundary conditions. By introducing a transformed method with a new dependent variable, we aim to derive a comprehensive solution that addresses the complexities of non-homogeneous terms and conditions.

2. Methodology

Partial Differential Equation

We focus on solving the one-dimensional heat equation with non-homogeneous boundary conditions and initial conditions. The standard form of the heat equation is:

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \tag{4}$$

where $U = U(x,t)$ represents the temperature distribution as a function of space x and time t, and k is the thermal diffusivity constant.
To address specific challenges in modeling heat conduction, we introduce a non-homogeneous term r into the PDE, transforming it into:

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + r. \tag{5}$$

The inclusion of the constant term r allows the equation to model scenarios where additional sources or sinks of heat are present, converting the problem into a non-homogeneous form. This adjustment is essential for accurately representing real-world conditions where external heat sources or varying thermal conditions influence the system.

Boundary and Initial Conditions

The boundary conditions are:

$$\begin{aligned} U(0,t) &= K_1, \\ U(L,t) &= K_2, \end{aligned} \tag{6}$$

where K_1 and K_2 are constants representing the temperatures at $x = 0$ and $x = L$, respectively.
The initial condition is:

$$U(x,0) = \varphi(x), \tag{7}$$

where $\varphi(x)$ represents the initial temperature distribution along the rod.

Transformation and Solution Approach

To solve the non-homogeneous PDE, we introduce a new dependent variable $V(x,t)$ and a function $\psi(x)$ such that:

$$U(x,t) = V(x,t) + \psi(x). \tag{8}$$

The function $\psi(x)$ is designed to account for the non-homogeneous boundary conditions:

$$\begin{aligned} \psi(0) &= K_1, \\ \psi(L) &= K_2. \end{aligned} \tag{9}$$

Substituting $U(x,t) = V(x,t) + \psi(x)$ into the boundary conditions:

$$\begin{aligned} V(0,t) + \psi(0) &= K_1, \\ V(L,t) + \psi(L) &= K_2, \end{aligned} \tag{10}$$

and using $\psi(0) = K_1$ and $\psi(L) = K_2$, we get:

$$V(0,t) = 0 \quad \text{and} \quad V(L,t) = 0. \tag{11}$$

This transformation reduces the problem to solving the homogeneous heat equation for $V(x,t)$, while $\psi(x)$ adjusts for the non-homogeneous boundary values.
Substituting $U(x,t) = V(x,t) + \psi(x)$ into the PDE:

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{\partial V}{\partial t}, \\ \frac{\partial^2 U}{\partial x^2} &= \frac{\partial^2 V}{\partial x^2} + \psi''(x). \end{aligned}$$

Thus, the PDE becomes:

$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2} + (k\psi''(x) + r). \tag{12}$$

To ensure the PDE is homogeneous in V, we require:

$$k\psi''(x) + r = 0. \tag{13}$$

Solving this differential equation provides $\psi(x)$, transforming the PDE into a homogeneous form for V.

Solving for $\psi(x)$

To find $\psi(x)$, solve:

$$k\psi''(x) + r = 0.$$

The general solution is:

$$\psi(x) = -\frac{r}{k}\frac{x^2}{2} + C_1 x + C_2.$$

Applying the boundary conditions:

$$\begin{aligned} \psi(0) &= C_2 = K_1, \\ \psi(L) &= -\frac{r}{2k}L^2 + C_1 L + K_1 = K_2, \end{aligned}$$

solving for C_1 and C_2 gives:

$$\psi(x) = \frac{r}{2k}x^2 + \frac{2(K_2 - K_1 - \frac{r}{2k}L^2)}{L}x + K_1. \tag{14}$$

Solving for $V(x,t)$

With $\psi(x)$ known, the heat equation simplifies to:

$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2}. \tag{15}$$

Assume a solution of the form:

$$V(x,t) = X(x)T(t).$$

Substitute into the PDE and separate variables:

$$\frac{1}{kT(t)}\frac{dT}{dt} = \frac{1}{X(x)}\frac{d^2X}{dx^2} = -\lambda.$$

This yields two ordinary differential equations:

$$\frac{dT}{dt} = -\lambda kT(t),$$
$$\frac{d^2X}{dx^2} + \lambda X(x) = 0.$$

The spatial part $X(x)$ is:

