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Research paper

Optimal strategies for investment and consumption: stochastic analysis with pontryagin's principle under economic uncertainty

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Abstract

This study investigates how stochastic optimization is applied to the management of a company's portfolio in order to maximize the expected utility of wealth over a given period. Inspired by Merton's research, this model involves random volatility in the financial markets, while maintaining a constant interest rate to take better account of real economic uncertainties. The aim is to formulate optimal investment and consumption strategies based on Pontryagin's maximum principle. Taking into account key factors such as economic growth and market volatility, as well as risk aversion in our financial considerations, we recommend an approach incorporating a quadratic penalty for excessive investment. This innovative method aims to adjust financial choices in line with economic fluctuations and ensure prudent management of the company's monetary resources. Finally, numerical simulations illustrate the influence of these factors on overall wealth, as well as on investment and consumption, underlining the importance of prudent portfolio management during periods of uncertainty.

Keywords: *Stochastic Optimization; Optimal Strategies; Stochastic Volatility; Pontryagin's Principle; Aversion; Investment; Consumption.*

1. Introduction

Portfolio management in an uncertain environment is a central issue in financial mathematics. The latter, introduced by Markowitz [1] is a central framework for stochastic optimization applied to investment and consumption decisions. Inspired by the pioneering work of Merton this [2] model explores optimal strategies in continuous time for maximizing the expected utility of wealth, based on the assumption that asset prices follow a geometric Brownian motion. However, unlike Merton [3] which assumes constant interest rates and volatilities, we incorporate stochastic financial market volatility while keeping interest rates constant. This allows us to take better account of real economic uncertainties, such as economic crises, political adjustments and exogenous shocks like the COVID-19 pandemic, which directly influence investment decisions.
The main objective is to maximize the investor's discounted expected utility over a given period, by applying Pontryagin's maximum principle in a stochastic framework. This framework offers a solution to the optimization problem through stochastic differential equations (SDEs) and adjoint equations, while taking into account unpredictable market fluctuations.
Investment and consumption decisions depend on several key economic parameters. For example, higher economic growth encourages bolder investment, while greater volatility encourages a more cautious approach. In addition, higher capital productivity encourages more significant investment. Risk aversion, as reflected in the coefficient θ for consumption, directly influences the investor's consumption decisions, while a quadratic penalty controlled by the parameter γ parameter limits excess investment. This distinction enables risk aversion to be modulated independently in consumption and investment regulation, thus ensuring optimal portfolio management.
Finally, numerical simulations are carried out to illustrate the impact of different economic parameters on wealth, investment and consumption. These simulations provide an in-depth exploration of optimal strategies in various economic scenarios, highlighting the importance of effective portfolio management in times of uncertainty.

2. Literature review

Stochastic optimization applied to investment and consumption decisions in an uncertain environment is a central topic in the mathematics of finance. It enables us to model the complex dynamics of financial markets and to propose appropriate strategies in the face of uncertainty. The foundations of this approach can be traced back to the work of Markowitz [1] who introduced modern portfolio management by maximizing expected returns for a given level of risk. Although revolutionary, this approach did not take into account temporal market fluctuations, which are essential for modeling continuous variations in financial asset prices.
Merton [3] then enriched this theory by developing a continuous-time optimization model for investment and consumption decisions, assuming constant interest rates and volatility. This assumption of constancy, while effective for initial modeling, limits the model's ability

to represent financial behavior in periods of increased volatility. To overcome this limitation, researchers have introduced stochastic elements to better capture market variations, particularly during crises or exogenous shocks.
The analytical framework of this work is inspired by the production and consumption model presented in Huyên Pham's book Optimisation et contrôle stochastique appliqués la finance, which explores in depth the applications of stochastic control to financial decisions. [4] Optimization and Stochastic Control Applied to Finance, which explores in depth the applications of stochastic control to financial decisions. In this model, investment and consumption decisions are optimized according to the stochastic evolution of markets, making it possible to study how uncertainty affects portfolio and production management. This reference provides an important foundation for the modeling adopted in our study.
Recent research has introduced stochastic volatility to better capture market uncertainties. For example, Benth [5] integrated Ornstein-Uhlenbeck-type volatility into a portfolio model, to represent variations in risk. In the same vein, Sandjo et al. [6] combined a constant expected return with stochastic volatility, while Zhang and Shreve [7] have shown that this approach improves predictions of optimal decisions in investment and consumption strategies under uncertainty. These studies underline the importance of stochastic volatility for allocation strategies adapted to uncertain markets.
Risk aversion is crucial for modeling investor behavior in the face of uncertainty. Campbell and Viceira [8] have highlighted the impact of risk aversion on consumption and asset allocation, while Epstein and Schneider [9] introduced long-term uncertainty to analyze risky investment choices. In a similar context, Drechsler and Yaron [10] have incorporated market uncertainty into long-term allocation decisions, underlining the importance of flexible risk aversion. In our model, based on Pham's framework, a concave utility function represents consumption risk aversion, adding a realistic dimension to portfolio management decisions.
To regulate risk-taking, several recent studies have introduced penalties for excessive investment. Korn and Kraft [11] have shown that penalty policies can control investment choices in a context of stochastic volatility. Our model, based on Pham's framework, applies a quadratic penalty to excessive investments. This unique approach favors more prudent management by limiting the risks associated with high investments during periods of large fluctuations. In addition, Ben-Tahar et al. [12] address optimal control in risk-averse consumption contexts by applying advanced stochastic control techniques to adjust strategies in response to frequent economic shocks.
The Pontryagin maximum principle is commonly used to determine optimal investment strategies under uncertainty. Kraft and Seifried [13] have shown that this principle, combined with stochastic volatility models, improves asset management in the face of financial shocks. Zhang and Li [14] have used this same principle for stochastic interest rates, demonstrating that adding dynamics to models enhances portfolio resilience. Although our model retains a constant interest rate, in line with Pham's framework, the application of Pontryagin's principle in a stochastic volatility framework provides a balanced combination of complexity and realism. Hirtle and Kelleher [15] also point out that dynamic optimization strategies, taking into account changing investor preferences, play a crucial role in periods of economic instability, which justifies the dynamic approach of our model.
This literature review highlights the advances made in stochastic optimization models and the contributions of various authors to the integration of dynamic risk factors. Based on Huyên Pham's model of production and consumption, our approach combines risk aversion for consumption and penalization for excessive investment in a context of stochastic volatility. It meets the needs of portfolio management in an uncertain context and makes a unique contribution to the current literature, offering a realistic representation of financial behavior in the face of economic uncertainty.

3. Modeling

We model the investor's decision-making process in a context of uncertainty, characterized by sources of volatility and risk.
Consider a finite time horizon $[0, \mathrm{T}]$ and a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ generated by two independent Brownian motions W_t^1 and W_t^2 representing two sources of uncertainty, and satisfying the usual conditions of being continuous on the right and complete. All stochastic processes are assumed to be welldefined and adapted to this filtered probability space.
Let's say an investor manages a company whose capital value K_t varies according to his investment decisions I_t. He seeks to maximize the value of the company by minimizing the costs associated with investment and consumption C_t while managing debt L_t which depends on the interest rate r and the rate of capital productivity P_t. In this new approach, we incorporate risk aversion, assuming that the investor prefers to avoid excessive fluctuations in wealth. Note that the (k_t, c_t) is a consumption investment strategy.
The main objective is to determine optimal controls k_t (relative investment rate) and c_t (relative consumption rate) so as to meet the investor's expectations of profit, while taking into account economic hazards and risk aversion.

3.1. System dynamics

The stochastic dynamics below describe how investment and consumption choices influence investor wealth and debt over time. At each point in time $t \in [0, T]$ the investor makes decisions to optimize these variables over the given time interval. The dynamics of the system are governed by the following stochastic differential equations (SDEs): for capital K_t debt L_t and net wealth X_t (net wealth = capital - debt).

Lemma 3.1: *The dynamics of the investor's net wealth when faced with investment and intermediate consumption, as well as an interest rate r economic growth rate μ and stochastic volatility rates $\sigma_1(t)$ and $\sigma_2(t)$ evolve according to the following dynamics:*

$$dX_t = X_t\left(k_t\left(\mu - r + \frac{b}{S_t}\right) + (r - c_t)\right)dt + k_t X_t \sigma_1 dW_t^1 + k_t \frac{X_t}{S_t}\sigma_2 dW_t^2 \quad (1)$$

Proof:
In this Lemma, we prove the investor's net wealth model, including both intermediate investment and intermediate consumption, is modeled by the following stochastic differential equation:
Net worth is the difference between the investor's total capital K_t and its debt L_t.
let:

$$X_t = K_t - L_t \quad (2)$$

The capital K_t varies according to investment decisions I_t while debt L_t evolves according to the interest rate r and capital productivity P_t. The stochastic differential equation for capital is given by:

$$dK_t = K_t \frac{dS_t}{S_t} + I_t dt \; K(0) = K_0 \tag{3}$$

With I_t is the investment rate and S_t is the unit price of capital. Using the equation given for dS_t:

$$dS_t = \mu S_t dt + \sigma_1 S_t dW_t^1 \; S(0) = S_0 \tag{4}$$

By replacing (4) in (3) we obtain:

$$dK_t = K_t(\mu S_t dt + \sigma_1 S_t dW_t^1) + I_t dt \tag{5}$$

Debt dynamics L_t is given by:

$$dL_t = rL_t dt - \frac{K_t}{S_t} dP_t + (I_t + C_t)dt \; L(0) = L_0 \tag{6}$$

With P_t capital productivity. Using the equation for dP_t :

$$dP_t = bdt + \sigma_2 dW_t^2 \; P(0) = P_0 \tag{7}$$

By replacing (7) in (6) this gives:

$$dL_t = rL_t dt - \frac{K_t}{S_t}(bdt + \sigma_2 dW_t^2) + (I_t + C_t)dt \tag{8}$$

Finally, by combining (8) and (5) we obtain the dynamics of net worth $X_t = K_t - L_t$ becomes:

$$\begin{cases} dX_t = dK_t - dL_t = \left(K_t \frac{dS_t}{S_t} + I_t dt\right) - \left(rL_t dt - \frac{K_t}{S_t} dP_t + (I_t + C_t)dt\right) & t \in [0, T] \\ \qquad X(0) = X_0 \end{cases} \tag{9}$$

Equation (9) shows how investment, intermediate consumption and stochastic factors (interest rates, volatility, capital productivity) influence net wealth. To understand investment and consumption decisions under uncertainty, we need to simplify stochastic dynamics. Net wealth X_t fluctuates with capital growth, interest rates and Brownian market movements. Investors adjust their decisions to maximize profits and minimize risks. For example, an increase in volatility σ_1 may prompt a reduction in the investment rate k_t. This method can be used to derive optimal strategies via Pontryagin's maximum principle.
Simplifying, we obtain (1):

a) The dynamics of net wealth X_t

$$dX_t = X_t\left(k_t\left(\mu - r + \frac{b}{S_t}\right) + (r - c_t)\right)dt + k_t X_t \sigma_1 dW_t^1 + k_t \frac{X_t}{S_t} \sigma_2 dW_t^2$$

With:

- $k_t = \frac{K_t}{X_t}$ investment rate relative to net worth, where $k_t \in [0, 1]$. This means that investment cannot exceed 100% of the production unit's net worth.
- $c_t = \frac{C_t}{X_t}$ consumption rate relative to net worth, where $c_t \in [0, 1]$ representing the proportion of net worth allocated to consumption.