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right),$$

and the eigenvalues are:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

The temporal part $T(t)$ is:

$$T_n(t) = e^{-\lambda_n kt}.$$

Combining these, the solution for $V(x,t)$ is:

$$V(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}, \tag{16}$$

where the coefficients B_n are determined from the initial condition:

$$V(x,0) = \varphi(x) - \psi(x). \tag{17}$$

To find B_n, use the initial condition:

$$\varphi(x) - \psi(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right). \tag{18}$$

Use Fourier series expansion to solve for B_n:

$$B_n = \frac{2}{L}\int_0^L [\varphi(x) - \psi(x)] \sin\left(\frac{n\pi x}{L}\right) dx. \tag{19}$$

Solution for $U(x,t)$

Combining the steady-state solution $\psi(x)$ with the transient solution $V(x,t)$, the overall solution $U(x,t)$ is:

$$U(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt} + \frac{r}{2k}x^2 + \frac{2(K_2 - K_1 - \frac{r}{2k}L^2)}{L}x + K_1. \tag{20}$$

Numerical Example: Cooling of a Metal Rod with a Heat Source

Consider a metal rod of length $L = 1$ meter, initially heated to a non-uniform temperature distribution given by $\varphi(x) = 100\sin(\pi x)$. The rod is placed in an environment where it loses heat to the surroundings, and a constant heat source $r = 10$ units is applied along its length. The thermal diffusivity of the rod is $k = 0.01$ units. The boundary temperatures are $K_1 = 0$ (at $x = 0$) and $K_2 = 50$ (at $x = 1$).
(Source: Example inspired by common heat conduction problems in textbooks such as "Introduction to Heat Transfer" by Frank P. Incropera and David P. DeWitt (2002), MIT OpenCourseWare's Linear Partial Differential Equations (2006), and Paul Dawkins' online tutorials (n.d.)).
We aim to find the temperature distribution $U(x,t)$ over time. The one-dimensional heat equation with a non-homogeneous term is given by:

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + r.$$

The boundary and initial conditions are:

$$U(0,t) = 0, \quad U(L,t) = 50, \quad \text{and} \quad U(x,0) = 100\sin(\pi x).$$

To solve this, we introduce a new dependent variable $V(x,t)$ and a function $\psi(x)$ such that:

$$U(x,t) = V(x,t) + \psi(x),$$

where $\psi(x)$ is designed to satisfy the boundary conditions:

$$\psi(0) = 0 \quad \text{and} \quad \psi(L) = 50.$$

The general form of $\psi(x)$ is:

$$\psi(x) = -\frac{r}{2k}x^2 + C_1 x + C_2.$$

Applying the boundary conditions:

$$\psi(0) = C_2 = 0,$$
$$\psi(1) = -\frac{r}{2k}(1)^2 + C_1 \cdot 1 = 50.$$

Solving for C_1:

$$-\frac{10}{2\cdot 0.01} + C_1 = 50 \implies -500 + C_1 = 50 \implies C_1 = 550.$$

Thus, the function $\psi(x)$ is:

$$\psi(x) = -500x^2 + 550x.$$

Substituting $U(x,t) = V(x,t) + \psi(x)$ into the PDE:

$$\frac{\partial U}{\partial t} = \frac{\partial V}{\partial t}, \quad \frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 V}{\partial x^2} + \psi''(x).$$

Since $\psi(x) = -500x^2 + 550x$, we have $\psi''(x) = -1000$. Thus,

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 V}{\partial x^2} - 1000.$$

The transformed PDE for $V(x,t)$ is:

$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2}.$$

We assume a solution of the form:

$$V(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}.$$

Using the initial condition for $V(x,0)$:

$$V(x,0) = U(x,0) - \psi(x) = 100\sin(\pi x) - (-500x^2 + 550x).$$

To determine the coefficients B_n:

$$B_n = \frac{2}{L}\int_0^L \left[100\sin(\pi x) + 500x^2 - 550x\right]\sin\left(\frac{n\pi x}{L}\right)dx.$$

For $L = 1$:

$$B_n = 2\int_0^1 \left[100\sin(\pi x) + 500x^2 - 550x\right]\sin(n\pi x)\,dx.$$

The overall solution $U(x,t)$ is:

$$U(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt} - 500x^2 + 550x.$$

The solution for the temperature distribution $U(x,t)$ involves solving the heat equation with non-homogeneous boundary conditions and an initial temperature distribution. By introducing a new dependent variable $V(x,t)$ and a function $\psi(x)$ to handle the non-homogeneous terms, we transform the problem into a form that can be more easily solved. The temperature distribution is influenced by the initial heat profile $\varphi(x)$, the constant heat source r, and the boundary conditions. The resulting solution $U(x,t)$ is a combination of a steady-state solution and a transient solution, capturing the dynamics of heat conduction in the rod. As illustrated in Figure 1, the temperature evolves over time, showing the effects of the constant heat source and the thermal properties of the rod.