Thus, the two control variables k_t and c_t belong to the closed interval $[0, 1]$ ensuring that investment and consumption are bounded by net worth X_t.
With:

- $\mu \in \mathbb{R}$ capital growth rate
- $r \in \mathbb{R}^+$ Risk-free interest rate
- $b \in \mathbb{R}^+$ capital productivity rate.
- $\sigma_1, \sigma_2 \in \mathbb{R}^+$Volatility due to Brownian motion W_t^1 and W_t^2
- W_t^1 and W_t^2 two independent Brownian motions defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.
- S_t the price of capital, which follows a stochastic dynamic.

The dynamics of X_t therefore depends on the relative investment rate k_t relative consumption rate c_t as well as uncertainties linked to the price of capital and productivity.

b) Risk aversion

We now introduce a utility function to reflect the investor's risk aversion. Utility is generally a concave function of consumption, for example a CRRA (Constant Relative Risk Aversion) function of the form:

$$U(C_t) = \frac{C_t^{1-\theta}}{1-\theta} \; \theta > 0 \; t \in [0, T] \tag{10}$$

With θ is the risk aversion coefficient its interpretation is as follows:

- $\theta < 1$ The investor is moderately risk-averse, accepting some volatility for higher returns.

- $\theta = 1$ This corresponds to a logarithmic utility function, signifying risk neutrality. The investor evaluates investment options solely on the basis of expected returns.
- $\theta > 1$ The investor is highly risk-averse, favoring safe investments even at the expense of potentially higher returns. The higher θ is high, the greater the risk aversion.

For investment, we use a quadratic penalty for excessive investment, modeled by the parameter γ which controls the investor's aversion to investment risk. This penalty is expressed as an investment cost in the form:

$$\frac{\gamma}{2}(k_t X_t)^2 \ \gamma > 0 \ t \in [0, T] \tag{11}$$

With $\gamma > 0$ is the investment risk aversion coefficient. The higher γ is higher, the more reluctant the investor is to make risky investments. This leads to a stronger aversion to increasing the volatility of the asset portfolio.

3.2. Stochastic retrograde differential equations (SRDE)

To optimize investment and consumption decisions, we define a cost or value function $Y(t, X_t, S_t)$ which follows a backward stochastic differential equation (SRDE):

$$dY_t = -f(t, X_t, S_t, k_t, c_t)dt + q_t^1 dW_t^1 + q_t^2 dW_t^2 \tag{12}$$

With terminal condition $Y_T = h(X_T)$. The variables q_t^1 and q_t^2 represent the stochastic shocks (Brownian motion) influencing the controls k_t and $c_t$determined by minimizing f.
The exogenous factor S_t follows the dynamic:

$$dS_t = \mu S_t dt + \sigma_1 S_t dW_t^1$$

This shows that S_t is affected by Brownian motion W_t^1 thus influencing the dynamics of S_t.

4. Cost function to minimize

The objective of the model is to minimize the instantaneous cost function $g(X_t, k_t, c_t)$ which represents the total cost of investment and consumption and applies to state and control variables over the time interval $[0, T]$. This cost function incorporates two key aspects: risk aversion for consumption, modeled by the parameter θ and a quadratic penalty on investment, controlled by the parameter γ. This quadratic penalty limits excessive investment, enabling prudent, measured management of the investor's capital.

$$g : \mathbb{R}^+ \times [0,1] \times [0,1] \longrightarrow \mathbb{R}$$

Hence the cost functional to be minimized is of the form:

$$J(k_t, c_t) = \mathbb{E}\left[\int_0^T g(X_t, k_t, c_t)dt + h(X_T)|\mathcal{F}_t\right] \tag{13}$$

With:

- $g(X_t, k_t, c_t)$ is the instantaneous cost function that includes consumption and investment costs.
- $h(X_T)$ represents the terminal cost or gain associated with final wealth X_T.

With risk aversion, the instantaneous cost function becomes:

$$g(X_t, k_t, c_t) = \frac{C_t^{1-\theta}}{1-\theta} + \frac{\gamma}{2} I_t^2 \tag{14}$$

In terms of relative consumption and investment rates, we have:

$$g(X_t, k_t, c_t) = \frac{(c_t X_t)^{1-\theta}}{1-\theta} + \frac{\gamma}{2}(k_t X_t)^2 \tag{15}$$

With:

- $c_t X_t$ represents consumption as a function of net wealth
- $k_t X_t$ represents investment as a function of net wealth.

In our case, the functional becomes:

$$J(k_t, c_t) = \mathbb{E}\left[\int_0^T e^{-\rho t}\left(\frac{(c_t X_t)^{1-\theta}}{1-\theta} + \frac{\gamma}{2}(k_t X_t)^2\right) dt|\mathcal{F}_t\right] \tag{16}$$

With $\gamma \in \mathbb{R}^+$penalizes high investments and $\rho \in \mathbb{R}^+$ is the discount rate. Optimal strategies k_t^* and c_t^* minimize this functional while respecting the dynamics of the system. The Pontryagin maximum principle is used to solve this optimal control problem.
We know that a strategy is optimal if:

$$J(k_t^*, c_t^*) = min_{(k_t, c_t) \in A} J(k_t, c_t) \tag{17}$$

To solve this optimal control problem, we use the Pontryagin maximum principle.