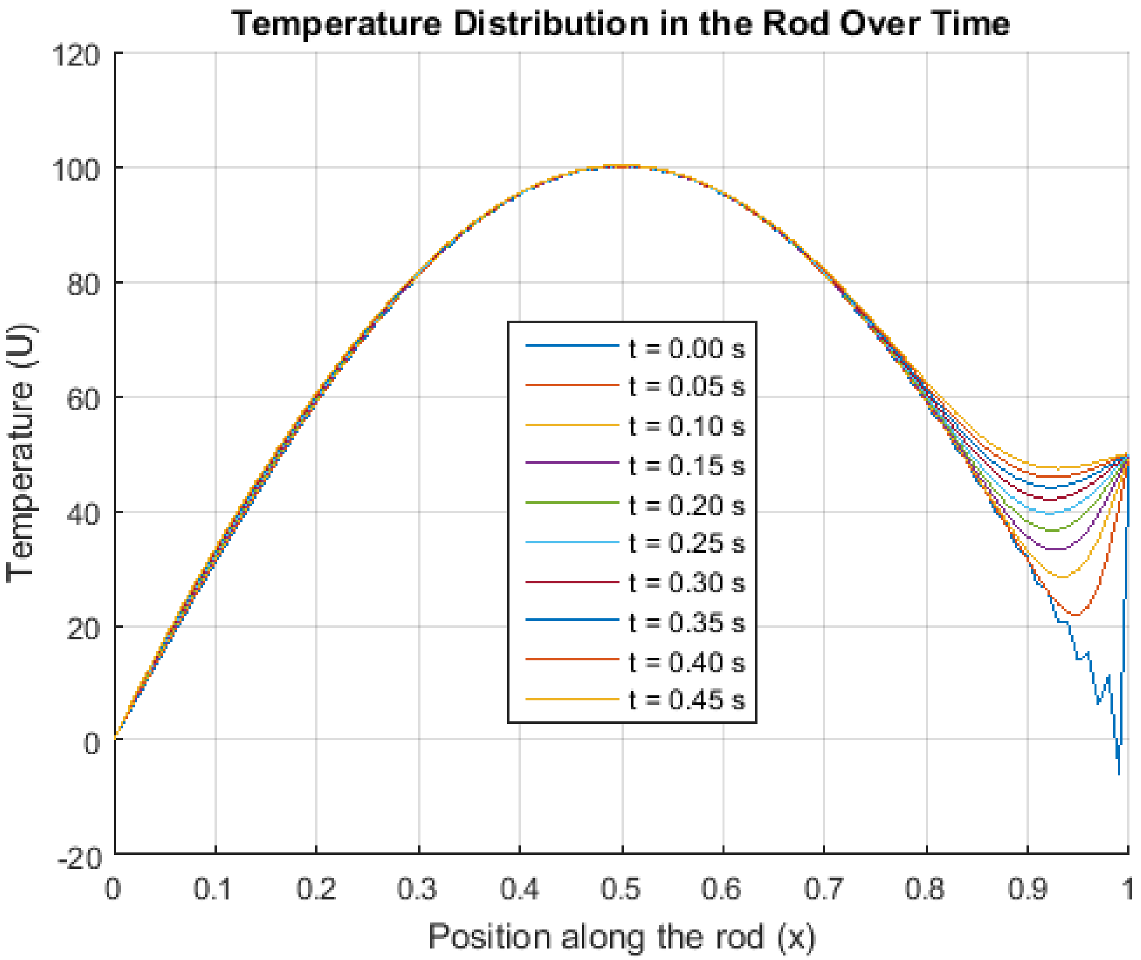

Figure 1: Temperature distribution $U(x,t)$ over time for the given problem.