5. Stochastic optimization

In this stochastic control framework, the Hamiltonian represents a central function that helps optimize decisions by integrating instantaneous cost and adjoint variables linked to future decisions. These adjoint variables directly influence investment and consumption decisions. The Hamiltonian is defined as:

$$\mathcal{H}(t, X_t, Y_t, , k_t, c_t, q_t^1, q_t^2) = f(t, X_t, S_t, k_t, c_t) + p_t . \mathbb{A}(X_t, k_t, c_t) + q_t^1 . k_t X_t \sigma_1 + q_t^2 . k_t \frac{X_t \sigma_2}{S_t}$$

With:

- p_t the assistant process associated with X_t
- q_t^1 and q_t^2 are the adjoint processes associated with Brownian terms.

$$\mathbb{A}(X_t, k_t, c_t) = X_t \left(k_t \left(\mu - r + \frac{b}{S_t} \right) + (r - c_t) \right) \tag{18}$$

By replacing (19) and the dynamics in $\mathcal{H}$ we obtain:

$$\mathcal{H}(t, X_t, S_t, P_t, k_t, c_t, q_t^1, q_t^2) = e^{-\rho t} \left(\frac{(c_t X_t)^{1-\theta}}{1-\theta} + \frac{\gamma}{2} (k_t X_t)^2 \right) + p_t \left(X_t \left(k_t \left(\mu - r + \frac{b}{S_t} \right) + (r - c_t) \right) \right) + q_t^1 k_t X_t \sigma_1 + q_t^2 k_t \frac{X_t \sigma_2}{S_t} \tag{19}$$

Adjunct variables p_t, q_t^1, q_t^2 follow backward differential equations (BDEs), and their role is crucial in accounting for economic fluctuations and market uncertainties. These variables capture the sensitivity of system states to stochastic shocks represented by Brownian motion W_t^1 and W_t^2.

5.1. Equation for pt

The adjoint variable p_t has the following dynamics:

$$dp_t = -\frac{\partial \mathcal{H}}{\partial X_t} dt + q_t^1 dW_t^2 + q_t^2 dW_t^2 \ t \in [0, T] \tag{20}$$

We get:

Step 1: Derive terms from H $e^{-\rho t} \frac{C_t^{-}}{1-\theta}$ with respect to X_t .

- First term: $e^{-\rho t} \frac{C_t^{1-\theta}}{1-\theta}$

$$\frac{\partial}{\partial X_t} \left(e^{-\rho t} \frac{(c_t X_t)^{1-\theta}}{1-\theta} \right) = e^{-\rho t} . \frac{(1-\theta)(c_t X_t)^{-\theta} . c_t}{1-\theta} = e^{-\rho t} c_t (c_t X_t)^{-\theta} \tag{21}$$

- Second term: $e^{-\rho t} . \frac{\gamma}{2} (k_t X_t)^2$

$$\frac{\partial}{\partial X_t} \left(e^{-\rho t} . \frac{\gamma}{2} (k_t X_t)^2 \right) = e^{-\rho t} . \frac{\gamma}{2} . 2 k_t^2 X_t = e^{-\rho t} \gamma k_t^2 X_t \tag{22}$$

- Third term: $p_t \left(X_t \left(k_t \left(\mu - r + \frac{b}{S_t} \right) + (r - c_t) \right) \right)$

$$\frac{\partial}{\partial X_t} \left(p_t \left(X_t \left(k_t \left(\mu - r + \frac{b}{S_t} \right) + (r - c_t) \right) \right) \right) = p_t \left(k_t \left(\mu - r + \frac{b}{S_t} \right) + (r - c_t) \right) \tag{23}$$

- Fourth term: $q_t^1 k_t X_t \sigma_1$

$$\frac{\partial}{\partial X_t} (q_t^1 k_t X_t \sigma_1) = q_t^1 k_t \sigma_1 \tag{24}$$

- Fifth term: $q_t^2 k_t \frac{X_t \sigma_2}{S_t}$

$$\frac{\partial}{\partial X_t} \left(q_t^2 k_t \frac{X_t \sigma_2}{S_t} \right) = q_t^2 k_t \frac{\sigma_2}{S_t} \tag{25}$$

Step 2: by grouping (21), (22) (23), (24), (25) we obtain

$$dp_t = -\left(e^{-\rho t} \left(c_t (c_t X_t)^{-\theta} + \gamma k_t^2 X_t \right) + p_t \left(k_t \left(\mu - r + \frac{b}{S_t} \right) + (r - c_t) \right) + q_t^1 k_t \sigma_1 + q_t^2 k_t \frac{\sigma_2}{S_t} + q_t^1 dW_t^1 + q_t^2 dW_t^2 \right) dt \ t \in \ [0, T] \tag{26}$$

This equation (26) describes the impact of X_t on system dynamics via adjoint covariables p_t, q_t^1, q_t^2 and controls k_t, c_t.

5.2. Equations for $\mathbf{q_t^1}, \mathbf{q_t^1}$ and terminal conditions

The adjoint co-variables q_t^1 and q_t^2 measure the Hamiltonian's sensitivity to stochastic shocks dW_t^1 and dW_t^2. They are calculated by deriving the Hamiltonian with respect to these processes. Thus, we obtain:

- $\frac{\partial H}{\partial dW_t^1} = k_t X_t \sigma_1$ (27)
- $\frac{\partial H}{\partial dW_t^2} = \frac{k_t X_t \sigma_2}{S_t}$ (28)

The adjoint co-variables depend on the control k_t state X_t volatilities σ_1 and σ_2 and the external factor S_t.
Terminal conditions set the value of co-variables at the final horizon T. For p_T we have:

$$p_T = \frac{\partial g(X_T)}{\partial X_T} \tag{29}$$

If no explicit function is given for $g(X_T)$ we often assume p_T indicating that there is no additional gain or cost to T. Similarly, for $q_T^1 = 0$ and $q_T^2 = 0$ if no shock is expected at the end.

5.3. Minimization condition and retrograde equations

Let's assume that the set of all admissible strategies is denoted by not A
Definition: an admissible strategy is a pair of adapted processes $(k_t, c_t) \in A$ such that the following conditions are satisfied:

1) The pair (k_t, c_t) is progressively t-measurable, i.e. both processes are adapted to filtration $\mathcal{F}_t$ and depend measurably on the information available at each instant t.
2) The expected values of the squared variables k_t (investment rate) and c_t (consumption rate) are finite for all $t > 0$.
3) $\mathbb{E}[|k_t|^2] < \infty, \mathbb{E}[|c_t|^2] < \infty \ \forall \ t > 0$

Furthermore, for all stochastic volatilities σ_1 and σ_2 it must be the case that:

$$\int_0^T k_t^2 \, \sigma_1(t) dt < \infty \ et \ \int_0^T c_t^2 \, \sigma_1(t) dt < \infty$$

This criterion imposes a limit on the scale of investment k_t and consumption c_t based on market volatilities, ensuring that strategies are not overly risky in highvolatility environments.

1) For any admissible pair (k_t, c_t) the wealth process X_t defined by the dynamic wealth equation in the article, with $X(0) = x_0 > 0$ has a unique solution for each trajectory.

To guarantee the consistency of the model, the existence and uniqueness of the solutions of the wealth EDS are ensured by assuming the regularity of the control functions and the continuity of the Brownian processes. Thus, for any pair of admissible strategies (k_t, c_t) respecting the measurability conditions, a unique solution of the wealth equation exists.
To minimize the cost function, Pontryagin's maximum conditions require that the partial derivatives of the Hamiltonian with respect to k_t and c_t to be zero.
Optimum investment strategy k_t^*:
The derivative of the Hamiltonian with respect to k_t^* gives:
Let's calculate $\frac{\partial \mathcal{H}}{\partial k_t}$ by deriving each term containing k_t separately.

1) Derivative of term : $e^{-\rho t} \frac{(c_t X_t)^{1-\theta}}{1-\theta}$

$$\frac{\partial}{\partial k_t}\left(e^{-\rho t} . \frac{\gamma}{2} (k_t X_t)^2 \right) = e^{-\rho t} \gamma k_t X_t^2 \tag{30}$$

2) Derivative of term : $p_t \left(X_t \left(k_t \left(\mu - r + \frac{b}{S_t} \right) + (r - c_t) \right) \right)$

$$\frac{\partial}{\partial k_t} \left(p_t \left(X_t \left(k_t \left(\mu - r + \frac{b}{S_t} \right) + (r - c_t) \right) \right) \right) = p_t X_t \left(\left(\mu - r + \frac{b}{S_t} \right) \right) \tag{31}$$

3) Derivative of term : $q_t^1 k_t X_t \sigma_1$

$$\frac{\partial}{\partial k_t} (q_t^1 k_t X_t \sigma_1) = q_t^1 X_t \sigma_1 \tag{32}$$

4) Derivative of term : $q_t^2 k_t \frac{X_t \sigma_2}{S_t}$

$$\frac{\partial}{\partial k_t} \left(q_t^2 k_t \frac{X_t \sigma_2}{S_t} \right) = q_t^2 \frac{X_t \sigma_2}{S_t} \tag{33}$$

The partial derivative equation with respect to $k_t$is obtained by combining $(30), (31), (32), (33)$ we have:

$$e^{-\rho t}\gamma k_t X_t^2 + p_t X_t\left(\left(\mu - r + \frac{b}{S_t}\right)\right) + q_t^1 X_t \sigma_1 + q_t^2 \frac{X_t \sigma_2}{S_t} \quad t \in [0, T] \tag{34}$$

Resolution for k_t
To find the optimal value of k_{t^*} we pose $\frac{\partial H}{\partial k_t} = 0$

$$e^{-\rho t}\gamma k_t X_t^2 + p_t X_t\left(\left(\mu - r + \frac{b}{S_t}\right)\right) + q_t^1 X_t \sigma_1 + q_t^2 \frac{X_t \sigma_2}{S_t} = 0$$

Simplifying the equation
Our aim is to solve this equation for k_t. To simplify, let's divide each term by X_t^2 (assuming that $X_t \neq 0$) to isolate k_t.

$$e^{-\rho t}\gamma k_t + \frac{p_t X_t\left(\left(\mu - r + \frac{b}{S_t}\right)\right)}{X_t} + \frac{q_t^1 \sigma_1}{X_t} + q_t^2 \frac{\sigma_2}{S_t X_t} = 0$$

By isolating k_t we obtain:

$$k_t^* = -\frac{1}{e^{-\rho t}.\gamma}\left[p_t\left(\left(\mu - r + \frac{b}{S_t}\right) + q_t^1 \sigma_1 + q_t^2 \frac{\sigma_2}{S_t}\right)\right] \quad t \in [0, T] \tag{35}$$

This strategy depends on the adjoint co-variables q_t^1, q_t^2, ρ the intertemporal discount rate and economic parameters: capital growth μ risk-free interest rate r capital productivity b and volatilities associated with stochastic shocks σ_1 and σ_2 to adjust investment according to profitability and risk.
Optimum consumption strategy c_t^*:
The derivative of the Hamiltonian with respect to c_t^*.
We will differentiate each of these terms in relation to c_t.

1) Derivative of term: $e^{-\rho t}\frac{(c_t X_t)^{1-\theta}}{1-\theta}$

$$\frac{\partial}{\partial c_t}\left(e^{-\rho t}\frac{(c_t X_t)^{1-\theta}}{1-\theta}\right) = e^{-\rho t}.\frac{\partial}{\partial c_t}\left(\frac{(c_t X_t)^{1-\theta}}{1-\theta}\right)$$

We get:

$$\frac{\partial}{\partial c_t}\left(\frac{(c_t X_t)^{1-\theta}}{1-\theta}\right) = \frac{1}{1-\theta}.(1-\theta)(c_t X_t)^{-\theta}.X_t = X_t^{1-\theta} c_t^{-\theta}$$

So...,

$$\frac{\partial}{\partial c_t}\left(\frac{(c_t X_t)^{1-\theta}}{1-\theta}\right) = e^{-\rho t}.X_t^{1-\theta} c_t^{-\theta} \tag{36}$$