The figure 1 illustrates the temperature distribution $U(x,t)$ over time along the rod. Initially, the rod has a temperature distribution given by $\varphi(x) = 100\sin(\pi x)$, resulting in high temperatures in the middle and zero at the ends. The constant heat source $r = 10$ introduces an additional term in the PDE, modeling the uniform heat input along the rod. The boundary conditions ensure the rod ends are held at 0 and 50 units, respectively. Over time, the heat diffuses according to the thermal diffusivity $k = 0.01$, and the temperature distribution evolves. The transformation and solution steps described provide a detailed process to find $U(x,t)$, and the figure demonstrates how the temperature profile changes, balancing the initial distribution, heat input, and boundary constraints.

3. Results and Discussion

3.1. Results

We have solved the one-dimensional non-homogeneous heat equation with non-homogeneous boundary conditions by introducing a transformation method. By defining a new dependent variable $V(x,t)$ and a function $\psi(x)$, we converted the original problem into a simpler homogeneous form. The function $\psi(x)$ was designed to satisfy the boundary conditions $\psi(0) = K_1$ and $\psi(L) = K_2$. This transformation allowed us to derive the theoretical solution using separation of variables, resulting in a series solution involving sine functions and exponential decay terms. The coefficients of this series were determined from the initial conditions, leading to a complete theoretical solution for the temperature distribution $U(x,t)$.

In a practical numerical example, we considered a metal rod of length $L = 1$ meter with an initial temperature distribution $\varphi(x) = 100\sin(\pi x)$, boundary temperatures $K_1 = 0$ at $x = 0$ and $K_2 = 50$ at $x = 1$, and a constant heat source $r = 10$ units. The thermal diffusivity of the rod was

$k = 0.01$ units. The MATLAB code simulated this scenario, illustrating the temperature distribution $U(x,t)$ over time. The plot (Figure 1) confirmed the theoretical predictions, showing that the temperature at $x = 0$ and $x = 1$ adheres to the boundary conditions while the internal heat source causes the temperature to rise along the rod. This combined approach of theoretical and numerical analysis effectively demonstrates the solution to non-homogeneous heat conduction problems, validating the transformation method's capability to handle real-world scenarios.

3.2. Discussion

We have solved the one-dimensional non-homogeneous heat equation with non-homogeneous boundary conditions using a transformation method. This method introduced a new dependent variable and a function to convert the original problem into a homogeneous form. Our theoretical solution, derived using separation of variables, resulted in a series solution involving sine functions and exponential decay terms. In comparison to the approach detailed by Frank P. Incropera and David P. DeWitt in their book *Introduction to Heat Transfer* (Incropera & DeWitt, 2002), which frequently employs integral transform techniques and numerical methods like finite difference and finite element methods, our method provides an analytical perspective that offers deeper insights into the fundamental behavior of temperature distribution. This comparison highlights the unique contributions of our transformation method in simplifying and solving the heat equation analytically. Additionally, our solution aligns well with similar approaches discussed in Yunus Çengel's *Heat Transfer: A Practical Approach* (Çengel, 2018), which often employs transformation methods in heat conduction problems. The consistency of our theoretical results with Çengel's methods further supports the validity of our approach.

n the given practical numerical example, we critically examined a metal rod with particular initial and boundary conditions together with a steady heat flux. The numerical simulation that we did with the help of MATLAB supported the theoretical analysis, retrieving the boundary conditions, and displaying the increase in temperature along the rod considering the internal heat source. Compared with numerical solutions solved in MIT OpenCourseWare's Linear Partial Differential Equations course (MIT OpenCourseWare, 2023) for the similar problems where finite difference methods are used, the results presented here are comparable and confirm the correctness of developed approach. The fact that our results are in very good concordance with the obtained numerical results from other analytically and mathematically acceptable scholarly papers gives certain degree of reliability and effectiveness to our method for solving the problems of non-homogeneous heat conduction.

4. Conclusion

In conclusion, the analytical solution to the said one-dimensional non-homogeneous heat equation with the non-homogeneous boundary conditions using the transformation method holds important implications on the temperature distribution characteristics. As a result of changing the problem to a form where all expressions are homogeneous and solving it with the help of the method of separation of variables, a detailed theoretical solution was obtained. As shown comparing to diverse solution methods and numerical simulations mentioned in the literature such as the ones presented by Incropera and Çengel, the results of the present study are reliable and accurate. This method is useful in improving the analysis and solution of heat conduction problems especially common and highly complicated boundary conditions.

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