Derivative of the term $-p_t X_t c_t$
We have:

$$\frac{\partial}{\partial c_t}(-p_t X_t c_t) = -p_t X_t \tag{37}$$

By combining (36) and (37) we obtain:

$$e^{-\rho t}.X_t^{1-\theta} c_t^{-\theta} - p_t X_t \tag{38}$$

The Pontryagin maximum condition implies that for c_t is optimal, we must have:

$$\frac{\partial H}{\partial c_t} = 0$$

So..,
$e^{-\rho t}.X_t^{1-\theta} c_t^{-\theta} - p_t X_t = 0$

Isolate $c_t^{-\theta}$:

$$c_t^{-\theta} = \frac{p_t X_t}{e^{-\rho t}.X_t^{1-\theta}} \tag{39}$$

(39) Perhaps simplify into:

$$c_t^{-\theta} = \frac{p_t}{e^{-\rho t}.X_t^{-\theta}} \tag{40}$$

Taking the reciprocal powers to isolate c_t we obtain:

$$c_t = \left(\frac{p_t}{e^{-\rho t}.X_t^{-\theta}}\right)^{\frac{1}{-\theta}} \tag{41}$$

$$c_t^* = \left(\frac{p_t e^{-\rho t}}{X_t^{\theta}}\right)^{\frac{1}{\theta}} \ t \in [0, T] \tag{42}$$

This result represents the optimal consumption strategy as a function of the adjoint covariate p_t discount rate ρ and wealth X_t adjusted for the risk aversion parameter θ.

5.4. Final system of equations

We therefore have the following system:

1) Forward equation for X_t :

$$dX_t = X_t\left(k_t\left(\mu - r + \frac{b}{S_t}\right) + (r - c_t)\right)dt + k_t X_t \sigma_1 dW_t^1 + k_t \frac{X_t}{S_t}\sigma_2 dW_t^2 \ t \in [0, T]$$

2) Retrograde equation for Y_t :

$$dY_t = -f(t, X_t, S_t, k_t, c_t)dt + q_t^1 dW_t^1 + q_t^2 dW_t^2$$

With

$$f(t, X_t, S_t, k_t, c_t) = e^{-\rho t}\left(\frac{(c_t X_t)^{1-\theta}}{1-\theta} + \frac{\gamma}{2}(k_t X_t)^2\right)$$

3) Adjunct equation for p_t :

$$dp_t = -\left[e^{-\rho t}\left(\frac{(c_t X_t)^{1-\theta}}{1-\theta} + \frac{\gamma}{2}(k_t X_t)^2\right) + p_t\left(k_t\left(\mu - r + \frac{b}{S_t}\right) + (r - c_t)\right) + k_t\sigma_1 q_t^1 + q_t^2\frac{k_t\sigma_2}{S_t}\right]dt + q_t^1 dW_t^1 + q_t^2 dW_t^2 \tag{43}$$

These three equations form the complete FBSDE system to be solved numerically.

5.5. Convexity and transversality

The cost functional $J(k_t, c_t)$ is convex, guaranteeing a unique optimal solution. Convexity is ensured by the quadratic term $\frac{\gamma}{2}k_t^2 X_t^2$ with $\gamma > 0$ and the linear term $c_t X_t$. The positive Hessian confirms this convexity, proving that the strategies do indeed minimize cost.
The transversality conditions ensure that the solution remains valid until the end of the time horizon. $[0, T]$. The terminal condition $p_T = g'(X_T)$ guarantees that the model takes into account the final costs or gains, without any loss of consistency over time. T.

6. Numerical example and simulations

This section analyzes how different parameters affect optimal investment and consumption strategies.

6.1. The effect of wealth X_t on investment k_t^* and consumption c_t^* optimal

Here, we analyze the impact of wealth on investment and consumption decisions. To do this, we:

$$\frac{\partial k_t^*}{\partial X_t} = \frac{p_t}{e^{-\rho t}\gamma}.\frac{\mu - r + \frac{b}{S_t}}{X^2}$$

And

$$\frac{\partial c_t^*}{\partial X_t} = -\frac{p_t e^{-\rho t}}{X_t^{\theta+1}}$$

Figure 1 below shows that wealth X_t positively influences the sensitivity of investment k_t^* and consumption c_t^*. As wealth increases, these variables are more influenced by
variations in X_t especially for low values of X_t where the impact is more marked.
When: $p_t = 0.5$; $\rho = 0.03$; $\gamma = 0.1$; $\mu = 0.05$; $r = 0.02$; $b = 0.1$; $S_t = 1.0$; $\theta = 2.0$ and 0 $X_t \in [0.1, 1.0]$

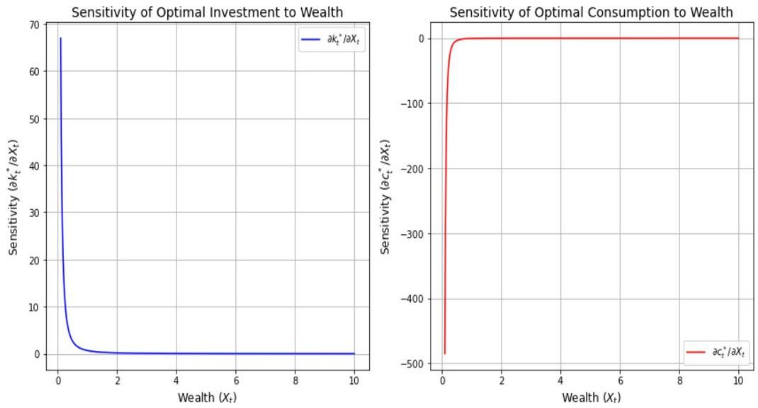

Fig. 1: The Effects of Wealth Sensitivity X_t on Optimal Investment k_t^* and Consumption c_t^*.

6.2. Effects of the expected return parameter μ on optimal investment k_t^* and optimal consumption c_t^*

Figure 2 below shows that the expected return μ positively influences the sensitivity of the optimal investment k_t^*. As expected return increases, investment sensitivity becomes more pronounced, encouraging the investor to increase his investment share. On the other hand, the sensitivity of consumption c_t^* remains relatively constant despite variations in μ suggesting that optimal consumption is less influenced by expected return.

When : $p_t = 0.5$; $\rho = 0.03$; $\gamma = 0.1$; $X_t = 5.0$; $r = 0.02$; $b = 0.1$; $S_t = 1.0$; $\theta = 2.0$ and $\mu \in [0.01, 1.0]$

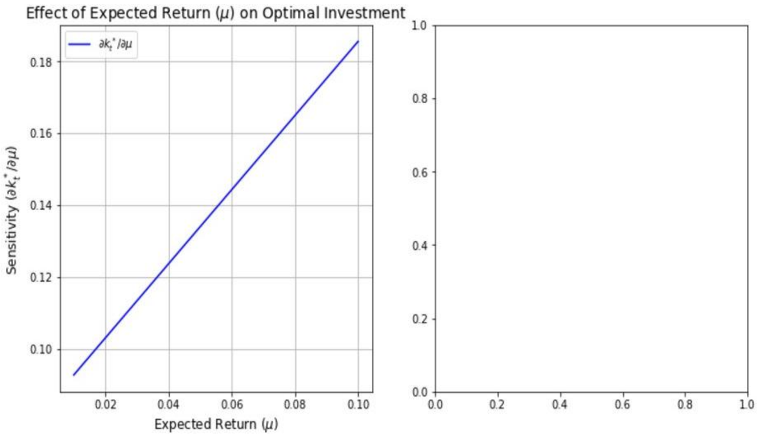

Fig. 2: Impact of Expected Return on Optimal Investment.

6.3. Effects of capital productivity b on optimal investment and consumption k_t^* and optimal consumption c_t^*

Figure 3 below shows that capital productivity b positively influences the sensitivity of optimal investment k_t^*. As b increases, the incentive to invest becomes stronger, as higher productivity makes investment more profitable. On the other hand, the sensitivity of consumption c_t^* is little affected by variations in b. This suggests that optimal consumption is relatively stable in the face of productivity changes, being determined more by accumulated wealth or personal preferences.

When: $p_t = 0.5$; $\rho = 0.03$; $\gamma = 0.1$; $X_t = 5.0$; $r = 0.02$; $\mu = 0.05$; $S_t = 1.0$; $\theta = 2.0$ and $b \in [0.01, 0.2]$

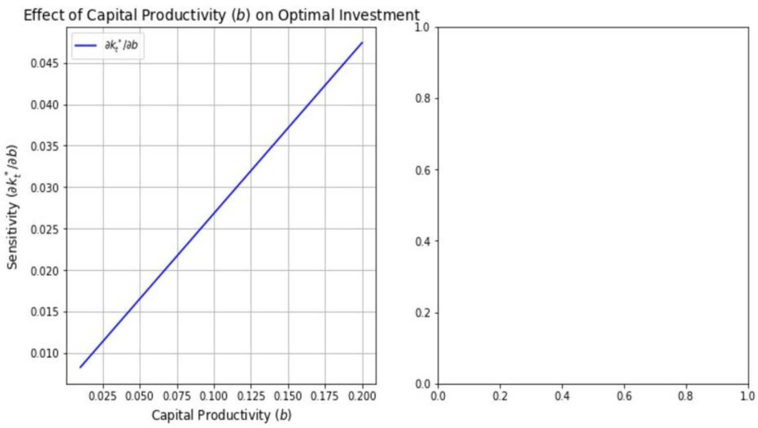

Fig. 3: Impact of Capital Productivity on Optimal Investment and Consumption.

6.4. The impact of risk aversion on consumption θ and the quadratic penalty for excessive investment γ

Figures 4 below illustrate that risk aversion has a negative θ negatively influences the sensitivity of optimal consumption c_t^*. Increased aversion reduces consumption, confirming that more cautious behavior leads to lower optimal consumption. Similarly, quadratic

penalization γ affects investment sensitivity k_t^*. Stronger penalization reduces optimal investment, favoring a more conservative approach and limiting the risks associated with excessive investment.
When: $p_t = 0.5$; $\rho = 0.03$; $\mu = 0.0.5$; $X_t = 5.0$; $r = 0.02$; $b = 0.1$; $S_t = 1.0$
$\theta \in [1.0, 3.0]$ and $\gamma \in [0.05, 0.]$

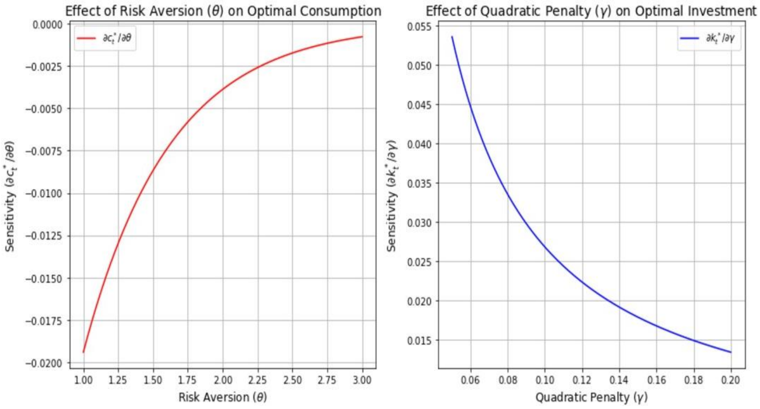

Fig. 4: The Impact of Penalization on Investment and Aversion on Consumption.

6.5. Impact of volatility on optimal investment k_t^* and optimal consumption c_t^*

Figures 5 and 6 below show the impact of different volatilities on investment and consumption. An increase in volatility σ_1 and σ_2 lead to a slight increase in optimal investment k_t^* reflecting cautious adaptation to return opportunities despite the risks. Conversely, optimal consumption c_t^* decreases with volatility, indicating conservative resource management in an uncertain environment. This behavior is consistent with heightened risk aversion, where priority is given to wealth preservation rather than immediate consumption. These results underline the importance of a balanced strategy in the face of economic fluctuations.
When: $p_t = 0.5$; $\rho = 0.03$; $\mu = 0.0.5$; $X_t = 5.0$; $r = 0.02$; $b = 0.1$; $S_t = 1.0$; $\theta = 2$; $\gamma = 0.1$; σ_1 *and* $\sigma_2 \in [0.1, 0.5]$

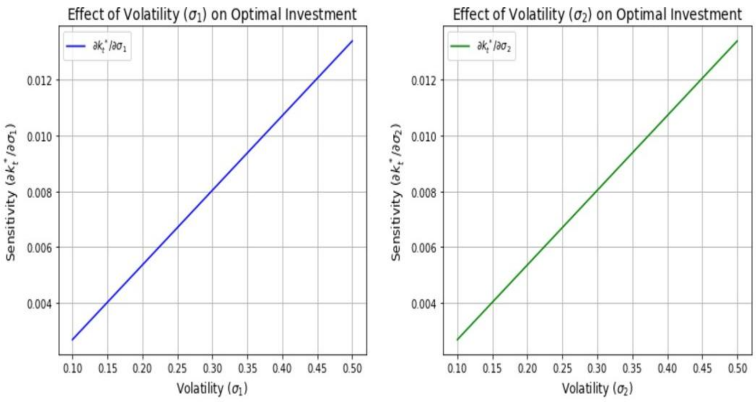

Fig. 5: Impact of Volatility σ_1 on Investment and Consumption.

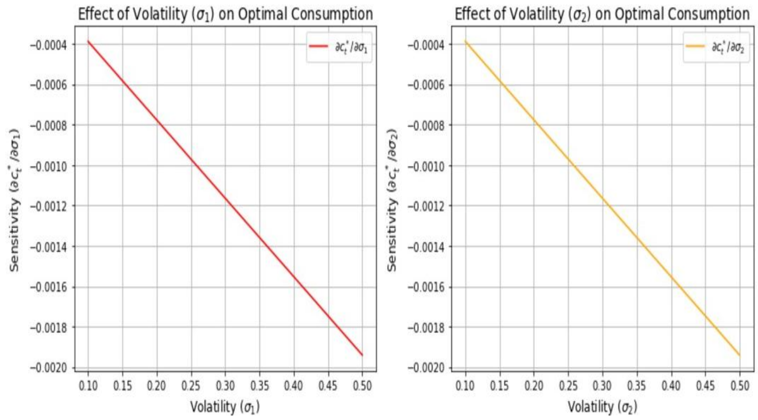

Fig. 6: Impact of Volatility on σ_2 on Investment and Consumption.

7. Conclusion and outlook

This paper proposes an innovative approach to portfolio management under uncertainty, using stochastic optimization based on Pontryagin's maximum principle. By incorporating stochastic volatility and a distinction between risk aversion for consumption and investment, our model offers a balanced solution for guiding financial decisions during periods of economic fluctuation. Numerical simulations demonstrate the importance of prudent and measured asset management, especially in the presence of variable volatilities.

For future work, it would be interesting to test the model in specific market environments, such as periods of financial crisis or phases of high volatility, in order to better assess its robustness. In addition, a theoretical extension could incorporate variable interest rates or other macroeconomic factors, such as exchange rate shocks or commodity price variations. Finally, a possible development would be to adapt the model for institutional portfolios or fund managers, taking into account regulatory constraints. These perspectives could reinforce the practical and theoretical relevance of the model in various financial contexts.

References

[1] Markowitz, H. (1952). "Portfolio Selection." The Journal of Finance, 7(1), 77-91. https://doi.org/10.1111/j.1540-6261.1952.tb01525.x.
[2] Merton, R. C. (1969). "Lifetime Portfolio Selection under Uncertainty: The Continuous Time Case." Review of Economics and Statistics, 51(3), 247-257. https://doi.org/10.2307/1926560.
[3] Merton, R. C. (1971). "Optimum Consumption and Portfolio Rules in a Continuous-Time Model." Journal of Economic Theory, 3(4), 373-413. https://doi.org/10.1016/0022-0531(71)90038-X.
[4] Pham, H. (2009). Optimization and stochastic control applied to finance. Springer.
[5] Benth, F. E. (2004). "On Stochastic Volatility Modeling in Energy Markets." Journal of Mathematical Finance, 8(2), 47-73.
[6] Sandjo, R., et al. (2015). "Optimal Portfolio Choice under Stochastic Volatility and Stochastic Interest Rate." Applied Mathematics and Computation, 254, 36-52.
[7] Zhang, Z., & Shreve, S. E. (2006). "Investment and Consumption under Stochastic Volatility." Finance and Stochastics, 10(1), 95-118.
[8] Campbell, J. Y., & Viceira, L. M. (2002). Strategic Asset Allocation: Portfolio Choice for Long-Term Investors. Oxford University Press. https://doi.org/10.1093/0198296940.001.0001.
[9] Epstein, L. G., & Schneider, M. (2016). "Ambiguity and Asset Markets." Annual Review of Financial Economics, 8(1), 355-383.
[10] Drechsler, I., & Yaron, A. (2011). "What's Vol Got to Do with It: Asset Prices and Volatility Risk." Journal of Financial Economics, 100(2), 241-265.
[11] Korn, R., & Kraft, H. (2004). "On Portfolio Optimization under Stochastic Interest Rates." Mathematical Finance, 14(4), 611-636. https://doi.org/10.1111/j.0960-1627.2004.00197.x.
[12] Ben-Tahar, I., Djellout, H., & Labidi, H. (2022). "Optimal InvestmentConsumption Strategies with Stochastic Volatility." Finance Research Letters, 45, 102315.
[13] Kraft, H., & Seifried, F. T. (2021). "Optimal Investment Strategies under Stochastic Volatility." Journal of Economic Dynamics and Control, 132, 104212.
[14] Zhang, D., & Li, S. (2020). "Optimal Consumption and Portfolio Choice with Stochastic Volatility." Economic Modelling, 90, 559-572.
[15] Hirtle, B., & Kelleher, S. (2019). "Portfolio Optimization and Institutional Risk Management." Journal of Risk and Financial Management, 12(1), 1-17.